Advances in Pure Mathematics, 2013, 3, 297-303
http://dx.doi.org/10.4236/apm.2013.32042 Published Online March 2013 (http://www.scirp.org/journal/apm)

Equivalence Problem of the Painlevé Equations

Sopita Khamrod
Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, Thailand
Email: kuntimak@nu.ac.th

Received November 1, 2012; revised January 17, 2013; accepted January 26, 2013

ABSTRACT

The manuscript is devoted to the equivalence problem of the Painlevé equations. Conditions which are necessary and sufficient for second-order ordinary differential equations $y''=F(x,y,y')$ to be equivalent to the first and second Painlevé equation under a general point transformation are obtained. A procedure to check these conditions is found.

Keywords: Equivalence Problem; Painlevé Equations; Point Transformation

1. Introduction

Many physical phenomena are described by differential equations. Ordinary differential equations play a significant role in the theory of differential equations. In the 19th century an important problem in analysis was the classification of ordinary differential equations [1-4]. One type of classification problem is an equivalence problem: a system of equations is equivalent to another system of equations if there exists an invertible change of the independent and dependent variables (point transformations) which transforms one system into another.

The six Painlevé equations (*PI-PVI*) are nonlinear second-order ordinary differential equations which are studied in many fields of Physics. These equations and their solutions, the Painlevé transcendent, play an important role in many areas of mathematics.

$$
\begin{aligned}
&PI: y''=6y^2+x,\\
&PII: y''=6y^3+xy+\alpha,\\
&PIII: y''=\frac{y'^2}{y}-\frac{y'}{x}+\frac{(\alpha y^2+\beta)}{x}+\gamma y^3+\frac{\delta}{y},\\
&PIV: y''=\frac{y'^2}{2y}-\frac{3y^3}{2}+4xy^2+2(x^2-\alpha)y+\frac{\beta}{y},\\
&PV: y''=\left(\frac{1}{2y}+\frac{1}{y-1}\right)y'^2-\frac{y'}{x}+\frac{(y-1)^2}{x^2}\left(\alpha y+\frac{\beta}{y}\right)+\frac{\gamma y}{x}+\frac{\delta y(y+1)}{y-1},\\
&PVI: y''=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y+1}+\frac{1}{y-x}\right)y'^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y'\\
&\qquad +\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\left(\frac{1}{2}-\delta\right)\frac{x(x-1)}{(x-1)^2}\right)
\end{aligned}
\tag{1}
$$

The Painlevé equations belongs to the class of equations of the form

$$
\begin{aligned}
&y''+a_1(x,y)y'^3+3a_2(x,y)y'^2\\
&+3a_3(x,y)y'+a_4(x,y)=0.
\end{aligned}
\tag{2}
$$

This form is conserved with respect to any change of the independent and dependent variables[1]

$$t=\varphi(x,y), u=\psi(x,y). \tag{3}$$

In fact, since under the change of variable (3) derivatives are changed by the formulae

[1]Point transformations are weaker than contact transformations. S. Lie showed that all second-order equations are equivalent with respect to contact transformations.

Copyright © 2013 SciRes.

$$u' = g(x, y, y') = \frac{D_x \psi}{D_x \varphi} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y},$$

$$\begin{aligned} u'' = P(x, y, y', y'') &= \frac{D_x g}{D_x \varphi} = \frac{g_x + y' g_y + y'' g_{y'}}{\varphi_x + y'\varphi_y} = \left(\varphi_x + y'\varphi_y\right)^{-3} \Big(y''\left(\varphi_x\psi_y - \varphi_y\psi_x\right) + y'^3\left(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y\right) \\ &+ y'^2\left(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2\left(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y\right)\right) \\ &+ y'\left(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2\left(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x\right)\right) + \varphi_x\psi_{xx} - \varphi_{xx}\psi_x \Big). \end{aligned} \tag{4}$$

Here

$$\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0,$$

subscript means a derivative, for example,

$$\varphi_x = \partial\varphi/\partial x, \psi_y = \partial\psi/\partial y. \tag{5}$$

