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Wright Type Hypergeometric Function and Its Properties

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ABSTRACT

Let s and z be complex variables, $\Gamma(s)$ be the Gamma function, and $(s)_v = \frac{\Gamma(s+v)}{\Gamma(s)}$ for any complex v be the generalized Pochhammer symbol. Wright Type Hypergeometric Function is defined (Virchenko *et al*. [1]), as: ${}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\,\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!}$, where $\tau \in \mathbb{R}_+ = (0,\infty)$; $a,b,c \in \mathbb{C}$; $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, $\mathrm{Re}(c)>0$; which is a direct generalization of classical Gauss Hypergeometric Function ${}_2F_1(a,b;c;z)$. The principal aim of this paper is to study the various properties of this Wright type hypergeometric function ${}_2R_1(a,b;c;\tau;z)$; which includes differentiation and integration, representation in terms of ${}_pF_q$ and in terms of Mellin-Barnes type integral. Euler (Beta) transforms, Laplace transform, Mellin transform, Whittaker transform have also been obtained; along with its relationship with Fox H-function and Wright hypergeometric function.

Keywords: Euler Transform; Fox H-Function; Wright Type Hypergeometric Function; Laplace Transform; Mellin Transform; Whittaker Transform; Wright Hypergeometric Function

1. Introduction and Preliminaries

The Gauss Hypergeometric Function is defined [2] as:

$$F(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\,k!}z^k, \text{; and} \quad (|z|<1, c \neq 0,-1,-2,\cdots) \tag{1.1}$$

The Generalized Hypergeometric Function, in a classical sense has been defined [3] by

$$\begin{aligned}{}_pF_q\left[\begin{matrix}a_1,\cdots,a_p;z\\ b_1,\cdots,b_q\end{matrix}\right] &= {}_pF_q\left[a_1,\cdots,a_p;b_1,\cdots,b_q;z\right]\\ &= \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!},\\ &(p=q+1, |z|<1);\end{aligned} \tag{1.2}$$

and no denominator parameter equal to zero or negative integer.

E. Wright [4] has further extended the generalization of the hypergeometric series in the following form

$$\begin{aligned}{}_p\Psi_q(z) &= \sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+\beta_1 n)\cdots\Gamma(\alpha_p+\beta_p n)}{\Gamma(\rho_1+\mu_1 n)\cdots\Gamma(\rho_q+\mu_q n)}\frac{z^n}{n!};\end{aligned} \tag{1.3}$$

where β_r and μ_t are real positive numbers such that

$$1+\sum_{t=1}^{q}\mu_t - \sum_{r=1}^{p}\beta_r > 0.$$

When β_r and μ_t are equal to 1, Equation (1.3) differs from the generalized hypergeometric function ${}_pF_q$ by a constant multiplier only.

The generalized form of the hypergeometric function has been investigated by Dotsenko [5], Malovichko [6] and one of the special cases considered by Dotsenko [5] as

$$ {}_2R_1^{\omega,\mu}(z) = {}_2R_1(a,b;c;\omega,\mu;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)} \frac{z^n}{n!} \quad (1.4) $$

and its integral representation expressed as

$$ {}_2R_1^{\omega,\mu}(z) = \frac{\Gamma(c)\mu}{\Gamma(c-b)\Gamma(b)} \cdot \int_0^1 t^{\mu b-1}\left(1-t^{\mu}\right)^{c-b-1}\left(1-zt^{\omega}\right)^{-a} \mathrm{d}t, \quad (1.5) $$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$. This is the analogue of Euler's formula for the Gauss's hypergeometric function [3]. In 2001 Virchenko *et al.* [1] defined the said Wright Type Hypergeometric Function by taking $\frac{\omega}{\mu} = \tau > 0$ in (1.4) as

$$ {}_2R_1^{\tau}(z) = {}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+\tau k)k!} z^k;\ \tau > 0, |z| < 1. \quad (1.6) $$

If $\tau = 1$, then (1.3) reduces to a Gauss's hyper- geometric function. Galue *et al.* [7] and Virchenko *et al.* [1] investigated some properties of the function ${}_2R_1(a,b;c;\tau;z)$.

The following well-known facts have been prepared for studying various properties of the function ${}_2R_1(a,b;c;\tau;z)$.

