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Commuting Structure Jacobi Operator for Real Hypersurfaces in Complex Space Forms

U-Hang Ki[1], Hiroyuki Kurihara[2]
[1]The National Academy of Science, Seoul, Korea
[2]Hachinohe National College of Technology, Aomori, Japan
Email: uhangki2005@yahoo.co.kr, kurihara-g@hachinohe-ct.ac.jp

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ABSTRACT

Let M be a real hypersurface of a complex space form with almost contact metric structure (ϕ,ξ,η,g). In this paper, we prove that if the structure Jacobi operator $R_\xi = R(\cdot,\xi)\xi$ is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with the shape operator, then M is a Hopf hypersurface. Further, if R_ξ is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with the Ricci tensor, then M is also a Hopf hypersurface provided that $\mathrm{Tr}R_\xi$ is constant.

Keywords: Complex Space Form; Hopf Hypersurface; Structure Jacobi Operator; Shape Operator; Ricci Tensor

1. Introduction

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature $4c \neq 0$ is called a complex space form, which is denoted by $M_n(c)$. So naturally there exists a Kähler structure J and Kähler metric g on $M_n(c)$. Now let us consider a real hypersurface M in $M_n(c)$. Then we also denote by g the induced Riemannian metric of M and by N a local unit normal vector field of M in $M_n(c)$. Further, A denotes by the shape operator of M in $M_n(c)$. Then, an almost contact metric structure (ϕ,ξ,η,g) of M is naturally induced from the Kähler structure of M as follows:

$$\phi X = (JX)^{\mathrm{T}}, \xi = -JN, \eta(X) = g(X,\xi), X \in TM,$$

where TM denotes the tangent bundle of M and $(\ )^{\mathrm{T}}$ the tangential component of a vector. The Reeb vector ξ is said to be *principal* if $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$. A real hypersurface is said to a *Hopf hypersurface* if the Reeb vector ξ of M is principal. Hopf hypersurfaces is realized as tubes over certain submanifolds in $P_n\mathbb{C}$, by using its focal map (see Cecil and Ryan [1]). By making use of those results and the mentioned work of Takagi [2,3], Kimura [4] proved the local classification theorem for Hopf hypersurfaces of $P_n\mathbb{C}$ whose all principal curvatures are constant. For the case $H_n\mathbb{C}$, Berndt [5] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant. Among the several types of real hypersurfaces appeared in Takagi's list or Berndt's list.

The Reeb vector field ξ plays an important role in the theory of real hypersurfaces in a complex space form $M_n(c)$. Related to the Reeb vector field ξ the Jacobi operator R_ξ defined by $R_\xi = R(\cdot,\xi)\xi$ for the curvature tensor R on a real hypersurface M in $M_n(c)$ is said to be a *structure Jacobi operator* on M. The structure Jacobi operator has a fundamental role in contact geometry. In [6], Cho and first author started the study on real hypersurfaces in complex space form by using the operator R_ξ. In particular the structure Jacobi operator has been studied under the various commutative condition [7-9]. For example, Pérez *et al.* [9] called that real hypersurfaces M has commuting structure Jacobi operator if $R_\xi R_X = R_X R_\xi$ for any vector field X on M, and proved that there exist no real hypersurfaces in $M_n(c)$ with commuting structure Jacobi operator. On the other hand Ortega *et al.* [10] have proved that there are no real hypersurfaces in $M_n(c)$ with parallel structure Jacobi operator R_ξ, that is, $\nabla_X R_\xi = 0$ for any vector field X on M. More generally, such a result has been extended by [11]. In this situation, if naturally leads us to be consider another condition weaker than parallelness. In the preceding work, we investigate the weaker condition ξ-parallelness, that is, $\nabla_\xi R_\xi = 0$ (cf. [8,12,13]).

In this paper we consider the notion of $\phi\nabla_\xi\xi$-parallel structure Jacobi operator R_ξ, that is, $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ for the vector $\phi\nabla_\xi\xi$ orthogonal to ξ. Further we invetigate the structure Jacobi operator is $\phi\nabla_\xi\xi$-parallel under the condition that the structure Jacobi operator commute with the shape operator or the Ricci tensor.