Since the Jacobian of the change of variables $\Delta \neq 0$, the equation

$$u'' + b_1(t,u)u'^3 + 3b_2(t,u)u'^2 + 3b_3(t,u)u' + b_4(t,u) = 0,$$

becomes (2), where

$$\begin{aligned} a_1 &= \Delta^{-1}\left(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y + \varphi_y^3 b_4 + 3\varphi_y^2\psi_y b_3 + 3\varphi_y\psi_y^2 b_2 + \psi_y^3 b_1\right), \\ a_2 &= \Delta^{-1}\Big(3^{-1}\left(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2\left(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y\right)\right) + \varphi_x\varphi_y^2 b_4 \\ &\quad + \varphi_y\left(2\varphi_x\psi_y + \varphi_y\psi_x\right)b_3 + \left(\varphi_x\psi_y^2 + 2\varphi_y\psi_x\psi_y\right)b_2 + \psi_x\psi_y^2 b_1\Big), \\ a_3 &= \Delta^{-1}\Big(3^{-1}\left(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2\left(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x\right)\right) + \varphi_x^2\varphi_y b_4 \\ &\quad + \left(\varphi_x^2\psi_y + 2\varphi_x\varphi_y\psi_x\right)b_3 + \left(2\varphi_x\psi_x\psi_y + \varphi_y\psi_x^2\right)b_2 + \psi_x^2\psi_y b_1\Big), \\ a_4 &= \Delta^{-1}\left(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x + \varphi_x^3 b_4 + 3\varphi_x^2\psi_x b_3 + 3\varphi_x\psi_x^2 b_2 + \psi_x^3 b_1\right). \end{aligned} \tag{6}$$

Two quantities play a major role in the study of Equation (5):

$$L_1 = -\frac{\partial\Pi_{11}}{\partial u} + \frac{\partial\Pi_{12}}{\partial t} - b_4\Pi_{22} - b_2\Pi_{11} + 2b_3\Pi_{12},$$

$$L_2 = -\frac{\partial\Pi_{12}}{\partial u} + \frac{\partial\Pi_{22}}{\partial t} - b_1\Pi_{11} - b_3\Pi_{22} + 2b_2\Pi_{12},$$

where

$$\Pi_{11} = 2\left(b_3^2 - b_2 b_4\right) + b_{3t} - b_{4u},$$

$$\Pi_{22} = 2\left(b_2^2 - 3b_1 b_3\right) + b_{1t} - b_{2u},$$

$$\Pi_{12} = b_2 b_3 - b_1 b_4 + b_{2t} - b_{3u}.$$

Under a point transformation (3) these components are transformed as follows [2]:

$$\begin{aligned} \tilde{L}_1 &= \Delta\left(L_1\varphi_x + L_2\psi_x\right), \\ \tilde{L}_2 &= \Delta\left(L_1\varphi_y + L_2\psi_y\right). \end{aligned} \tag{7}$$

Here tilde means that a value corresponds to system (2): the coefficients b_i are exchanged with a_i, the variables t and u are exchanged with x and y, respectively.

S. Lie showed that any equation with $L_1 = 0$ and $L_2 = 0$ is equivalent to the equation $u'' = 0$. For the Painlevé equations $L_1 \neq 0$ and $L_2 = 0$.

R. Liouville [2] also found other relative invariants, for example,

$$\begin{aligned} v_5 &= L_2\left(L_1 L_{2t} - L_2 L_{1t}\right) + L_1\left(L_2 L_{1u} - L_1 L_{2u}\right) - b_1 L_1^3 \\ &\quad + 3b_2 L_1^2 L_2 - 3b_3 L_1 L_2^2 + b_4 L_2^3, \end{aligned}$$

and

$$\begin{aligned} w_1 &= L_1^{-4}\Big(-L_1^3\left(\Pi_{12}L_1 - \Pi_{11}L_2\right) + R_1\left(L_1^2\right)_t \\ &\quad - L_1^2 R_{1t} + L_1 R_1\left(b_3 L_1 - b_4 L_2\right)\Big), \end{aligned}$$

where

$$R_1 = L_1 L_{2t} - L_2 L_{1t} + b_2 L_1^2 - 2b_3 L_1 L_2 + b_4 L_2^2.$$

For the Painlevé equations $v_5 = 0$ and $w_1 = 0$ [5]. Up to now, the equivalence problem has been solved in a form more convenient for testing only for (*PI*) and (*PII*) equations, by using an explicit point change of variables was given in [6].

The manuscript is devoted to solving the problem of describing all second-order differential equations

$y'' = F(x, y, y'')$ which are equivalent with respect to point transformations (3) to the first and second Painlevé equation (*PI*) and (*PII*). Example of the first Painlevé equation (*PI*) is presented.

Necessary and sufficient conditions for an equation $y'' = F(x, y, y'')$ to be equivalent to (*PI*) and (*PII*) are obtained. As was noted, some of the necessary conditions are [5]:

$$\frac{\partial^4 F}{\partial y^4} = 0, v_5 = 0 \text{ and } w_1 = 0.$$

Other conditions are also expressed in terms of relations for the coefficients of Equation (5).

