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# Resolvable Spaces and Compactifications

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## ABSTRACT

This paper deals with spaces such that their compactification is a resolvable space. A characterization of space such that its one point compactification (resp. Wallman compactification) is a resolvable space is given.

**Keywords:** Resolvable Space; Alexandroff Compactification; Wallman Compactification

## 1. Introduction

In 1943, Hewitt [1] has introduced the notion of resolvable space as follows: A topological space is said to be resolvable if it has two disjoint dense subsets. Hence a topological space $X$ is resolvable if and only if $X$ is written as a union of two disjoint dense subsets. Hewitt in [1] has also called a topological space $X$ maximally irresolvable if each dense subset of $X$ is open. Nowadays, maximally irresolvable spaces are called submaximal spaces.

Recently, Belaid *et al.* [2], were interested in spaces such that their compactifications are submaximal. They proved that if $X$ is a topological space and $K(X)$ is a compactification of *X*, then the following statements are equivalent:

1) $K(X)$ is submaximal.

2) For each dense subset *D* of *X*, the following properties hold:

a) *D* is co-finite in *K(X)*;

b) for each $x \in K(X) - D$, $\{x\}$ is closed.

It is clear that a compactification of resolvable spaces is resolvable. Hence the following question is natural:

"Characterize spaces *X* such that a compactification $K(X)$ of *X* is a resolvable space?"

The first section is devoted to a brief study of spaces *X* such that their compactification is a resolvable space. The particular case of the one-point compactification is given.

The purpose of the second section is to give an intrinsic topological characterization of spaces *X* such that the Wallman compactification $wX$ of *X* is a resolvable space.

## 2. Resolvable Space and Compactifications

First, recall that a compactification of a topological space $X$ is a couple $(K(X), e)$, where $K(X)$ is a compact space and $e : X \to K(X)$ is a continuous embedding (*e* is a continuous one-to-one map and induces a homeomorphism from $X$ onto $e(X)$) such that $e(X)$ is a dense subspace of $K(X)$. When a compactification $(K(X), e)$ of $X$ is given, $X$ will be identified with $e(X)$ and assumed to be dense in $K(X)$.

Let us give some basic facts about space such that its compactification is a resolvable space.

**Lemma 2.1** *Let X be a topological space*, $K(X)$ *be a compactification of X and* $A$ *be a subset of* $K(X)$. *If X is an open set of* $K(X)$, *then the following statements are equivalent*:

1) *A is a dense subset of* $K(X)$;

2) $A \cap X$ *is a dense subset of* $X$.

*Proof.* 1)$\Rightarrow$ 2) Let $O$ be an open set of $X$. Since $X$ is an open set of $K(X)$, $O$ is an open set of $K(X)$. Hence $O \cap A \neq \varnothing$. Thus $O \cap (A \cap X) \neq \varnothing$; so that $A \cap X$ is a dense set of $X$.

2)$\Rightarrow$ 1) Let $U$ be an open set of $K(X)$. Since $U \cap X$ is a non-empty open set of $X$, $(A \cap X) \cap (U \cap X) \neq \varnothing$. Then $U \cap A \neq \varnothing$. Therefore $A$ is a dense set of $K(X)$.

An immediate consequence of Lemma 1.1 is the following.

**Proposition 2.2** *Let X be a topological space and* $K(X)$ *be a compactification of* $X$. *If X is an open set of* $K(X)$, *then the following statements are equivalent*:

1) $X$ *is resolvable*;

2) $K(X)$ is resolvable.

Let us recall the construction of the one-point compactification: For any non-compact space $X$ the one-point compactification of $X$ is obtained by adding one extra point $\infty$ (called a point at infinity) and defining the open sets of $\tilde{X} = X \bigcup \{\infty\}$ to be the open sets of $X$ together with the sets of the form $O \bigcup \{\infty\}$, where $O$ is an open set of $X$ such that $X - O$ is a closed compact set of $X$. The one point compactification $\tilde{X}$ of $X$ is also called the Alexandroff compactification of $X$ [3].

The following result characterizes space such that its one point compactification is a resolvable space. Its proof follows immediately from Proposition 2.2; thus it is omitted.

**Proposition 2.3** *Let X be a non-compact topological space. Then the following statements are equivalent*:

1) The one-point compactification $\tilde{X}$ of $X$ is resolvable;

2) $X$ is resolvable.