This paper consists of two parts. In the first part of this paper, we prove that if the structure Jacobi operator $R_\xi = R(\cdot,\xi)\xi$ is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with the shape operator, then M is a Hopf hypersurface (see Theorem 1 in Section 4). In the second part of this paper, we prove that if R_ξ is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with the Ricci tensor, then M is also a Hopf hypersurface provided that $\mathrm{Tr}R_\xi$ is constant (see Theorem 2 in Section 5).

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

2. Fundamental Facts of Real Hypersurface

In this section the elemental factors of a real hypersurface are recalled. Let M be a real hypersurface in a complex space form $M_n(c)$ with constant holomorphic sectional curvature $4c, c \neq 0$ and N be a unit normal vector field on M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric $\tilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX,Y)N, \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$ and A is the shape operator of M in $M_n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, JN = -\xi,$$

for the complex structure J of $M_n(c)$. We call ξ the Reeb vector field. Then we may see that the aggregate (ϕ,ξ,η,g) is an almost contact metric structure on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi,$$
$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$\eta(\xi) = 1, \phi\xi = 0, \eta(X) = g(X,\xi)$$

for any vector fields X and Y on M. From Kähler condition $\tilde{\nabla}J = 0$, and making use of Gauss and Weingarten formulas, we obtain

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi \tag{2.1}$$

and

$$\nabla_X \xi = \phi AX \tag{2.2}$$

for any vector fields X,Y tangent to M.

The equations of Gauss and Codazzi are respectively given by the following:

$$\begin{aligned} &R(X,Y)Z \\ &= c\big(g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X \\ &\quad - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\big) \\ &\quad + g(AY,Z)AX - g(AX,Z)AY \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &(\nabla_X A)Y - (\nabla_Y A)X \\ &= c\big(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\big), \end{aligned} \tag{2.4}$$

where R denotes the curvature tensor of M.

In what follows, to write our formulas in convention forms, we denote by $\alpha = \eta(A\xi), \beta = \eta(A^2\xi), \gamma = \eta(A^3\xi)$ and $h = TrA$, and for a function f we denote by ∇f the gradient vector field of f.

If we put $U = \nabla_\xi \xi$, then U is orthogonal to the Reeb vector field ξ. We get

$$\phi U = -A\xi + \alpha\xi, \tag{2.5}$$

which shows that $g(U,U) = \beta - \alpha^2$. Thus we easily verify that ξ is a principal curvature vector, that is $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$.

From Gauss Equation (2.3), the Ricci tensor S of M is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X \tag{2.6}$$

for any vector field X on M.

If $A\xi - g(A\xi,\xi)\xi \neq 0$, then we can put

$$A\xi = \alpha\xi + \mu W, \tag{2.7}$$

where W is a unit vector field orthogonal to ξ. Then by (2.2) we see that $U = \mu\phi W$ and hence $g(U,U) = \mu^2$. So we have

$$\mu^2 = \beta - \alpha^2. \tag{2.8}$$

In this paper, we basically use the technical computations with the orthogonal triplet $\{\xi,U,W\}$ and their associated scalars α,β and μ.

Using (2.2) and (2.7), it is seen that

$$\mu g(\nabla_X W,\xi) = g(AU,X), \tag{2.9}$$

$$g(\nabla_X \xi,U) = \mu g(AW,X). \tag{2.10}$$

Now, differentiating (2.5) covariantly along M and making use of (2.1), (2.2) and (2.4), we find

$$\begin{aligned} (\nabla_X A)\xi = &-\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi \\ &- A\phi AX + \alpha\phi AX, \end{aligned} \tag{2.11}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{2.12}$$

By the definition of U, (2.2) and (2.12), it is verified that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{2.13}$$

From the Gauss Equation (2.3) the structure Jacobi operator R_ξ is given by

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= c\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \end{aligned} \tag{2.14}$$

for any vector field X on M.