- *Euler* (*Beta*) *transform* (Sneddon [8]):

The Euler transform of the function $f(z)$ is defined as

$$ B\{f(z):a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)\mathrm{d}z. \quad (1.7) $$

- *Laplace transform* (Sneddon [8]):

The Laplace transform of the function $f(z)$ is defined as

$$ L\{f(z)\} = \int_0^{\infty} \mathrm{e}^{-sz} f(z)\mathrm{d}z. \quad (1.8) $$

- *Mellin transform* (Sneddon [8]):

The Mellin transform of the function $f(x)$ is defined as

$$ M\left[f(x);s\right] = \int_0^{\infty} x^{s-1} f(x)\mathrm{d}x = f^*(s), \quad \mathrm{Re}(s) > 0, \quad (1.9) $$

then

$$ f(x) = M^{-1}\left[f^*(s);x\right] = \frac{1}{2\pi\mathrm{i}} \int_L f^*(s) x^{-s}\mathrm{d}s. \quad (1.10) $$

- Wright generalized hypergeometric function (Srivastava and Manocha [9]), denoted by ${}_p\Psi_q$, is defined as

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\cdots,(\alpha_p,A_p); & z\\ (\beta_1,B_1),\cdots,(\beta_q,B_q); & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i k)}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j k)} \frac{z^k}{k!} \quad (1.11) $$

$$ = H_{p,q+1}^{1,p}\left[-z\middle| \begin{matrix}(1-\alpha_1,A_1),\cdots,(1-\alpha_p,A_p)\\ (0,1),(1-\beta_1,B_1),\cdots,(1-\beta_q,B_q)\end{matrix}\right], \quad (1.12) $$

where $H_{p,q}^{m,n}\left[z\middle| \begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ (b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right]$ denotes the Fox *H*-function [10].

2. Basic Properties of the Function ${}_2R_1(a,b;c;\tau;z)$

Theorem 2.1

If $a,b,c \in \mathbb{C}; \mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, \mathrm{Re}(c) > 0; \tau \in \mathbb{N}$ then

$$ c \cdot {}_2R_1(a,b;c;\tau;z) = c \cdot {}_2R_1(a,b;c+1;\tau;z) + \tau z \frac{\mathrm{d}}{\mathrm{d}z} {}_2R_1(a,b;c+1;\tau;z) \quad (2.1.1) $$

$$ {}_2R_1(a,b;c-\tau;\tau;z) - {}_2R_1(a,b-1;c-\tau;\tau;z) = a\tau z \frac{\Gamma(c-\tau)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a+1)_k \Gamma(b-1+\tau+\tau k)}{\Gamma(c+\tau k)} \frac{z^k}{k!}, \quad (2.1.2) $$

$(b \neq 1)$.

In particular,

$$ \left[{}_2F_1(a,b;c-1;z) - {}_2F_1(a,b-1;c-1;z)\right] \frac{(c-1)}{az} = {}_2F_1(a+1,b;c;z) \quad (2.1.3) $$

Proof.

$$ \begin{aligned} &\tau z \frac{\mathrm{d}}{\mathrm{d}z} {}_2R_1(a,b;c+1;\tau;z) \\ &= \tau z \frac{\Gamma(c+1)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+1+\tau k)} \frac{k \cdot z^{k-1}}{k!} \\ &= \frac{c\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+\tau k)} \frac{z^k}{k!} - \frac{c\Gamma(c+1)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+1+\tau k)} \frac{z^k}{k!} \\ &= c \cdot {}_2R_1(a,b;c;\tau;z) - c \cdot {}_2R_1(a,b;c+1;\tau;z), \end{aligned} $$

which is the (2.1.1).

Now,

$$_2R_1\left(a,b;c-\tau;\tau;z\right)-{}_2R_1\left(a,b-1;c-\tau;\tau;z\right)$$

$$=\frac{\Gamma(c-\tau)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\,\Gamma(b+\tau k)}{\Gamma(c-\tau+\tau k)}\frac{z^k}{k!}$$

$$-\frac{\Gamma(c-\tau)}{\Gamma(b-1)}\sum_{k=0}^{\infty}\frac{(a)_k\,\Gamma(b-1+\tau k)}{\Gamma(c-\tau+\tau k)}\frac{z^k}{k!}$$

$$=\frac{\Gamma(c-\tau)}{\Gamma(b-1)}$$

$$\cdot\sum_{k=0}^{\infty}\frac{(a)_k\,\Gamma(b-1+\tau k)}{\Gamma(c-\tau+\tau k)}\frac{z^k}{k!}\left[\frac{(b-1+\tau k)}{(b-1)}-1\right]$$