The method of the study is similar to [7-9]. It uses analysis of compatibility of an over determined system of partial differential equations.

2. Equations Equivalent to the Painlevé Equations

This section studies Equation (5) which are equivalent to the first and second Painlevé equation (*PI*) and (*PII*). Since any equation of (1) belongs to the type of equation (2), the necessary condition for an equation $y'' = F(x, y, y'')$ to be equivalent to the first and second Painlevé equation (*PI*) and (*PII*) are that it has to be of the same type. Since $v_5 = 0$ and $w_1 = 0$ are relative invariants with respect to (3), they are also necessary condition.

2.1. The First Painlevé Equation (*PI*)

For obtaining sufficient conditions one has to find conditions for the coefficients $b_1(t,u), b_2(t,u), b_3(t,u), b_4(t,u)$ which guarantee existence of the functions $\varphi(x, y), \Psi(x, y)$ transforming the coefficient of Equation (6) into the coefficients of equations (*PI*).

Also note that the the first Painlevé equation has the coefficients are

$$\begin{aligned} &a_1(x, y) = 0, a_2(x, y) = 0, \\ &a_3(x, y) = 0, a_4(x, y) = -6y^2 - x. \end{aligned} \tag{8}$$

Without loss of generality it is assumed that $L_1 \neq 0$. Since for Equation (8), the value $\tilde{L}_2 = 0$, and hence, the functions $\varphi(x, y)$ and $\Psi(x, y)$ satisfy the equation

$$\varphi_y L_1 + \psi_y L_2 = 0. \tag{9}$$

Substituting these coefficients into (6), one obtains over determined system of partial differential equations.

$$\psi_{yy} L_1^2 + \psi_y^2 \left(3b_4 L_2^2 - 6b_3 L_1 L_2 + 3b_2 L_1^2 - 2L_{1t} L_2 + 2L_{2t} L_1\right) = 0, \tag{10}$$

$$\begin{aligned} &2\psi_{xy} L_1^2 - \Delta_{1x} \psi_y \Delta_1^{-1} L_1^2 + \psi_y \Delta_1 \left(L_{1t} - 3b_4 L_2 + 3b_3 L_1\right) \\ &+\psi_x \psi_y \left(6b_4 L_2^2 - 12b_3 L_1 L_2 + 6b_2 L_1^2 - 4L_{1t} L_2 + L_{1u} L_1 + 3L_{2t} L_1\right) = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &\psi_{xx} L_1^2 - \Delta_{1x} \psi_x \Delta_1^{-1} L_1^2 + b_4 \Delta_1^2 + \psi_x \Delta_1 \left(L_{1t} - 3b_4 L_2 + 3b_3 L_1\right) \\ &+\psi_x^2 \left(3b_4 L_2^2 - 6b_3 L_1 L_2 + 3b_2 L_1^2 - 2L_{1t} L_2 + L_{1u} L_1 + L_{2t} L_1\right) + \psi_y L_1^2 \left(6y^2 + x\right) = 0, \end{aligned} \tag{12}$$

where $\Delta_1 = \varphi_x L_1 + \psi_x L_2$. Notice that

$$L_1 \Delta_{1y} = \psi_y \Delta_1 \left(L_{1u} - L_{2t}\right). \tag{13}$$

From Equations (10)-(12) one can find the derivatives

$$\psi_{yy} = L_1^{-2} \psi_y^2 \left(2L_{1t} L_2 - 2L_{2t} L_1 - 3b_4 L_2^2 + 6b_3 L_1 L_2 - 3b_2 L_1^2\right), \tag{14}$$

$$L_1^2 \psi_{xx} = 2\psi_{xy} \psi_x \psi_y^{-1} L_1^2 - b_4 \Delta_1^2 - \psi_y L_1^2 \left(6y^2 + x\right) + \psi_x^2 \left(3b_4 L_2^2 - 6b_3 L_1 L_2 + 3b_2 L_1^2 - 2L_{1t} L_2 + 2L_{2t} L_1\right), \tag{15}$$

$$L_1^2 \Delta_{1x} = 2\psi_{xy} \psi_y^{-1} \Delta_1 L_1^2 + \Delta_1^2 \left(L_{1t} - 3b_4 L_2 + 3b_3 L_1\right) + \psi_x \Delta_1 \left(L_{1u} L_1 - 4L_{1t} L_2 + 3L_{2t} L_1 + 6b_4 L_2^2 - 12b_3 L_1 L_2 + 6b_2 L_1^2\right). \tag{16}$$