Let Ω be the open subset of M defined by

$$\Omega = \{p \in M; \mu(p) \neq 0\}.$$

At each point of Ω, the Reeb vector field ξ is not principal. That is, ξ is not an eigenvector of the shape operator A of M if $\Omega \neq \emptyset$.

In what follows we assume that Ω is not an empty set in order to prove our main theorem by reductio ad absurdum, unless otherwise stated, all discussion concerns the set Ω.

3. Real Hypersurfaces Satisfying $R_\xi S = SR_\xi$

Let M be a real hypersurface in $M_n(c), c \neq 0$ satisfying $R_\xi S = SR_\xi$, which means that the Ricci tensor S of type (1,1) and the structure Jacobi operator R_ξ commute to each other. Then by (2.6) and (2.14) we have

$$\begin{aligned} &g(R_\xi(Y), SX) - g(R_\xi(X), SY) \\ &= g(A^3\xi, Y)g(A\xi, X) - g(A^3\xi, X)g(A\xi, Y) \\ &\quad - g(A^2\xi, Y)g(hA\xi - c\xi, X) \\ &\quad + g(A^2\xi, X)g(hA\xi - c\xi, Y) \\ &\quad - ch(g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)), \end{aligned}$$

which shows that

$$\alpha A^3\xi = (\alpha h - c)A^2\xi + (\gamma - \beta h + ch)A\xi + c(\beta - h\alpha)\xi. \tag{3.1}$$

Combining above two equations and using (2.7), we obtain

$$\begin{aligned} &\mu(g(A^2\xi, Y)w(X) - g(A^2\xi, X)w(Y)) \\ &= \beta(\eta(Y)g(A\xi, X) - \eta(X)g(A\xi, Y)), \end{aligned}$$

where a 1-form w is defined by $w(X) = g(W, X)$ for any vector field X. Putting $Y = A\xi$ in this, we find

$$\mu^2 g(A^2\xi, X) = \mu\gamma w(X) - \beta\alpha g(A\xi, X) + \beta^2\eta(X),$$

which shows that

$$\mu^2 A^2\xi = (\gamma - \beta\alpha)A\xi + (\beta^2 - \alpha\gamma)\xi. \tag{3.2}$$

Comparing (3.1) with (3.2), we find

$$(h - \rho)(\beta - \rho\alpha - c) = 0 \tag{3.3}$$

on Ω, where we have put $\mu^2\rho = \gamma - \beta\alpha$ and $\mu^2(\beta - \rho\alpha) = \beta^2 - \alpha\gamma$. So (3.2) becomes on Ω

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi, \tag{3.4}$$

which together with (2.7) yields

$$AW = \mu\xi + (\rho - \alpha)W \tag{3.5}$$

and hence

$$A^2W = \rho AW + (\beta - \rho\alpha)W. \tag{3.6}$$

Now, differentiating (3.5) covariantly along Ω, we find

$$\begin{aligned} &(\nabla_X A)W + A\nabla_X W \\ &= (X\mu)\xi + \mu\nabla_X\xi + X(\rho - \alpha)W + (\rho - \alpha)\nabla_X W. \end{aligned} \tag{3.7}$$

By taking the inner product with W in the last equation, we obtain

$$g((\nabla_X A)W, W) = -2g(AU, X) + X\rho - X\alpha \tag{3.8}$$

since W is a unit vector field orthogonal to ξ. We also have by applying ξ to (3.7) and making use of (2.9)

$$\mu g((\nabla_X A)W, \xi) = (\rho - 2\alpha)g(AU, X) + \mu(X\mu), \tag{3.9}$$

which together with the Codazzi Equation (2.4) gives

$$\mu(\nabla_W A)\xi = (\rho - 2\alpha)AU - 2cU + \mu\nabla\mu, \tag{3.10}$$

$$\mu(\nabla_\xi A)W = (\rho - 2\alpha)AU - cU + \mu\nabla\mu. \tag{3.11}$$

Putting $X = \xi$ in (3.8) and using (3.11), we obtain

$$W\mu = \xi\rho - \xi\alpha =: \xi\lambda, \tag{3.12}$$

where we have put $\lambda = w(AW)$. Differentiating (3.4) covariantly and using (2.2) we find