$$=\frac{\Gamma(c-\tau)}{(b-1)\Gamma(b-1)}\sum_{k=1}^{\infty}\frac{(a)_k\,\Gamma(b-1+\tau k)}{\Gamma(c-\tau+\tau k)}\frac{z^k}{(k-1)!}\tau$$

$$=\frac{a\tau z}{(b-1)}$$

$$\cdot\left(\frac{\Gamma(c-\tau)}{\Gamma(b-1)}\sum_{k=1}^{\infty}\frac{(a+1)_{k-1}\,\Gamma(b-1+\tau+\tau(k-1))}{\Gamma(c+\tau(k-1))}\frac{z^{k-1}}{(k-1)!}\right)$$

$$=\tau z\frac{a}{(b-1)}\frac{\Gamma(c-\tau)}{\Gamma(b-1)}$$

$$\cdot\sum_{k=0}^{\infty}\frac{(a+1)_k\,\Gamma(b-1+\tau+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!}$$

$$=a\tau z\frac{\Gamma(c-\tau)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a+1)_k\,\Gamma(b-1+\tau+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!},$$

This is the proof of (2.1.2).

For $c\neq 1$ and substituting $\tau=1$ in above result, this will immediately leads to particular case (2.1.3). □

Theorem 2.2 1) If

$a,b,c,\delta\in\mathbb{C};$

$\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,\mathrm{Re}(c)>0,\mathrm{Re}(\delta)>0$

and $\tau\in\mathbb{N}$ then

$$\frac{\Gamma(c+\delta)}{\Gamma(\delta)}\cdot\int_0^1 u^{c-1}(1-u)^{\delta-1}\,{}_2R_1\left(a,b;c;\tau;zu^{\tau}\right)\mathrm{d}u=\Gamma(c)\,{}_2R_1\left(a,b;c+\delta;\tau;z\right)\tag{2.2.1}$$

2) If

$a,b,c,\delta,\lambda\in\mathbb{C};$

$\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,\mathrm{Re}(c)>0,\mathrm{Re}(\delta)>0$

and $\tau\in\mathbb{N}$ then

$$\frac{\Gamma(c+\delta)}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{c-1}\,{}_2R_1\left(a,b;c;\tau;\lambda(s-t)^{\tau}\right)\mathrm{d}s=(x-t)^{\delta+c-1}\,\Gamma(c)\,{}_2R_1\left(a,b;c+\delta;\tau;\lambda(x-t)^{\tau}\right).\tag{2.2.2}$$

3)

$a,b,c\in\mathbb{C};$

$\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,\mathrm{Re}(c)>0$

and $\tau\in\mathbb{N}$ then

$$\int_0^z t^{c-1}\,{}_2R_1\left(a,b;c;\tau;\omega t^{\tau}\right)\mathrm{d}t=\frac{z^c}{c}\,{}_2R_1\left(a,b;c+1;\tau;\omega z^{\tau}\right).\tag{2.2.3}$$

In particular,

$$\int_0^z t^{c-1}\,{}_2F_1\left(a,b;c;\omega t\right)\mathrm{d}t=\frac{z^c}{c}\,{}_2F_1\left(a,b;c+1;\omega z\right).\tag{2.2.4}$$

Proof.

1)

$$\frac{\Gamma(c+\delta)}{\Gamma(\delta)}\int_0^1 u^{c-1}(1-u)^{\delta-1}\,{}_2R_1\left(a,b;c;\tau;zu^{\tau}\right)\mathrm{d}u$$

$$=\frac{\Gamma(c+\delta)}{\Gamma(\delta)}$$

$$\cdot\int_0^1 u^{c-1}(1-u)^{\delta-1}\left(\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\,\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{\left(zu^{\tau}\right)^k}{k!}\right)\mathrm{d}u$$

$$=\frac{\Gamma(c+\delta)}{\Gamma(\delta)}$$

$$\cdot\sum_{k=0}^{\infty}\left(\frac{(a)_k\,\Gamma(b+\tau k)\Gamma(c)}{\Gamma(b)\Gamma(c+\tau k)}\frac{z^k}{k!}\int_0^1 u^{c+\tau k-1}(1-u)^{\delta-1}\mathrm{d}u\right)$$