Taking the mixed derivatives $\left(\Psi_{xx}\right)_{yy} = \left(\Psi_{yy}\right)_{xx}$, one obtains

$$\psi_y \Delta_1^2 + 12L_1 = 0. \tag{17}$$

Differentiating this equation with respect to *x* and *y*, and substituting Ψ_y found from Equation (17), one gets

$$5\psi_{xy} \Delta_1^2 L_1 - 12\psi_x \left(12\left(b_4 L_2^2 - 2b_3 L_1 L_2 + b_2 L_1^2\right) - 7L_{1t} L_2 + L_{1u} L_1 + 6L_{2t} L_1\right) - 12\Delta_1 \left(L_{1t} - 6b_4 L_2 + 6b_3 L_1\right) = 0, \tag{18}$$

$$3b_4 L_2^2 - 6b_3 L_1 L_2 + 3b_2 L_1^2 - 3L_{1t} L_2 - L_{1u} L_1 + 4L_{2t} L_1 = 0. \tag{19}$$

Finding the derivatives: L_{2u} from the equation $v_5 = 0$, L_{2tt} from the equation $w_1 = 0$, and L_{2t} from (19), and composing

Copyright © 2013 SciRes.

the equations

$$\left(L_{2t}\right)_t - L_{2tt} = 0, \left(L_{2t}\right)_u - \left(L_{2u}\right)_t = 0,$$

one can find the derivatives

$$\begin{aligned} L_{1tu} = \Big(&4b_4^2L_2^3 - 18b_4b_3L_1L_2^2 + 60b_4b_2L_1^2L_2 + 80b_4b_1L_1^3 - 3b_4L_{1t}L_2^2 - 36b_3^2L_1^2L_2 - 90b_3b_2L_1^3 - 12b_3L_{1t}L_1L_2 \\ &+ 30b_3L_{1u}L_1^2 - 15b_2L_{1t}L_1^2 + 20b_{4u}L_1^2L_2 + 80b_{3u}L_1^3 - 100b_{2t}L_1^3 - L_{1t}^2L_2 + 25L_{1t}L_{1u}L_1 + 2KL_2\Big)\Big/\left(20L_1^2\right), \end{aligned} \tag{20}$$

$$\begin{aligned} L_{1uu} = \Big(&-b_4^2L_2^4 + 12b_4b_3L_1L_2^3 + 40b_4b_1L_1^3L_2 + 2b_4L_{1t}L_2^3 - 36b_3^2L_1^2L_2^2 + 120b_3b_1L_1^4 - 12b_3L_{1t}L_1L_2^2 - 135b_2^2L_1^4 \\ &+ 30b_2L_{1u}L_1^3 - 20b_1L_{1t}L_1^3 + 40b_{3u}L_1^3L_2 - 20b_{2t}L_1^3L_2 + 60b_{2u}L_1^4 - 80b_{1t}L_1^4 - L_{1t}^2L_2^2 + 25L_{1u}^2L_1^2 + 2KL_2^2\Big)\Big/\left(20L_1^3\right), \end{aligned} \tag{21}$$

where

$$\begin{aligned} K = &3b_4^2L_2^2 - 6b_4b_3L_1L_2 - 105b_4b_2L_1^2 + 9b_4L_{1t}L_2 - 15b_4L_{1u}L_1 + 108b_3^2L_1^2 + 6b_3L_{1t}L_1 \\ &- 10b_{4t}L_1L_2 - 50b_{4u}L_1^2 + 60b_{3t}L_1^2 + 10L_{1tt}L_1 - 12L_{1t}^2. \end{aligned} \tag{22}$$

Since of (14), (15) and (18) all second order derivatives of the function $\Psi(x,y)$ can be found, then one can compose the equations $\left(\Psi_{xy}\right)_x - \left(\Psi_{xx}\right)_y = 0$ and $\left(\Psi_{xy}\right)_y - \left(\Psi_{yy}\right)_x = 0$, which are reduced to the only equation

$$\Delta_1^2K - 600L_1^4y = 0. \tag{23}$$

The equation $\left(L_{1tt}\right)_u - \left(L_{1tu}\right)_t = 0$ gives

$$\begin{aligned} &2L_1\left(K_uL_1 - K_tL_2\right) + 3K\left(b_4L_2^2 - 2b_3L_1L_2 + b_2L_1^2\right) \\ &+5K\left(L_{1t}L_2 - L_{1u}L_1\right) + 100L_1^5 = 0. \end{aligned} \tag{24}$$

