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Inverse Problems for Dynamic Systems: Classification and Solution Methods

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ABSTRACT

The inverse problems for motions of dynamic systems of which are described by system of the ordinary differential equations are examined. The classification of such type of inverse problems is given. It was shown that inverse problems can be divided into two types: synthesis inverse problems and inverse problems of measurement (recognition). Each type of inverse problems requires separate approach to statements and solution methods. The regularization method for obtaining of stable solution of inverse problems was suggested. In some cases, instead of recognition of inverse problems solution, the estimation of solution can be used. Within the framework of this approach, two practical inverse problems of measurement are considered.

Keywords: Inverse Problems; Dynamic Systems; Classification; Regularization; Estimation

1. Introduction

Many important practical problems related to mechanical systems, economical characteristics etc. can be reduced to investigation of inverse problems for dynamic systems [1-3].

For the authentic forecast of motion of dynamic systems it is necessary to use the adequate mathematical description of physical process as an example [4,5]. One way to obtain such mathematical description is the solution of an inverse problems [4,5]. Besides, it is important that this description was steady against small changes of initial data of an inverse problem. The decisions can be accepted on the basis of forecast within conditions of uncertainty.

Side by side with problems of the motion forecast of dynamical systems there exist the practical inverse problems by the purpose of which solutions is the definition or estimation of real properties of dynamic systems or real loads on these systems [2,3,6,7]. The problems of technical diagnostics [3], medical diagnostics [8], diagnostics of the economic characteristics [9], study of the real external loads on dynamic systems [10,11] may be included into such inverse problems. Some problems related to obtaining of the decisions are also reduced to the solution of inverse problems for dynamic systems [7]. The equations of inverse problems can be linear, nonlinear, as well as in partial derivatives, with delay etc.

So it is necessary to take classification of inverse problems to obtain more convenient statements of problems, more effective methods of study and more right interpretation of approximate solutions.

The given paper is limited to consideration only linear inverse problems for dynamic systems with the concentrated parameters.

Practically all linear inverse problems for dynamic systems of such type can be reduced to the solution of Fredgolm (Volltera) integral equation of the first kind [12]:

$$\int_a^b K(x,\tau) z(\tau) \mathrm{d}\tau = u(x), \ c \le x \le d, \tag{1}$$

where $K(x,\tau)$ is given kern, $u(x)$ is given function.

Equation (1) can be represented in the form

$$\tilde{A}z = \tilde{u}, \tag{2}$$

where $\tilde{A}$ is continuous operator, $z \in Z$ is the function to be found, $\tilde{u} \in U$ is given function.

2. Classification of Inverse Problems

Let's assume, that the functional spaces Z, U are metric spaces.

The function $\tilde{u}$ in Equation (2) is presented approximately. In some cases $\tilde{u}$ is defined from experiment [4,5], in other cases this function represents the approximation of the given function [12,15].

The error of initial function $\tilde{u}$ is given

$$\rho_U\left(\tilde{u}, u_{ex}\right) \le \delta, \tag{3}$$

where u_{ex} is exact initial function, $\rho_U\left(\cdot,\cdot\right)$ is distance between arguments.

In most cases practical problems determine such functional spaces Z,U that the operator $\tilde{A}$ is compact operator [13]. Such property of the operator $\tilde{A}$ represents significant difficulties in the solution of Equation (2), as the inverse operator $\tilde{A}^{-1}$ is not continuous.

The set of the possible solutions of Equation (2) for this reason is unbounded and is defined as

$$Q_\delta = \left\{z : z \in Z, \rho_U\left(\tilde{u}, \tilde{A}z\right) \le \delta\right\}. \tag{4}$$

In a number of inverse problems for dynamic systems an ultimate goal of research is the definition of such solution of Equation (2) which satisfies Equation (2) with accuracy of the initial data δ. In other words, the function $\tilde{z}$ from set of the possible solutions Q_δ is determined only: $\tilde{z} \in Q_\delta$. Other purposes in the solution of an inverse problem are not pursued. For example, in inverse problems [5,14] a ultimate goal is the obtaining of the adequate mathematical description of motion of dynamic system for the purposes of reception of the authentic forecast of motion. Any function from set of the possible solutions $\tilde{z} \in Q_\delta$ together with the operator $\tilde{A}$ satisfies the specified requirements $\rho_U\left(\tilde{u}, \tilde{A}\tilde{z}\right) \le \delta$. If instead operator $\tilde{A}$ is used an exact operator A_{ex} with function $\tilde{z}$, then even the similar result will not be received with guarantee.

The change of size of the initial data error δ does not change a situation. The investigation of problem solution depending on decrease of initial data error δ has no sense as the limiting function does not contain any additional information.

It is necessary to distinguish two opportunities:

1) function of initial data $\tilde{u}$ is obtained as a result of approximation of experimental measurements u_{exp} [4, 5];

2) function $\tilde{u}$ of initial data $\tilde{u}$ is obtained as a result of approximation some dependence which are given a priori u_g [15].