$$=\Gamma(c)\left\{\frac{\Gamma(c+\delta)}{\Gamma(b)}\sum_{k=0}^{\infty}\left(\frac{(a)_k\,\Gamma(b+\tau k)}{\Gamma(c+\delta+\tau k)}\frac{z^k}{k!}\right)\right\}$$

$$=\Gamma(c)\,{}_2R_1\left(a,b;c+\delta;\tau;z\right).$$

which concludes the proof of (2.2.1). □

2)

$$\frac{\Gamma(c+\delta)}{(x-t)^{\delta-1}\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{c-1}\,{}_2R_1\left(a,b;c;\tau;\lambda(s-t)^\tau\right)\mathrm{d}s$$
$$=\frac{\Gamma(c+\delta)}{\Gamma(\delta)}\int_t^x\left(\frac{(x-t)-(s-t)}{(x-t)}\right)^{\delta-1}(s-t)^{c-1}\,{}_2R_1\left(a,b;c;\tau;\lambda(s-t)^\tau\right)\mathrm{d}s$$
$$=\frac{\Gamma(c+\delta)}{\Gamma(\delta)}\int_0^1(1-u)^{\delta-1}u^{c-1}(x-t)^{c-1}\,{}_2R_1\left(a,b;c;\tau;\lambda\left(u(x-t)\right)^\tau\right)(x-t)\mathrm{d}u,$$
$$\left(\text{applying the transformation formula } u=\frac{s-t}{x-t}\right)$$
$$=\frac{\Gamma(c)}{\Gamma(b)}\frac{\Gamma(c+\delta)}{\Gamma(\delta)}\int_0^1(1-u)^{\delta-1}u^{c-1}(x-t)^c\left(\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{\lambda^k u^{\tau k}(x-t)^{\tau k}}{k!}\right)\mathrm{d}u$$
$$=(x-t)^c\frac{\Gamma(c+\delta)}{\Gamma(\delta)}\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{\lambda^k(x-t)^{\tau k}}{k!}\left(\int_0^1(1-u)^{\delta-1}u^{c+\tau k-1}\mathrm{d}u\right)$$
$$=(x-t)^c\Gamma(c)\left(\frac{\Gamma(c+\delta)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\delta+\tau k)}\frac{\lambda^k(x-t)^{\tau k}}{k!}\right)$$
$$=(x-t)^c\Gamma(c)\,{}_2R_1\left(a,b;c+\delta;\tau;\lambda(x-t)^\tau\right).$$

Therefore,

$$\frac{\Gamma(c+\delta)}{\Gamma(\delta)}\int_t^x(x-s)^{\delta-1}(s-t)^{c-1}\,{}_2R_1\left(a,b;c;\tau;\lambda(s-t)^\tau\right)\mathrm{d}s=(x-t)^{\delta+c-1}\Gamma(c)\,{}_2R_1\left(a,b;c+\delta;\tau;\lambda(x-t)^\tau\right).$$

Which is the proof of (2.2.2). □

3)

$$\int_0^z t^{c-1}\,{}_2R_1\left(a,b;c;\tau;\omega t^\tau\right)\mathrm{d}t=\int_0^z t^{c-1}\left(\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{\left(\omega t^\tau\right)^k}{k!}\right)\mathrm{d}t=\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{\omega^k}{k!}\left(\int_0^z t^{c+\tau k-1}\mathrm{d}t\right)$$
$$=\frac{z^c}{c}\left\{\frac{\Gamma(c+1)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+1+\tau k)}\frac{\left(\omega z^\tau\right)^k}{k!}\right\}=\frac{z^c}{c}\,{}_2R_1\left(a,b;c+1;\tau;\omega z^\tau\right).$$

This leads the proof of (2.2.3).

On putting $\tau=1$, in the above expression immediately leads to (2.2.4). □

Theorem 2.3

If $a,b,c\in\mathbb{C};\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,\mathrm{Re}(c)>0,$ then

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\left[z^{c-1}\,{}_2R_1\left(a,b;c;\tau;\omega z^\tau\right)\right]=z^{c-m-1}\frac{\Gamma(c)}{\Gamma(c-m)}\,{}_2R_1\left(a,b;c-m;\tau;\omega z^\tau\right)\tag{2.3.1}$$

Proof.