Differentiating Equation (23) with respect to x, one obtains[2]

$$\begin{aligned} &250\psi_xL_1^7y \\ &+\Delta_1L_1^2y\left(6b_4KL_2 - 6b_3KL_1 - 5K_tL_1 + 14L_{1t}K\right) = 0. \end{aligned}$$

From this equation one can find the derivative

$$\psi_x = \left(\Delta_1\left(-6b_4KL_2 + 6b_3KL_1 + 5K_tL_1 - 14L_{1t}K\right)\right)\Big/\left(250L_1^5\right).$$

Notice that the equations $\left(\Psi_x\right)_y - \left(\Psi_y\right)_x = 0$ and $\Psi_{xy} - \left(\Psi_x\right)_y = 0$ Are satisfied, and the equation $\Psi_{xx} - \left(\Psi_x\right)_x = 0$ becomes

$$Qy^2 + x = 0, \tag{25}$$

where

$$\begin{aligned} Q = 6K^{-2}\Big(&-390b_4^2KL_2^2 + 780b_4b_3KL_1L_2 \\ &+ 2850b_4b_2KL_1^2 + 300b_4K_tL_1L_2 - 1170b_4L_{1t}KL_2 \\ &+ 450b_4L_{1u}KL_1 - 5000b_4L_1^5 - 3240b_3^2KL_1^2 \\ &- 300b_3K_tL_1^2 + 720b_3L_{1t}KL_1 + 400b_{4t}KL_1L_2 \\ &+ 1400b_{4u}KL_1^2 - 1800b_{3t}KL_1^2 - 100K_{tt}L_1^2 \\ &+ 600K_tL_{1t}L_1 - 840L_{1t}^2K + 29K^2\Big). \end{aligned} \tag{26}$$

Because of (25), the function $Q(t,u) \neq 0$. Differentiating (25) with respect to x and y, one gets

$$\Delta_1y^2R - 5K^2L_1^2 = 0, \tag{27}$$

$$K\left(L_1Q_u - Q_tL_2\right) - 100QL_1^4 = 0, \tag{28}$$

where

$$R = K\left(4QK\left(3b_4L_2 - 3b_3L_1 + 7L_{1t}\right) - 5L_1\left(Q_tK + 2K_tQ\right)\right). \tag{29}$$

Differentiating Equation (27) with respect to x and y one obtains the only equation

$$R_tL_1 - R\left(7L_{1t} + 3b_4L_2 - 3b_3L_1\right) = 0. \tag{30}$$

Finding the function Δ_1 from (27), and substituting it into (23), (16), (13) one gets

$$24y^5R^2 - K^5 = 0. \tag{31}$$

Notice that

$$\begin{aligned} &4L_1\left(L_1R_u - R_tL_2\right) + 25R\left(L_{1t}L_2 - L_{1u}L_1\right) \\ &+15R\left(b_4L_2^2 - 2b_3L_1L_2 + b_2L_1^2\right) = 0. \end{aligned} \tag{32}$$

Thus, the necessary and sufficient conditions for equation $y'' = F(x,y,y'')$ to be equivalent to the first Painlevé equation are: the equation has to be of the form (5) with the coefficients $b_i(t,u), (i = 1,2,3,4)$ satisfying the conditions[3] $v_5 = 0$, (19)-(21), (24), (28) and (32), where the functions $K(t,u), R(t,u)$ and $Q(t,u)$ are defined by Equations (22), (26), (29). The transformation is defined by (25) and (31).

2.2. The Second Painlevé Equation (*PII*)

Similar to the first Painlevé equation one can study the second Painlevé equation. Painlevé equation (*PII*) has the coefficients are

$$\begin{aligned} &a_1(x,y) = 0, a_2(x,y) = 0, \\ &a_3(x,y) = 0, a_4(x,y) = -\left(2y^3 + xy + \alpha\right). \end{aligned} \tag{33}$$

Substituting these coefficients into (6), one obtains over determined system of partial differential equations.

[2]The derivative with respect to y is equal to zero.