In the first case the solution of an inverse problem can be used at synthesis of the adequate mathematical description of dynamic systems [4,5].

In the second case the solution of an inverse problem allows to receive the approximate control of dynamic system for obtaining of given motion of this system with accuracy δ [15]. Sometimes such inverse problems are named as problems of output restoration of dynamic systems [11].

The important characteristic in the solution of inverse problems is the size of an error of the solution depending on size of an error of the initial data. However, in this case the function in relation of which the error must be calculated is not defined. If such function represents the exact solution $\tilde{z}_{ex}$ of the equation $\tilde{A}z = u_{ex}$, then in this case the size of an error of inverse problem solution (in relation to function $\tilde{z}_{ex}$) does not have any meaning. Any function from unbounded set of the possible solutions together with the operator $\tilde{A}$ gives the necessary result at further use. The size of the specified error can reach unnatural sizes for engineering calculations (100% - 10000%).

The inverse problems of such type in some works are named as ***inverse problems of synthesis*** [11,12].

Some expansions of a class of inverse problems of such type on a case when the operator $\tilde{A}$ in the equation can be selected from the beforehand given class of the operators $\tilde{A} \in K_A$ is examined in works [11,12]. Various non-standard statements of inverse problems in this case are possible [16]. The basic purpose of research of inverse problems in the specified statements is the research of the solutions at the various additional requirements: the unitary solution, steadiest solution, most convenient solution, optimum solution for the purposes of the forecast etc. [16]. The inverse problems of such type are named as ***inverse problems of synthesis for a class of models*** [11,16].

The qualitative distortion of the solution of an inverse problem of synthesis can be caused by an uncontrollable error in the initial data [17].

Essentially, other situation arises in research of inverse problems, when result of the solution of an inverse problem is the obtaining of the information about the exact solution of an inverse problem (2) [7,18]. The problems of technical diagnostics, medical diagnostics, acceptance of the decisions in conditions under uncertainties may be included into such problems [1,2,3,7,10].

At the solution of inverse problems of such type it is necessary to have the information about the error of the initial data $\tilde{A}$, $\tilde{u}$ in relation to the exact initial data A_{ex}, u_{ex}. For a case of metric spaces $Z,U, z \in Z, \tilde{u} \in U$ of such information will be the size of an error:

$$\rho_A\left(\tilde{A}, A_{ex}\right) \le \frac{\sup\limits_{z\in Z} \rho_U\left(\tilde{A}z, A_{ex}z\right)}{\rho_Z\left(z,0\right)} = h, \qquad \rho_Z\left(z,0\right) \ne 0, \quad \rho_U\left(u_{ex}, \tilde{u}\right) \le \delta. \tag{5}$$

The set of possible solutions such inverse problem (with account of operator $\tilde{A}$ error) is defined as:

$$Q_{\delta,h} = \left\{z : z \in Z, \rho_U\left(\tilde{A}z, \tilde{u}\right) \le h\rho_U\left(z,0\right) + \delta\right\}. \tag{6}$$

The set $Q_{\delta,h}$ is unbounded for the same reason and obviously includes the set Q_δ.

It is impossible to use a priori given function as initial function $\tilde{u}$ here.

The any element $z \in Q_{\delta,h}$ does not represent interest from the point of view of obtaining of the information about the exact solution of Equation (2). Besides, here is not represented the opportunity to study the solution behavior at reduction of error of the initial data size. The size δ is determined by quality of the measuring equipments and cannot be changed.

The basic difficulty at study of the solutions of such inverse problem consists in absence of the information about properties of the exact operator A_{ex}. Therefore, the error h of the operator $\tilde{A}$ in relation to the exact operator A_{ex} can be defined rather approximately with big overestimate. On the basis of this it is problematic the consideration of limiting transitions at $\delta \to 0, h \to 0$. It is necessary to interpret the approximate solutions of inverse problems at the fixed sizes of initial data error.

It is clearly evident that the error of the solution of an inverse problem of such type has decisive meaning.

The inverse problems of the specified type are named as ***inverse problems of measurements*** (interpretation).

Such a classification of inverse problems is not common and can be replaced by a different classification. However, this classification is useful because it avoids some methodological errors.

3. Methods of Solution

It is well known that the inverse problems are unstable with respect to small changes of the initial data and for their solution are used special regularized algorithms [13, 19].

Let us assume that functional spaces Z, U are Banach spaces. The error of function $\tilde{u}$ from the function u_g has the size:

$$\left\| \tilde{u} - u_g \right\|_U \le \delta . \tag{7}$$

The set of possible solutions of synthesis inverse problems for fixed operator $\tilde{A}$ is denoted by Q_δ: $Q_\delta = \left\{ z : z \in Z,\ \left\| \tilde{A}z - \tilde{u} \right\|_U \le \delta \right\}$.