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\left[z^{c-1}\,{}_2R_1\left(a,b;c;\tau;\omega z^\tau\right)\right]=\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\left[\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{\omega^k z^{\tau k+c-1}}{k!}\right]$$
$$=\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{\omega^k(\tau k+c-1)(\tau k+c-2)\cdots(\tau k+c-m)z^{\tau k+c-m-1}}{k!}$$
$$=z^{c-m-1}\frac{\Gamma(c)}{\Gamma(c-m)}\left\{\frac{\Gamma(c-m)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c-m+\tau k)}\frac{\left(\omega z^\tau\right)^k}{k!}\right\}=z^{c-m-1}\frac{\Gamma(c)}{\Gamma(c-m)}\,{}_2R_1\left(a,b;c-m;\tau;\omega z^\tau\right).$$

This establishes (2.3.1).

3. Representation of Wright Type Hypergeometric Function ${}_2R_1(a,b;c;\tau;z)$ in Terms of the Function ${}_pF_q$

Using the definition ${}_2R_1(a,b;c;\tau;z)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!}$, and taking $\tau=q\in\mathbb{N}$ we have

$$
\begin{aligned}
&{}_2R_1(a,b;c;q;z)\\
&=\sum_{k=0}^{\infty}\frac{(a)_k(b)_{qk}}{(c)_{qk}}\frac{z^k}{k!}=\sum_{k=0}^{\infty}\frac{(a)_k\prod_{i=1}^{q}\left(\frac{b+i-1}{q}\right)_k q^{qk}}{\prod_{j=1}^{q}\left(\frac{c+j-1}{q}\right)_k q^{qk}}\frac{z^k}{k!}\\
&={}_{q+1}F_q\left[\begin{matrix}a,\frac{b}{q},\frac{b+1}{q},\cdots,\frac{b+q-1}{q};\\ \frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\end{matrix}\ z\right] \qquad (3.1)\\
&={}_{q+1}F_q\left[\begin{matrix}a,\Delta(q;b);\\ \Delta(q;c);\end{matrix}\ z\right],
\end{aligned}
$$

where $\Delta(q;b)$ is a q-tuple $\frac{b}{q},\frac{b+1}{q},\cdots,\frac{b+q-1}{q}$; $\Delta(q;c)$ is a q-tuple $\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q}$.

Convergence criteria for generalized hyperfeometric function

$$
{}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\cdots,\alpha_p;\\ \beta_1,\beta_2,\cdots,\beta_q;\end{matrix}\ z\right]=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{z^k}{k!}:
$$

1) If $p\le q$, the function ${}_pF_q$ converges for all finite z.

2) If $p=q+1$, the function ${}_pF_q$ converges for $|z|<1$ and diverges for $|z|>1$.

3) If $p>q+1$, the function ${}_pF_q$ is divergent for $|z|\ne 0$.

4) If $p=q+1$, the function ${}_pF_q$ is absolutely convergent on the circle $|z|=1$ if

$$
\mathrm{Re}\left(\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i\right)>0.
$$

4. Mellin-Barnes Integral Representation of ${}_2R_1(a,b;c;\tau;z)$

Theorem 4.1 *Let*

$$
\tau\in\mathbb{R}_+=(0,\infty);a,b,c\in\mathbb{C};\mathrm{Re}(a)>0,\\
\mathrm{Re}(b)>0,\mathrm{Re}(c)>0.
$$

Then ${}_2R_1(a,b;c;\tau;z)$ *is represented by the Mellin-Barnes integral*

$$
\begin{aligned}
&{}_2R_1(a,b;c;\tau;z)\\
&=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\frac{1}{2\pi i}\int_L\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\tau s)}{\Gamma(c-\tau s)}(-z)^{-s}\,ds
\end{aligned}
\qquad (4.1.1)
$$

where $|\arg(z)|<\pi$; *the contour of integration beginning at* $-i\infty$ *and ending at* $+i\infty$, *and intended to separate the poles of the integrand at* $s=-k,\ k=0,1,\cdots$ *to the left and all the poles at* $s=n+a,\ n=0,1,\cdots$ *as well as* $s=\frac{n+b}{\tau},\ n=0,1,\cdots$ *to the right.*

Proof. We shall use the sum of residues at the poles $s=-k,\ k=0,1,\cdots$ to obtain the integral of (4.1.1).