[3]Recall that Equation (20) is obtained from the equation $w_1 = 0$.

$$\psi_{yy}L_1^2+\psi_y^2\left(3b_4L_2^2-6b_3L_1L_2+3b_2L_1^2-2L_{1t}L_2+2L_{2t}L_1\right)=0, \tag{34}$$

$$2\psi_{xy}L_1^2-\Delta_{1x}\psi_y\Delta_1^{-1}L_1^2+\psi_y\Delta_1\left(L_{1t}-3b_4L_2+3b_3L_1\right)+\psi_x\psi_y\left(6b_4L_2^2-12b_3L_1L_2+6b_2L_1^2-4L_{1t}L_2+L_{1u}L_1+3L_{2t}L_1\right)=0, \tag{35}$$

$$\begin{aligned}&\psi_{xx}L_1^2-\Delta_{1x}\psi_x\Delta_1^{-1}L_1^2+b_4\Delta_1^2+\psi_x\Delta_1\left(L_{1t}-3b_4L_2+3b_3L_1\right)\\&+\psi_x^2\left(3b_4L_2^2-6b_3L_1L_2+3b_2L_1^2-2L_{1t}L_2+L_{1u}L_1+L_{2t}L_1\right)+\psi_yL_1^2\left(2y^3+xy+\alpha\right)=0,\end{aligned} \tag{36}$$

where $\Delta_1=\varphi_xL_1+\psi_xL_2$. Notice that

$$L_1\Delta_{1y}=\psi_y\Delta_1\left(L_{1u}-L_{2t}\right). \tag{37}$$

From Equations (34)-(36) one can find the derivatives

$$\psi_{yy}=L_1^{-2}\psi_y^2\left(2L_{1t}L_2-2L_{2t}L_1-3b_4L_2^2+6b_3L_1L_2-3b_2L_1^2\right), \tag{38}$$

$$L_1^2\psi_{xx}=2\psi_{xy}\psi_x\psi_y^{-1}L_1^2-b_4\Delta_1^2-\psi_yL_1^2\left(2y^3+xy+\alpha\right)+\psi_x^2\left(3b_4L_2^2-6b_3L_1L_2+3b_2L_1^2-2L_{1t}L_2+2L_{2t}L_1\right), \tag{39}$$

$$L_1^2\Delta_{1x}=2\psi_{xy}\psi_y^{-1}\Delta_1L_1^2+\Delta_1^2\left(L_{1t}-3b_4L_2+3b_3L_1\right)+\psi_x\Delta_1\left(L_{1u}L_1-4L_{1t}L_2+3L_{2t}L_1+6b_4L_2^2-12b_3L_1L_2+6b_2L_1^2\right). \tag{40}$$

Taking the mixed derivatives $\left(\Psi_{xx}\right)_{yy}=\left(\Psi_{yy}\right)_{xx}$, one obtains

$$\psi_y\Delta_1^2+12L_1y=0. \tag{41}$$

Differentiating this equation with respect to x and y, and substituting Ψ_y found from Equation (41), one gets

$$\begin{aligned}&5\psi_{xy}\Delta_1^2L_1-12y\left(\Delta_1\left(L_{1t}-6b_4L_2+6b_3L_1\right)\right.\\&\left.+\psi_x\left(12b_4L_2^2-24b_3L_1L_2+12b_2L_1^2-7L_{1t}L_2+L_{1u}L_1+6L_{2t}L_1\right)\right)=0,\end{aligned} \tag{42}$$

$$\Delta_1-12Ky=0, \tag{43}$$

where the function $K\left(t,u\right)$ is defined by the formula

$$K^2=\left(12L_1\right)^{-1}\left(3b_4L_2^2-6b_3L_1L_2+3b_2L_1^2-3L_{1t}L_2-L_{1u}L_1+4L_{2t}L_1\right). \tag{44}$$

Since $\Delta_1\neq0$, then $K\neq0$. Hence, Equations (42) and (43) define $\Delta_1=12Ky$ and the derivative Ψ_{xy}. Thus, all second-order derivatives $\Psi_{xx},\Psi_{xy},\Psi_{yy}$ and the derivative Ψ_{yy} of the function $\Psi\left(x,y\right)$ are defined.