The solution of following extreme problem can be accepted as stable solution of synthesis inverse problem:

$$\Omega[\tilde{z}] = \inf_{z \in Q_\delta \cap Z_1} \Omega[z], \tag{8}$$

where $\Omega[z]$ is stabilizing functional which is defined on Z_1 (set Z_1 is everywhere dense into Z) [13].

The obtaining of function $\tilde{z}$ in a synthesis inverse problem is important. The operator $\tilde{A}$ together with solution $\tilde{z}$ provides the stable adequate mathematical description of process [4,5].

In some cases the "simplest" solution can be chosen as solution of synthesis inverse problem [20].

In inverse problems of interpretation it is necessary additionally to take into account the inaccuracy of operator $\tilde{A}$ with respect to the exact operator A_{ex} [11, 13].

Let us suppose that the characteristic of an error of the operator $\tilde{A}$ is given if the operator A_{ex} is linear operator:

$$\left\| \tilde{A} - A_{ex} \right\|_{Z \to U} \le h . \tag{9}$$

The set of possible solution of Equation (1) is necessary to extend to set $Q_{\delta,h}$ taking into account the inaccuracy of the operator $\tilde{A}$:

$$Q_{\delta,h} = \left\{ z : z \in Z, \left\| \tilde{A}z - u_\delta \right\|_U \le h \left\| z \right\|_Z + \delta \right\}. \tag{10}$$

The algorithm of the solution of the incorrect problem with approximate operator was proposed in work [19] which is based on Tikhonov regularization method [13].

The statement of such interpretation inverse problem can be formulated for obtaining of the stable solution as follows: it is necessary to find an element $z_{est} \in Q_{\delta,h}$ on which the greatest lower bound of some stabilizing functional $\Omega[z]$ is reached

$$\inf_{z \in Q_{\delta,h} \cap Z_1} \Omega[z] = \Omega[z_{est}], \tag{11}$$

where Z_1 is subset of Z, on subset Z_1 has been defined stabilizing functional $\Omega[z]$, the set Z_1 is everywhere dense in Z [13].

Sometimes in inverse problems of measurement it is enough to find the value $\Omega[z_{est}]$ only.

One of the important characteristics for the specified algorithm is the size h of an error. The obtaining of h represents significant difficulties, as the exact operator A_{ex} is unknown.

As a result of the solution of interpretation inverse problem it is necessary to accept some approximation $\tilde{z}$ to the exact solution z_{ex} of Equation (1) or its estimation z_{est} in the beforehand certain sense [7,10].

The functional $\Omega[z]$ can characterize the chosen property of the exact solution (for example, smoothness). The approximate solution will give the estimation from below of exact solution on a degree of smoothness. If a functional $\Omega[z]$ characterizes a deviation of the approximate solution from the given function z_{ap}, then the solution of an extreme problem (11) will give function from set $Q_{\delta,h} \cap Z_1$ closest to function z_{ap}. Thus, it is obvious that z_{ap} should not belong to the set $Q_{\delta,h} \cap Z_1$.

The estimation of a deviation of the operator $\tilde{A}$ from exact operator A_{ex} cannot be done effectively during a consideration of interpretation problems.

For overcoming the specified difficulties it is offered to accept the following ***hypothesis***: for the exact solution z_{ex} of the equation $A_{ex}z = u_{ex}$ the inequality is valid

$$\Omega[z_{ex}] \ge \Omega[\tilde{z}], \tag{12}$$

where $\tilde{z}$ is regularized solution of Equation (1) with approximate operator $\tilde{A}$ and approximate initial data

$\tilde{u}$, $\Omega[z]$ is stabilizing functional [13].

Theorem. If $\Omega[z]$ is stabilizing functional [13] then estimation $\Omega[\tilde{z}]$ of exact solution exists and is stable with respect to small change of initial data.

The offered ***hypothesis*** is not supposed to use the size of inaccuracy h of the operator $\tilde{A}$ from the exact operator A_{ex} in the solution of inverse problems of interpretation.

The satisfaction of an inequality (12) is obvious if the operators $A_{ex}, \tilde{A}$ are linear. For the nonlinear operator A_{ex} (that in the greater degree corresponds to a reality) the inequality (12) can be proved by properties of the approximate operators which are used in calculations [21].

Use of the offered hypothesis allows to receive various objective estimations of the exact solution z_{ex} of inverse problems such as (1) that is important in recognition problems [7,10]. Moreover, the size h is not used in calculations. At $\tilde{A} \to A_{ex}$ the estimation of function z_{ex} will be more exact. For definition of parameter regularization it is possible to use the usual discrepancy method [13] where the value h is absent.

The proposed approach was used to solve two practical inverse problems of measurement [7,10].

Offered algorithm can be used also for estimation of real unknown parameters of physical processes by identification method.

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