$$
\begin{aligned}
&{}_2R_1(a,b,c,\tau;z)\\
&=\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!}\\
&=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!}\\
&=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\sum_{k=0}^{\infty}(-1)^k\frac{1}{\Gamma(1+k)}\frac{\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\Gamma(a+k)(-z)^k
\end{aligned}
\qquad (4.1.2)
$$

Now,

$$
\begin{aligned}
&=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\frac{1}{2\pi i}\int_L\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\tau s)(-z)^{-s}}{\Gamma(c-\tau s)}ds\\
&=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\sum_{k=0}^{\infty}\underset{s=-k}{res}\left[\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\tau s)(-z)^{-s}}{\Gamma(c-\tau s)}\right]\\
&=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\\
&\cdot\sum_{k=0}^{\infty}\lim_{s\to-k}\left(\frac{\pi(s+k)}{\sin\pi s}\frac{1}{\Gamma(1-s)}\Gamma(a-s)\frac{\Gamma(b-\tau s)}{\Gamma(c-\tau s)}(-z)^{-s}\right)\\
&=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\sum_{k=0}^{\infty}(-1)^k\frac{1}{\Gamma(1+k)}\frac{\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\Gamma(a+k)(-z)^k
\end{aligned}
\qquad (4.1.3)
$$

(4.1.2) and (4.1.3) completes the proof of (4.1.1). □

5. Integral Transforms of ${}_2R_1(a,b;c;\tau;z)$

In this section we discussed some useful integral transforms like Euler transforms, Laplace transform, Mellin transform and Whittaker transform.

Theorem 5.1 (*Euler* (*Beta*) *transforms*).

$$\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\,{}_2R_1\left(a,b;c;\tau;xz^{\sigma}\right)\mathrm{d}z = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(a)\Gamma(b)}\,{}_3\Psi_2\left[\begin{matrix}(a,1),(b,\tau),(\alpha,\sigma); & x\\ (c,\tau),(\alpha+\beta,\sigma); & \end{matrix}\right], \tag{5.1.1}$$

where $a,b,c,\alpha,\beta,\tau,\sigma\in\mathbb{C};\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,$ $\mathrm{Re}(c)>0,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\tau)>0,\mathrm{Re}(\sigma)>0.$

Proof.

$$\begin{aligned}
&\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\,{}_2R_1\left(a,b;c;\tau;xz^{\sigma}\right)\mathrm{d}z\\
&=\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\left(\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(c)\Gamma(b+\tau k)}{\Gamma(c+\tau k)\Gamma(b)}\frac{\left(xz^{\sigma}\right)^k}{k!}\right)\mathrm{d}z\\
&=\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(c)\Gamma(b+\tau k)}{\Gamma(c+\tau k)\Gamma(b)}\frac{x^k}{k!}\left(\int_0^1 z^{\sigma k+\alpha-1}(1-z)^{\beta-1}\,\mathrm{d}z\right)\\
&=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{\Gamma(\alpha+\sigma k)\Gamma(\beta)}{\Gamma(\alpha+\beta+\sigma k)}\frac{x^k}{k!}\\
&=\frac{\Gamma(c)\Gamma(\beta)}{\Gamma(a)\Gamma(b)}\,{}_3\Psi_2\left[\begin{matrix}(a,1),(b,\tau),(\alpha,\sigma); & x\\ (c,\tau),(\alpha+\beta,\sigma); & \end{matrix}\right],
\end{aligned}$$

This is the proof of (5.1.1). □

Remark: Putting $\tau=1$ in (5.1.1), we get

$$\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\,{}_2F_1\left(a,b;c;xz^{\sigma}\right)\mathrm{d}z = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(a)\Gamma(b)}\,{}_3\Psi_2\left[\begin{matrix}(a,1),(b,1),(\alpha,\sigma); & x\\ (c,1),(\alpha+\beta,\sigma); & \end{matrix}\right]. \tag{5.1.2}$$

Taking $\tau=\sigma$, $\alpha=c$ and substituting γ in place of the notation β; (5.1.1) reduces to

$$\int_0^1 z^{c-1}(1-z)^{\gamma-1}\,{}_2R_1\left(a,b;c;\sigma;xz^{\sigma}\right)\mathrm{d}z = \beta(c,\gamma)\,{}_2R_1\left(a,b;c+\gamma;\sigma,x\right) \tag{5.1.3}$$