$$\begin{aligned} &g((\nabla_X A)A\xi, Y) + g(A(\nabla_X A)\xi, Y) \\ &+ g(A^2\phi AX, Y) - \rho g(A\phi AX, Y) \\ &= (X\rho)g(A\xi, Y) + \rho g((\nabla_X A)\xi, Y) \\ &\quad + X(\beta - \rho\alpha)\eta(Y) + (\beta - \rho\alpha)g(\phi AX, Y), \end{aligned} \tag{3.13}$$

which together with (2.4) and (2.12) implies that

$$(\nabla_\xi A)A\xi = \rho AU - cU + \frac{1}{2}\nabla\beta.$$

If we replace X by ξ in (3.13) and make use of (2.4), (2.12) and the last equation, then we get

$$\begin{aligned} &3A^2U - 2\rho AU + (\alpha\rho - \beta - c)U \\ &= (\xi\rho)A\xi + \xi(\beta - \alpha\rho)\xi - A\nabla\alpha + \rho\nabla\alpha - \frac{1}{2}\nabla\beta. \end{aligned} \tag{3.14}$$

Now, we define a 1-form u by $u(X) = g(U, X)$

for any vector field X, it is, using (2.4) and (3.13), seen that

$$\begin{aligned}
&c\left(u(Y)\eta(X)-u(X)\eta(Y)\right)+2c(\rho-\alpha)g(\phi Y,X)\\
&-g\left(A^2\phi AX,Y\right)+g\left(A^2\phi AY,X\right)\\
&+2\rho g(\phi AX,AY)\\
&-(\beta-\rho\alpha)\left(g(\phi AY,X)-g(\phi AX,Y)\right)\\
&=g\left(AY,(\nabla_X A)\xi\right)-g\left(AX,(\nabla_Y A)\xi\right)\\
&+(Y\rho)g(A\xi,X)-(X\rho)g(A\xi,Y)\\
&+Y(\beta-\rho\alpha)\eta(X)-X(\beta-\rho\alpha)\eta(Y).
\end{aligned} \tag{3.15}$$

If replace X by μW to both sides of (3.15) and take account of (2.12), (3.5), (3.6), (3.9) and (3.10), then we obtain

$$\begin{aligned}
&(3\alpha-2\rho)A^2U+2\left(\rho^2+\beta-2\rho\alpha+c\right)AU\\
&+(\rho-\alpha)(\beta-\rho\alpha-2c)U\\
&=\mu A\nabla\mu+(\rho\alpha-\beta)\nabla\alpha-\frac{1}{2}(\rho-\alpha)\nabla\beta\\
&+\mu^2\nabla\rho-\mu(W\rho)A\xi-\mu W(\beta-\rho\alpha)\xi.
\end{aligned} \tag{3.16}$$

Differentiating (2.14) covariantly along Ω, we find

$$\begin{aligned}
&g\left((\nabla_X R_\xi)Y,Z\right)\\
&=g\left(\nabla_X(R_\xi Y)-R_\xi(\nabla_X Y),Z\right)\\
&=-c\left(\eta(Z)g(\nabla_X\xi,Y)+\eta(Y)g(\nabla_X\xi,Z)\right)\\
&+(X\alpha)g(AY,Z)+\alpha g\left((\nabla_X A)Y,Z\right)\\
&-\eta(AZ)\left\{g\left((\nabla_X A)\xi,Y\right)+g(A\phi AX,Y)\right\}\\
&-\eta(AY)\left\{g\left((\nabla_X A)\xi,Z\right)+g(A\phi AX,Z)\right\}.
\end{aligned}$$