## 3. Resolvable Space and Wallman Compactification

First, recall that the Wallman compactification of $T_1$-space was introduced, in 1938, by Wallman [4] as follows:

Let $\mathcal{P}$ be a class of subsets of a topological space $X$ which is closed under finite intersections and finite unions.

A $\mathcal{P}$-filter on $X$ is a collection $\mathcal{F}$ of nonempty elements of $\mathcal{P}$ with the properties:

1) $\mathcal{F}$ is closed under finite intersections;

2) $P_1 \in \mathcal{F}, P_1 \subseteq P_2$ implies $P_2 \in \mathcal{F}$.

A $\mathcal{P}$-ultrafilter is a maximal $\mathcal{P}$-filter. When $\mathcal{P}$ is the class of closed sets of $X$, then the $\mathcal{P}$-filters are called closed filters.

The points of the Wallman compactification $wX$ of a space $X$ are the closed ultrafilters on $X$. For each closed set $D \subseteq X$, define $D^*$ to be the set
$D^* = \{\mathcal{F} \in wX \mid D \in \mathcal{F}\}$. Thus
$\mathcal{C} = \{D^* \mid D \text{ is a closed set of } X\}$ is a base for the closed sets of a topology on $wX$.

Let $U$ be an open set of $X$, we define
$U^* = \{\mathcal{F} \in wX \mid F \subseteq U \text{ for some } F \text{ in } \mathcal{F}\}$, it is easily seen that the class $\{U^* \mid U \text{ is an open set of } X\}$ is a base for open sets of the topology of $wX$. The following properties of $wX$ are frequently useful:

**Proposition 3.1** *Let $X$ be a $T_1$-space and $wX$ the Wallman compactification of $X$. Then the following statements hold*:

1) $wX$ is a $T_1$-space;

2) For $x \in X$ and
$\Phi(x) = \{F \mid F \text{ is a closed set of } X \text{ and } x \in F\}$. Then $\Phi$

is an embedding of $X$ into $wX$ ($\Phi(x)$ will be identified to $x$).

3) If $U$ is an open set of $X$, then
$wX - U^* = (X - U)^*$.

4) If $U_1$ and $U_2$ are two open sets of $X$, then $(U_1 \bigcap U_2)^* = U_1^* \bigcap U_2^*$ and $(U_1 \bigcup U_2)^* = U_1^* \bigcup U_2^*$.

Recall that Kovar in [5] has characterized space with finite Wallman compactification remainder as following:

**Proposition 3.2** *Let $X$ be a $T_1$-space. Then the following statements are equivalent*:

1) $\mathrm{Card}(wX - X) = n$;

2) There exists a collection of $n$ pairwise disjoint non-compact closed sets of $X$ and every family of non-compact pairwise disjoint closed sets of $X$ contain at most $n$ elements.

The following proposition follows immediately from Proposition 3.2 and Proposition 3.1-1).

**Proposition 3.3** *Let X be a $T_1$-space and $n \in \mathbb{N}$ such that every family of non-compact pairwise disjoint closed sets of X contains at most $n$ elements. Then X is resolvable if and only if $wX$ is resolvable.*

The following lemma has been given in [2] as Remark 4.5 and Remark 4.9.

**Lemma 3.4** *Let $X$ be a $T_1$-space. Then the following properties hold*:

1) If $F$ is a closed non-compact subset of $X$, then there exists $\mathcal{F} \in wX - X$ such that $F \in \mathcal{F}$.

2) $\mathcal{F} \in wX - X$. Then for each $F \in \mathcal{F}$, $F$ is a non-compact closed set of $X$.

The following result is an immediate consequence of Lemma 3.4.

**Corollary 3.5** *Let $X$ be a $T_1$-space, $wX$ be the Wallman compactification of X and $U$ be an open set of X. Then the following statements are equivalent*:

1) $U \subsetneq U^*$;

2) There exists a non compact closed set $F$ of $X$ such that $F \subseteq U$.

Now, we are in a position to give a characterization of spaces such that their Wallman compactification is resolvable.

**Theorem 3.6** *Let $X$ be a $T_1$-space. Then the following statements are equivalent*:

1) The Wallman compactification $wX$ of $X$ is resolvable;

2) There exist two disjoint subsets $D_1$ and $D_2$ of $X$ such that:

a) $X = D_1 \bigcup D_2$.

b) For $i \in \{1, 2\}$ and for each non empty open set $O \subseteq D_i$, there exists a non compact closed set $F$ of $X$ such that $F \subseteq O$.