Also, considering $\sigma=\tau$ and $\beta=c$ in (5.1.1), with replacement of z by $(1-z)$ at ${}_2R_1$, we get

$$\int_0^{\infty} z^{\alpha-1}(1-z)^{c-1}\,{}_2R_1\left(a,b;c;\tau;x(1-z)^{\tau}\right)\mathrm{d}z = \beta(\alpha,c)\,{}_2R_1\left(a,b;\alpha+c;\tau,x\right). \tag{5.1.4}$$

Theorem 5.2 (*Laplace transform*).

$$\int_0^{\infty} \mathrm{e}^{-sz}z^{\alpha-1}\,{}_2R_1\left(a,b;c;\tau;xz^{\sigma}\right)\mathrm{d}z = \frac{s^{-\alpha}\Gamma(c)}{\Gamma(a)\Gamma(b)}\,{}_3\Psi_1\left[\begin{matrix}(a,1),(b,\tau),(\alpha,\sigma); & \dfrac{x}{s^{\sigma}}\\ (c,\tau); & \end{matrix}\right], \tag{5.2.1}$$

where $a,b,c,\alpha,\tau,\sigma\in\mathbb{C};\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,$ $\mathrm{Re}(c)>0,\mathrm{Re}(\alpha)>0,\mathrm{Re}(s)>0,\mathrm{Re}(\tau)>0,$ $\mathrm{Re}(\sigma)>0$ and $\left|\dfrac{x}{s^{\sigma}}\right|<1.$

Proof.

$$\begin{aligned}
&\int_0^{\infty} \mathrm{e}^{-sz}z^{\alpha-1}\,{}_2R_1\left(a,b;c;\tau;xz^{\sigma}\right)\mathrm{d}z\\
&=\int_0^{\infty}\mathrm{e}^{-sz}z^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(c)\Gamma(b+\tau k)}{\Gamma(c+\tau k)\Gamma(b)}\frac{\left(xz^{\sigma}\right)^k}{k!}\right)\mathrm{d}z\\
&=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{x^k}{k!}\left(\int_0^{\infty}\mathrm{e}^{-sz}z^{\sigma k+\alpha-1}\mathrm{d}z\right)\\
&=\frac{\Gamma(c)s^{-\alpha}}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)\Gamma(\sigma k+\alpha)}{\Gamma(c+\tau k)k!}\left(\frac{x}{s^{\sigma}}\right)^k\\
&=\frac{s^{-\alpha}\Gamma(c)}{\Gamma(a)\Gamma(b)}\,{}_3\Psi_1\left[\begin{matrix}(a,1),(b,\tau),(\alpha,\sigma); & \dfrac{x}{s^{\sigma}}\\ (c,\tau); & \end{matrix}\right],
\end{aligned}$$

This is the proof of (5.2.1). □

Theorem 5.3 (*Mellin transform*).

$$\int_0^{\infty} t^{s-1}\,{}_2R_1\left(a,b;c;\tau;-\omega t\right)\mathrm{d}t = \frac{\Gamma(c)\Gamma(s)\Gamma(a-s)\Gamma(b-\tau s)}{\Gamma(a)\Gamma(b)\omega^{s}\Gamma(c-\tau s)}, \tag{5.3.1}$$

where $a,b,c,\tau,s\in\mathbb{C};\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,$ $\mathrm{Re}(c)>0,\mathrm{Re}(\tau)>0,\mathrm{Re}(s)>0.$

Proof. Putting $z=-\omega t$ in (4.1.1), we get

$$\begin{aligned}
&{}_2R_1\left(a,b;c;\tau;-\omega t\right)\\
&=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\frac{1}{2\pi\mathrm{i}}\int_L\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\tau s)}{\Gamma(c-\tau s)}(\omega t)^{-s}\,\mathrm{d}s\\
&=\frac{1}{2\pi\mathrm{i}}\int_L\left(\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\tau s)}{\Gamma(c-\tau s)}\omega^{-s}\right)t^{-s}\mathrm{d}s\\
&=\frac{1}{2\pi\mathrm{i}}\int_L f^{*}(s)t^{-s}\mathrm{d}s,
\end{aligned} \tag{5.3.2}$$

where, $f^{*}(s)=\dfrac{\Gamma(c)\Gamma(s)\Gamma(a-s)\Gamma(b-\tau s)}{\Gamma(a)\Gamma(b)\omega^{s}\Gamma(c-\tau s)}.$

Using (1.9), (1.10), and (5.3.2) immediately lead to (5.3.1). □

Theorem 5.4 (*Whittaker transform*).