In the following we assume that M satisfies $\nabla_{\phi\nabla_\xi\xi}R_\xi=0$. Then we have $\nabla_W R_\xi=0$ on Ω because of (2.5) and (2.7). Putting $X=W$ in the last equation and using (2.2), we have

$$\begin{aligned}
&-c\left(\eta(Z)g(\phi AW,Y)+\eta(Y)g(\phi AW,Z)\right)\\
&+(W\alpha)g(AY,Z)+\alpha g\left((\nabla_W A)Y,Z\right)\\
&-\eta(AZ)\left(g\left((\nabla_W A)\xi,Y\right)+g(A\phi AW,Y)\right)\\
&-\eta(AY)\left(g\left((\nabla_W A)\xi,Z\right)+g(A\phi AW,Z)\right)=0
\end{aligned} \tag{3.17}$$

because of $\nabla_W R_\xi=0$. If we replace Y by ξ and make use of (2.12) and (3.5), then we obtain

$$\alpha A\phi AW+c\phi AW=0. \tag{3.18}$$

Remark 1. $\alpha\neq 0$ *on* Ω.

If not, then we have $\alpha=0$, and then we restrict our arguments on such a place. From (3.18) we have $\phi AW=0$, which together with (3.5) yields $\rho=0$ and hence (3.5) reformed as $AW=\mu\xi$. But, it is, using (2.8) and (3.12), that $W\beta=0$. So (3.16) turns out to be

$$2(\beta+c)AU=\frac{1}{2}A\nabla\beta, \tag{3.19}$$

where we have used (2.8) and $\rho=\alpha=0$.

On the other hand (3.17) is reduced to

$$\eta(AY)g\left((\nabla_W A)\xi,Z\right)+\eta(AZ)g\left((\nabla_W A)\xi,Y\right)=0$$

because of (3.18) with $\alpha=0$. If we replace Y by W and take account of (2.7), (2.8) and (3.10), then we obtain $(\nabla_W A)\xi=0$. Thus (3.10) becomes $\mu\nabla\mu=2cU$ and consequently $(1/2)\nabla\beta=2cU$ and hence $\xi\beta=0$. Accordingly (3.19) reformed as $\beta AU=0$ and thus $AU=0$. Using these facts, (3.14) is reduced to $(1/2)\nabla\beta=(\beta+c)U$. This contradicts the fact that $\nabla\beta=4cU$. Therefore $\alpha\neq 0$ on Ω is proved.

If we make use of (3.18) and Remark 1, then (3.17) reformed as

$$\begin{aligned}
&\alpha(\nabla_W A)X\\
&=-(W\alpha)AX+g(A\xi,X)(\nabla_W A)\xi\\
&+g\left((\nabla_W A)\xi,X\right)A\xi\\
&-\frac{c}{\alpha}\mu\left(w(X)\phi AW+g(\phi AW,X)W\right).
\end{aligned}$$

Using (3.5) and (3.10), we can write the last equation as

$$\begin{aligned}
&\alpha(\nabla_W A)X\\
&=-(W\alpha)AX-\frac{c}{\alpha}\lambda\left(w(X)U+u(X)W\right)\\
&+\frac{1}{\mu}\left\{(\rho-2\alpha)AU-2cU+\mu\nabla\mu\right\}g(A\xi,X)\\
&+\frac{1}{\mu}g\left((\rho-2\alpha)AU-2cU+\mu\nabla\mu,X\right)A\xi.
\end{aligned} \tag{3.20}$$

If we put $X=W$ in (3.20) and make use of (2.8), (3.8) and (3.12), then we obtain

$$\begin{aligned}
&\frac{1}{2}\nabla\beta-\alpha\nabla\rho\\
&=c\left(2+\frac{\lambda}{\alpha}\right)U-\rho AU+(W\alpha)AW-(\xi\lambda)A\xi.
\end{aligned} \tag{3.21}$$

Taking inner product W to this, and using (3.5) and (3.12), we find

$$\frac{1}{2}W\beta-\alpha(W\rho)=(\rho-\alpha)W\alpha-\mu(W\mu),$$

which together with (2.8) implies that

$$W\beta=\alpha(W\rho)+\rho(W\alpha). \tag{3.22}$$

If we take the inner product ξ to (3.21) and make use of (2.7) and (3.5), then we have $\xi\beta=2\mu(W\alpha)+2\alpha(\xi\alpha)$, which connected to (2.8) gives

$$\xi\mu = W\alpha. \tag{3.23}$$

From (2.8) we have $2\mu(W\mu) = W\beta - 2\alpha(W\alpha)$, which together with (3.12) and (3.22) yields

