*Advances in Pure Mathematics*, 2013, 3, 331-334
http://dx.doi.org/10.4236/apm.2013.33047 Published Online May 2013 (http://www.scirp.org/journal/apm)

# An Elementary Proof of the Mean Inequalities

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Received November 24, 2012; revised December 30, 2012; accepted February 3, 2013



## ABSTRACT

In this paper we will extend the well-known chain of inequalities involving the Pythagorean means, namely the harmonic, geometric, and arithmetic means to the more refined chain of inequalities by including the logarithmic and identric means using nothing more than basic calculus. Of course, these results are all well-known and several proofs of them and their generalizations have been given. See [1-6] for more information. Our goal here is to present a unified approach and give the proofs as corollaries of one basic theorem.

**Keywords:** Pythagorean Means; Arithmetic Mean; Geometric Mean; Harmonic Mean; Identric Mean; Logarithmic Mean

## 1. Pythagorean Means

For a sequence of numbers $x = \{x_1, x_2, \cdots, x_n\}$ we will let

$$AM(x_1, x_2, \cdots, x_n) = AM(x) = \frac{\sum_{j-1}^{n} x_j}{n}$$

$$GM(x_1, x_2, \cdots, x_n) = GM(x) = \prod_{j=1}^{n} x_j^{1/n}$$

and

$$HM(x_1, x_2, \cdots, x_n) = HM(x) = \frac{n}{\sum_{j=1}^{n} \frac{1}{x_j}}$$

to denote the well known *arithmetic*, *geometric*, and *harmonic means*, also called the *Pythagorean means* $(PM)$.

The Pythagorean means have the obvious properties:

1) $PM(x_1, x_2, \cdots, x_n)$ is independent of order

2) $PM(x, x, \cdots, x) = x$

3) $PM(bx_1, bx_2, \cdots, bx_n) = bPM(x_1, x_2, \cdots, x_n)$

4) $PM(x_1, x_2)$ is always a solution of a simple equation. In particular, the arithmetic mean of two numbers $x_1$ and $x_2$ can be defined via the equation

$$AM - x_1 = x_2 - AM$$

The harmonic mean satisfies the same relation with reciprocals, that is, it is a solution of the equation

$$\frac{1}{x_1} - \frac{1}{HM} = \frac{1}{HM} - \frac{1}{x_2}$$

The geometric mean of two numbers $x_1$ and $x_2$ can be visualized as the solution of the equation

$$\frac{x_1}{GM} = \frac{GM}{x_2}$$

1) $GM = \sqrt{(AM)(HM)}$

2) $HM\left(x_1, \frac{1}{x_1}\right) = \frac{1}{AM\left(x_1, \frac{1}{x_1}\right)}$

3) $(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2$

This follows because

$$\frac{AM(x_1, x_2, \cdots, x_n)}{HM(x_1, x_2, \cdots, x_n)} \geq 1$$

## 2. Logarithmic and Identric Means

The *logarithmic mean* of two non-negative numbers $x_1$ and $x_2$ is defined as follows:

$$LM(0, x_2) = LM(x_1, 0) = 0$$

$$LM(x_1, x_1) = x_1$$

and for positive distinct numbers $x_1$ and $x_2$

$$LM(x_1,x_2)=\frac{x_2-x_1}{\ln x_2-\ln x_1}$$

The following are some basic properties of the logarithmic means:

1) Logarithmic mean $LM(a,b)$ can be thought of as the mean-value of the function $f(x)=\ln x$ over the interval $[a,b]$.

2) The logarithmic mean can also be interpreted as the area under an exponential curve.