$$\int_0^\infty t^{\rho-1}e^{-\frac{1}{2}pt}W_{\lambda,\mu}(pt)\,{}_2R_1(a,b;c;\tau;\omega t^\delta)\mathrm{d}t$$

$$=\frac{p^{-\rho}\Gamma(c)}{\Gamma(a)\Gamma(b)}{}_4\Psi_2\left[\begin{matrix}(a,1),(b,\tau),\left(\frac{1}{2}\pm\mu+\rho,\delta\right);\frac{\omega}{p^\delta}\\(c,\tau),(1-\lambda+\rho,\delta)\end{matrix}\right],\quad(5.4.1)$$

where $a,b,c,\tau,\rho,\delta,p\in\mathbb{C};\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,$ $\mathrm{Re}(c)>0,\mathrm{Re}(\tau)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(\delta)>0.$

Proof. To obtain Whittaker transform, we use the following integral:

$$\int_0^\infty e^{-\frac{t}{2}}t^{v-1}W_{\lambda,\mu}(t)\mathrm{d}t=\frac{\Gamma\left(\frac{1}{2}+\mu+v\right)\Gamma\left(\frac{1}{2}-\mu+v\right)}{\Gamma(1-\lambda+v)},$$

where $\mathrm{Re}(v\pm\mu)>-\frac{1}{2}.$

Substituting $pt=v$ on the L.H.S. of (5.4.1), it reduces to

$$\int_0^\infty\left(\frac{v}{p}\right)^{\rho-1}e^{-\frac{1}{2}v}W_{\lambda,\mu}(v)\,{}_2R_1\left(a,b;c;\tau;\omega\left(\frac{v}{p}\right)^\delta\right)\frac{1}{p}\mathrm{d}v$$

$$=\frac{\Gamma(c)p^{-\rho}}{\Gamma(a)\Gamma(b)}\sum_{k=0}^\infty\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\left(\frac{\omega}{p^\delta}\right)^k\cdot\frac{1}{k!}\left\{\int_0^\infty e^{-\frac{1}{2}v}v^{\delta k+\rho-1}w_{\lambda,\mu}(v)\mathrm{d}v\right\}$$

$$=\frac{\Gamma(c)p^{-\rho}}{\Gamma(a)\Gamma(b)}\sum_{k=0}^\infty\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\cdot\left\{\frac{\Gamma\left(\frac{1}{2}+\mu+\rho+\delta k\right)\Gamma\left(\frac{1}{2}-\mu+\rho+\delta k\right)}{\Gamma(1-\lambda+\rho+\delta k)}\right\}\frac{1}{k!}\left(\frac{\omega}{p^\delta}\right)^k$$

$$=\frac{\Gamma(c)p^{-\rho}}{\Gamma(a)\Gamma(b)}{}_4\Psi_2\left[\begin{matrix}(a,1),(b,\tau)\left(\frac{1}{2}\pm\mu+\rho,\delta\right);\ \frac{\omega}{p^\delta}\\(c,\tau)(1-\lambda+\rho,\delta);\end{matrix}\right].$$

This completes the proof of (5.4.1).

6. Relationship with Some Known Special Functions (Fox H-Function, Wright Hypergeometric Function)

6.1. Relationship with Fox H-Function

Using (4.1.1), we get

$${}_2R_1(a,b;c;\tau;z)$$
$$=\frac{1}{2\pi i}\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\int_L\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\tau s)}{\Gamma(c-\tau s)}(-z)^{-s}\mathrm{d}s$$
$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}H_{2,2}^{1,2}\left[-z\left|\begin{matrix}(1-a,1),(1-b,\tau)\\(0,1),(1-c,\tau)\end{matrix}\right.\right].$$

6.2. Relationship with Wright Hypergeometric Function

The Generalized Hypergeometric Function ${}_2R_1(a,b;c;\tau;z)$ as in (1.3) is

$${}_2R_1^\tau(z)={}_2R_1(a,b;c;\tau;z)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^\infty\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)k!}z^k;\ \tau>0,|z|<1.\quad(6.2.1)$$

From (1.11) and (6.2.1) yields

$${}_2R_1^\tau(z)={}_2R_1(a,b;c;\tau;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}{}_2\Psi_1\left[\begin{matrix}(a,1),(b,\tau);z\\(c,\tau);\end{matrix}\right].\quad(6.2.2)$$

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