*Proof.* 1) $\Rightarrow$ 2) Since $wX$ is a resolvable space, there exist two disjoint dense sets $A_1$ and $A_2$ of $wX$ such that $wX$ is the union of $A_1$ and $A_2$. Set $D_1 = X \bigcap A_1$ and $D_2 = X \bigcap A_2$.

Let $i \in \{1,2\}$ and $O$ be a non empty open set of $X$ such that $O \subseteq D_i$. Set $j \in \{1,2\}$ such that $j \neq i$. Since $A_j$ is a dense subset of $wX$, $O^* \cap A_j \neq \varnothing$. Now,

$$O^* \cap \left( A_j \cap X \right) = O \cap D_j = \varnothing \quad \text{implies that}$$

$O^* \cap \left( A_j \cap wX - X \right) \neq \varnothing$. It follows that there exists

$\mathcal{F} \in O^* \cap \left( A_j \cap wX - X \right)$, and thus $O \subsetneq O^*$. According to Corollary C4 there exists a non compact closed set $F$ of $X$ such that $F \in \mathcal{F}$ and $F \subseteq O$.

$2) \Rightarrow 1)$ Let $D_1, D_2$ be two disjoint subsets of $X$ satisfying the condition b) and such that $X = D_1 \cup D_2$. Let $i \neq j$ in $\{1,2\}$ and we define

$A_i = D_i \cup \{\mathcal{F} \in wX - X \mid \text{there exists } F \in \mathcal{F}$

and an open set $O$ of $X$ such that $F \subseteq O \subseteq D_j\}$.

It is immediate that $A_i \cap A_j = \varnothing$.

Now, let $U$ be a open set of $X$. We consider two cases:

Case 1: $U \cap D_i \neq \varnothing$. Then $U \cap A_i \neq \varnothing$. So $U^* \cap A_i \neq \varnothing$.

Case 2: $U \cap D_i = \varnothing$. Then $U \subseteq D_j$. By condition b), there exists a non compact closed $F$ of $X$ such that $F \subseteq U$. Let $\mathcal{F} \in wX - X$ such that $F \in \mathcal{F}$. Hence $\mathcal{F} \in U^* \cap A_i$. Thus $U^* \cap A_i \neq \varnothing$.

Therefore $A_i$ is a dense set of $wX$; so that $wX$ is a resolvable space.

**Example 3.7** *Let $\mathbb{Q}$ be the set of all rational numbers equipped with the natural topology $T_{\mathbb{Q}}$. Let $X = \mathbb{Q} \times \{0\} \cup \mathbb{Q} \times \{1\}$ equipped with the topology*

$T = \{U \times \{0\} \mid U \in T_{\mathbb{Q}}\} \cup \{\mathbb{V} \times \{1\} \mid V \in T_{\mathbb{Q}}\}$. *It is immediate that the topological space $X$ satisfies the condition* 2)

*of the Theorem* 2.6. *Then $wX$ is a resolvable space.*
The previous result incites us to ask the following question.

**Question 3.8** *Let X be a space. We denote by $\beta_\omega X$ (resp. $\beta X$) the $T_0$-compactifcation of X introduced by Herrlich in* [6] (*resp. the Stone Cech compactification*). *When is $\beta_\omega X$ (resp. $\beta X$) a resolvable space?*

## 4. Acknowledgements

## REFERENCES

[1] E. Hewitt, "A Problem of Set Theoretic Topology," *Duke Mathematical Journal*, Vol. 10, No. 2, 1943, pp. 309-333. doi:10.1215/S0012-7094-43-01029-4

[2] K. Belaid, L. Dridi and O. Echi, "Submaximal and Door Compactifications," *Topology and Its Applications*, Vol. 158, No. 15, 2011, pp. 1969-1975. doi:10.1016/j.topol.2011.06.039

[3] J. L. Kelly, "General Topology," D. Van. Nostrand Company, Inc., Princeton, 1955.

[4] H. Wallman, "Lattices and Topological Spaces," *Annals of Mathematics*, Vol. 39, No. 1, 1938, pp. 112-126. doi:10.2307/1968717

[5] M. M. Kovar, "Which Topological Spaces Have a Weak Reflection in Compact Spaces?" *Commentationes Mathematicae Universitatis Carolinae*, Vol. 36, No. 3, 1995, pp. 529-536.

[6] H. Herrlich, "Compact $T_0$-Spaces and $T_0$-Compactifications," *Applied Categorical Structures*, Vol. 1, No. 1, 1993, pp. 111-132. doi:10.1007/BF00872990