Since

$$\int_0^1 x^{1-t}y^t \mathrm{d}t = x\int_0^1\left(\frac{y}{x}\right)^t \mathrm{d}t=\frac{x-y}{\ln x-\ln y}$$

We also have the identity

$$LM(x,y)=\int_0^1 x^{1-t}y^t\mathrm{d}t$$

Using this representation it is easy to show that

$$LM(cx,cy)=cLM(x,y)$$

1) We have the identity

$$\frac{LM(x^2,y^2)}{LM(x,y)}=AM(x,y)$$

which follows easily:

$$\frac{LM(x^2,y^2)}{LM(x,y)}=\frac{x^2-y^2}{2(\ln x-\ln y)}\div\frac{x-y}{\ln x-\ln y}$$
$$=\frac{x+y}{2}$$

To define the logarithmic mean of positive numbers $x_0,x_1,\cdots,x_n$, we first recall the definition of divided differences for a function $f(x)$ at points $x_0,x_1,\cdots,x_n$, denoted as

$$[x_0,x_1,\cdots,x_n]f$$

For $0\le m\le n$

$$[x_m]f=f(x_m)$$

and for $0\le m\le n-1-j$ and $1\le j\le n-1$,

$$[x_m,\cdots,x_{m+j}]f$$
$$=\frac{[x_{m+1},\cdots,x_{m+j}]f-[x_m,\cdots,x_{m+j-1}]f}{x_{m+j}-x_m}$$

We now define

$$LM(x_0,\cdots,x_n)$$
$$=\left((-1)^{n+1}\cdot n\cdot[x_0,\cdots,x_n]\ln\right)^{-1/n}$$

So for example for $n = 2$, we get

$$LM(x_0,x_1,x_2)$$
$$=\sqrt{\frac{(x_0-x_1)(x_1-x_2)(x_2-x_0)}{2((x_2-x_1)\ln x_0-(x_2-x_0)\ln x_1+(x_0-x_1)\ln x_2)}}$$

The *identric mean* of two distinct positive real numbers $x_1,x_2$ is defined as:

$$IM(x_1,x_2)=\frac{1}{e}\left(\frac{x_2^{x_2}}{x_1^{x_1}}\right)^{\frac{1}{x_2-x_1}}$$

with $IM(x_1,x_1)=x_1$.

The slope of the secant line joining the points $(a,f(a))$ and $(b,f(b))$ on the graph of the function $f(x)=x\ln x$ is the natural logarithm of $IM(a,b)$.

It can be generalized to more variables according by the mean value theorem for divided differences.

## 3. The Main Theorem

**Theorem 1.** *Suppose* $f:[a,b]\to\Re$ *is a function with a strictly increasing derivative. Then*

$$\int_a^b f(t)\mathrm{d}t$$
$$<\frac{b-a}{2}\left[f(s)-s\left(\frac{f(b)-f(a)}{b-a}\right)+\frac{bf(b)-af(a)}{b-a}\right]$$

*for all* $s<t$ *in* $[a,b]$.

*Let* $s_0$ *be defined by the equation*

$$f'(s_0)=\frac{f(b)-f(a)}{b-a}$$

*Then,*

$$\int_a^b f(t)\mathrm{d}t$$
$$<\frac{b-a}{2}\left[f(s_0)-s_0\left(\frac{f(b)-f(a)}{b-a}\right)+\frac{bf(b)-af(a)}{b-a}\right]$$

*is the sharpest form of the above inequality.*

***Proof.*** By the Mean Value Theorem, for all $s,t$ in $[a,b]$, we have

$$\frac{f(t)-f(s)}{t-s}<f'(u)$$

for some $u$ between $s$ and $t$. Assuming without loss of generality $s<t$, by the assumption of the theorem we have

$$f(t)-f(s)<(t-s)f'(t)$$

Integrating both sides with respect to $t$, we have

$$\int_a^b f(t)\mathrm{d}t<(b-a)f(s)+bf(b)-af(a)$$
$$-s[f(b)-f(a)]-\int_a^b f(t)\mathrm{d}t$$

and the inequality of the theorem follows.

Let us now put

$$g(s)=(b-a)f(s)+bf(b)-af(a)$$
$$-s\left[f(b)-f(a)\right]-\int_a^b f(t)\,\mathrm{d}t$$

Note that

$$g'(s)=(b-a)f'(s)-\left[f(b)-f(a)\right]$$

Moreover, since

$$g(b)=g(a)=(b-a)\left[f(a)+f(b)\right]-\int_a^b f(t)\,\mathrm{d}t$$

there exists an $s_0$ in $(a,b)$ such that $g'(s_0)=0$.

Since $f'$ is strictly increasing, we have

$$g'(s)>g'(s_0)=0$$

for $s>s_0$
and

$$g'(s)<g'(s_0)=0$$

for $s<s_0$

Thus, $s_0$ is a minimum of $g$ and $g(s_0)\le g(s)$ for all $s \in [a,b]$.