Substituting the expression of Δ_1 into Equations (37) and (40), one obtains

$$\begin{aligned}&4L_1\left(K_uL_1-K_tL_2\right)\\&-3K\left(L_{1u}L_1-L_{1t}L_2+12K^2L_1+b_4L_2^2-2b_3L_1L_2+b_2L_1^2\right)=0,\end{aligned} \tag{45}$$

$$3KL_1\psi_x=y\left(9\left(L_{1t}-b_4L_2+b_3L_1\right)-5K^{-1}K_tL_1\right). \tag{46}$$

Equations (41) and (46) define all first-order derivatives Ψ_x,Ψ_y of the function $\Psi\left(x,y\right)$. Since the second-order derivatives $\Psi_{xx},\Psi_{xy},\Psi_{yy}$ have been found, one needs to check the conditions

$$\left(\psi_x\right)_x=\psi_{xx},\left(\psi_x\right)_y=\psi_{xy},\left(\psi_y\right)_x=\psi_{xy},\left(\psi_y\right)_y=\psi_{yy}.$$

All these conditions are satisfied except the first one, which becomes

$$\begin{aligned}&4y^3K\left(60K_{tt}L_1+4K_t\left(51\left(b_3L_1-b_4L_2\right)+36L_{1t}-50K^{-1}K_tL_1\right)+9b_4K\left(L_{1u}-3b_2L_1\right)+99b_4KL_1^{-1}L_2\left(L_{1t}-b_4L_2+2b_3L_1\right)\right.\\&\left.-36K\left(L_{1tt}+3b_3L_{1t}-b_{4t}L_2+b_{3t}L_1+9b_4K^2+2b_3^2L_1\right)\right)+L_1^3\left(2y^3+yx+\alpha\right)=0.\end{aligned} \tag{47}$$

Differentiating (47) with respect to y and excluding x by using (47), one obtains

$$\begin{aligned}&144y^3\left(2KL_1\left(K_{tt}L_1+2K_tL_{1t}-3b_4K_tL_2+3b_3K_tL_1\right)-6K_t^2L_1^2\right.\\&\left.+K^2\left(3b_4L_{1t}L_2-3b_3L_{1t}L_1-L_{1tt}L_1+b_{4t}L_1L_2-b_{4u}L_1^2-3b_4^2L_2^2+6b_4b_3L_1L_2-3b_4b_2L_1^2-12b_4K^2L_1\right)\right)-L_1^4\left(4y^3-\alpha\right)=0.\end{aligned} \tag{48}$$

Excluding the variable α from (47) by using (48), Equation (47) becomes

$$2y^2Q-x=0, \tag{49}$$

Copyright © 2013 SciRes.

where

$$Q = L_1^{-4}\Big(8L_1\big(3K_{tt}KL_1 - 4K_t^2L_1 - 3b_4K_tKL_2 + 3b_3K_tKL_1\big) \\ -18K^2\big(b_4^2L_2^2 - 2b_4b_3L_1L_2 + 9b_4b_2L_1^2 - b_4L_{1t}L_2 + b_4L_{1u}L_1 + 12b_4K^2L_1 - 8b_3^2L_1^2 + 4b_{4u}L_1^2 - 4b_{3t}L_1^2\big)\Big) - 3. \quad (50)$$

Differentiating (49) with respect to x and y, one gets, respectively,

$$2y^3\Big(\big(Q_tL_2 - Q_uL_1\big)\big(3b_4KL_2 - 3b_3KL_1 + 5K_tL_1 - 3L_{1t}K\big) + 36Q_tK^3L_1\Big) - 3K^2L_1^2 = 0, \quad (51)$$

$$Q_tL_2 - Q_uL_1 + 24QK^2 = 0. \quad (52)$$

Since $KL_1 \neq 0$, the coefficient with y^3 in (51) is not equal to zero. Hence, Equations (49) and (51) define the variable x and y. Equation (48) becomes

$$18K^2\big(2L_{1tt} - 21b_4b_2L_1 - 3b_4L_{1t}L_1^{-1}L_2 - 3b_4L_{1u} + 24b_3^2L_1 - 12b_4K^2 - 2b_{4t}L_2 - 10b_{4u}L_1 + 12b_{3t}L_1 + 3b_4^2L_1^{-1}L_2^2 - 6b_4b_3L_2 + 6b_3L_{1t}\big) \\ +6K\big(12K_t + \alpha QL_1\big)\big(b_4L_2 - b_3L_1 - L_{1t}\big) + 120K_t^2L_1 - 6Q_t\alpha KL_1^2 + QL_1^2\big(20K_t\alpha - 3L_1\big) - 8L_1^3 = 0. \quad (53)$$

Remaining equations are obtained by differentiating (51) with respect to x and y. Excluding from them x and y these equations are reduced to the equation

$$-36Q_{tt}K^2L_1^2 + 12Q_tKL_1\big(22K_tL_1 - 15b_3KL_1 + 15b_4KL_2 - 12L_{1t}K - \alpha QL_1^2\big) - 2Q\big(6K\big(L_{1t} - b_4L_2 + b_3L_1\big) - 10K_tL_1\big)^2 \\ -4\alpha Q^2L_1^2\big(6K\big(b_3L_1 - b_4L_2 + L_{1t}\big) - 10K_tL_1\big) + 864b_4QK^4L_1 - QL_1^4 - Q^2L_1^4 = 0. \quad (54)$$

Thus, the necessary and sufficient conditions for an equation $y'' = F(x, y, y'')$ which can be transformed to the second Painlevé equations are: this equation has to be of the form (5), where the coefficients satisfy the equations $v_5 = 0$, $w_1 = 0$, (45), (52)-(54), where the functions $K(t,u)$ and $Q(t,u)$ are defined by Equations (44) and (50). The transformation of the Equation (5) into the second Painlevé equation (*PII*) is defined by Equations (49) and (51).

3. Example of the Results

Example. The following equation is equivalent to the first Painlevé equation (*PI*)

$$u'' + (1/t)u' - u - u^2/2 - 392/\big(625t^4\big) = 0.$$

This equation has to be of the form (5) with the coefficients

$$b_1 = 0, b_2 = 0, b_3 = 1/(3t), b_4 = -u - u^2/2 - 392/\big(625t^4\big).$$

satisfying the conditions

$$L_1 = -1, L_2 = 0, K = 2s/t^2, Q = 6\times5^4t^4s^{-2}, R = 6\times10^4t^{-1},$$

where $s = 25t^2(u+1) - 4$. Equations (19)-(21), (24), (28) and (32) are satisfied and Equations (25) and (31) become $x = -Qy^2, y = 3^{-3}\times10^{-8}t^{-8}s$. The changes of variable are the following:

$$t = (1/10)(1/1944x)^{1/12}, u = \sqrt{6/x}\,y + 16\times(3)^{1/3}\sqrt{6x} - 1.$$

4. Conclusion

The necessary and sufficient conditions that an equation of the form $y'' = F(x, y, y'')$ to be equivalent to the first and second Painlevé equation under a general point transformation are obtained. As was noted some of the necessary conditions are $v_5 = 0$ and $w_1 = 0$. Other conditions are also expressed in terms of relations for the coefficients of Equation (2). A procedure to check these conditions is found. Since intermediate calculations in the equivalence problem are cumbersome, computer algebra system have become an important computational tool.

5. Acknowledgements

This research is supported by Commission on Higher Education and the Thailand Research Fund under Grant. No. MRG 4980154, Naresuan University and Suranaree University of Technology.

REFERENCES

[1] S. Lie, "Klassifikation und Integration von Gewonlichen Differentialgleichungen Zwischen x,y, Die Eine Gruppe von Transformationen Gestaten. III," *Archiv for Matematik og Naturvidenskab*, Vol. 8, No. 4, 1883, pp. 371-427.

[2] R. Liouville, "Sur les Invariants de Certaines Equations Differentielles et sur Leurs Applications," *Journal de l'École Polytechnique*, Vol. 59, 1889, pp. 7-76.

[3] A. M. Tresse, "Détermination des Invariants Ponctuels de

l'Équation Différentielle Ordinaire du Second Ordre $y''=\omega(x,y,y')$," Preisschriften der Furstlichen Jablonowski'schen Gesellschaft XXXII, Leipzig, 1896.

[4] E. Cartan, "Sur les Variétés à Connexion Projective," *Bulletin de la Société Mathématique de France*, Vol. 52, 1924, pp. 205-241.

[5] M. V. Babich and L. A. Bordag, "Projective Differential Geometrical Structure of the Painleve Equations," *Journal of Differential Equations*, Vol. 157, No. 2, 1999, pp. 452-485.

[6] V. V. Kartak, "Explicit Solution of the Problem of Equivalence for Some Painleve Equations," *CUfa Math Journal*, Vol. 1, No. 3, 2009, pp. 1-11.

[7] N. H. Ibragimov, "Invariants of a Remarkable Family of Nonlinear Equations," *Nonlinear Dynamics*, Vol. 30, No. 2, 2002, pp. 155-166. doi:10.1023/A:1020406015011

[8] N. H. Ibragimov and S. V. Meleshko, "Linearization of Third-Order Ordinary Differential Equations by Point Transformations," *Archives of ALGA*, Vol. 1, 2004, pp. 71-93.

[9] N. H. Ibragimov and S. V. Meleshko, "Linearization of Third-Order Ordinary Differential Equations by Point Transformations," *Journal of Mathematical Analysis and Applications*, Vol. 308, No. 1, 2005, pp. 266-289. doi:10.1016/j.jmaa.2005.01.025