## 4. Applications to Mean Inequalities

We will extend the well-known chain of inequalities

$$HM(a,b)\le GM(a,b)\le A(a,b)$$

to the more refined

$$HM(a,b)\le GM(a,b)\le LM(a,b)$$
$$\le IM(a,b)\le A(a,b)$$

using nothing more than the mean value theorem of differential calculus. All of these are strict inequalities unless, of course, the numbers are the same, in which case all means are equal to the common value of the two numbers.

Let us now assume that $0<a<b$.

Let us let $f(t)=\dfrac{1}{t^2}$. The condition of the Theorem 1 is satisfied. Solving the equation

$$-\frac{2}{s_0^3}=\frac{\frac{1}{b^2}-\frac{1}{a^2}}{b-a}$$

we find

$$s_0=(abH)^{1/3}$$

where $H=HM(a,b)$.

Hence the left-hand side of the inequality becomes

$$\frac{b-a}{2}\left[\frac{1}{(abH)^{2/3}}+\frac{2}{(abH)^{2/3}}-\frac{1}{ab}\right]$$

Thus we have

$$\frac{b-a}{ab}<\frac{b-a}{2}\left[\frac{3}{(abH)^{2/3}}-\frac{1}{ab}\right]$$

implying

$$H^2<ab$$

or

$$HM(a,b)<GM(a,b)$$

Let us let $f(t)=\dfrac{1}{t}$. The condition of Theorem 1 is satisfied. We can easily compute the $s_0$ of the theorem from the equation

$$-\frac{1}{s_0^2}=\frac{\frac{1}{b}-\frac{1}{a}}{b-a}$$

as

$$s_0=\sqrt{ab}$$

Our inequality becomes

$$\ln b-\ln a<\frac{b-a}{2}\left[\frac{1}{\sqrt{ab}}-\sqrt{ab}\left(-\frac{1}{ab}\right)\right]$$

Implying,

$$\frac{\ln b-\ln a}{b-a}<\frac{1}{\sqrt{ab}}$$

that is

$$GM(a,b)<LM(a,b)$$

Now let $f(t)=-\ln t$. Again the condition of Theorem 1 is satisfied. The $s_0$ of the theorem can be computed from the equation

$$-\frac{1}{s_0}=\frac{-\ln b+\ln a}{b-a}$$

as

$$s_0=L$$

where $L=LM(a,b)$

Since

$$\int_a^b -\ln t\,\mathrm{d}t=(b-a)-(b\ln b-a\ln a)$$
$$=(b-a)-\ln\left(\frac{b^b}{a^a}\right)$$

Thus,

$$\int_a^b -\ln t\,\mathrm{d}t=-(b-a)\ln I$$

where $I=IM(a,b)$.

Consequently our inequality becomes

$$-(b-a)\ln I < \frac{b-a}{2}\left[-\ln L - L\left(-\frac{1}{L}\right) - \ln(eI)\right]$$

implying

$$-\ln I < -\ln L$$

that is,

$$LM(a,b) < IM(a,b)$$

Finally, let us put $f(t) = t\ln t$. Again the condition of Theorem 1 is satisfied. Since in this case

$$\frac{f(b)-f(a)}{b-a} = \frac{b\ln b - a\ln a}{b-a} = 1 + \ln I$$

the $s_0$ of the theorem can be computed as

$$s_0 = I$$

The right-hand side of the inequality becomes

$$\frac{b-a}{2}\left[-I + \frac{b^2\ln b - a^2\ln a}{b-a}\right]$$

The integral on the left-hand side of our inequality yields

$$\int_a^b t\ln t\,\mathrm{d}t = \frac{1}{2}\left(b^2\ln b - a^2\ln a\right) - \frac{1}{4}\left(b^2 - a^2\right)$$

implying

$$-\frac{1}{2}(b+a) < -I$$

or

$$IM(a,b) < AM(a,b)$$

Thus, we now have for $a \neq b$

$$HM(a,b) < GM(a,b) < LM(a,b)$$
$$< IM(a,b) < AM(a,b)$$

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