$$\alpha(W\lambda) = 2\mu(\xi\lambda) - \lambda(W\alpha). \tag{3.24}$$

4. Real Hypersurfaces Satisfying $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$ and $R_\xi A = AR_\xi$

Let M be a real hypersurface in $M_n(c), c \neq 0$ satisfying $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$. We have from (2.14)

$$\begin{aligned} &g(R_\xi Y, AX) - g(R_\xi X, AY) \\ &= g(A^2\xi, Y) g(A\xi, X) - g(A^2\xi, X) g(A\xi, Y) \\ &\quad - c\big(g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)\big). \end{aligned}$$

In the following we assume that $R_\xi A = AR_\xi$. Then we have from above equation

$$A^2\xi = \rho A\xi + c\xi, \tag{4.1}$$

which shows that

$$\beta = \rho\alpha + c. \tag{4.2}$$

Substituting (4.1) into the first equation of section 3, we find $R_\xi S - SR_\xi = 0$. Thus, all relationships (3.3)-(3.22) with $\beta = \rho\alpha + c$ are established on Ω. Combining (3.5) to (3.18), we obtain $\lambda(\alpha AU + cU) = 0$. So we have

$$\alpha AU + cU = 0. \tag{4.3}$$

In fact, if not, then we have $\lambda = 0$, that is, $\rho = \alpha$. Therefore, (3.5) and (4.2) are reduces respectively to $AW = \mu\xi$ and $\mu^2 = c$. So (3.16) becomes

$$\alpha A^2 U + 4cAU = -\mu(W\alpha)A\xi$$

on this subset. On the other hand, if we take the inner product W to (3.20) and make use of (3.8) and $\mu^2 = c$, then we obtain $\alpha AU = cU + \mu(W\alpha)\xi$. Comparing this with last equation, we verify that $U = 0$, a contradiction, Therefore (4.3) is established on whole space.

Because of (4.2) and (4.3), we can write (3.21) as

$$\frac{1}{2}(\alpha\nabla\rho - \rho\nabla\alpha) = -c\left(1 + \frac{2\rho}{\alpha}\right)U + (\xi\lambda)A\xi - (W\alpha)AW, \tag{4.4}$$

which shows that

$$\alpha^2(U\rho) - \rho\alpha(U\alpha) = -2c(2\rho + \alpha)\mu^2. \tag{4.5}$$

Using (4.3), we can also write (3.20) as

$$\begin{aligned} &\alpha(\nabla_W A)X \\ &= -(W\alpha)AX - \frac{c\lambda}{\alpha}\big(u(X)W + w(X)U\big) \\ &\quad + g\left(\nabla\mu - \frac{c\rho}{\alpha\mu}U, X\right)A\xi + g(A\xi, X)\left(\nabla\mu - \frac{c\rho}{\alpha\mu}U\right). \end{aligned}$$

Replacing X by U in this and using (4.3), we find

$$\alpha(\nabla_W A)U = (U\mu)A\xi + \frac{c}{\alpha}\big((W\alpha)U - \lambda\mu^2 W - \mu\rho A\xi\big). \tag{4.6}$$

If we take the inner product U to (3.7) and take account of (2.4), (2.10) and (4.3), then we obtain

$$\begin{aligned} &(\alpha\lambda + c)g(\nabla_X W, U) \\ &= \alpha g\big((\nabla_W A)X, U\big) + c\alpha\mu\eta(X) - \alpha\mu^2 g(AW, X), \end{aligned}$$

which together with (2.8), (4.2) and (4.6) yields

$$\begin{aligned} &\mu^2 g(\nabla_X W, U) \\ &= g\Big(X, (U\mu)A\xi + \frac{c}{\alpha}\big((W\alpha)U - \lambda\mu^2 W - \mu\rho A\xi\big) \\ &\quad + c\alpha\mu\xi - \alpha\mu^2 AW\Big). \end{aligned} \tag{4.7}$$

Putting $X = U$ in this, we have

$$g(\nabla_U W, U) = \frac{c}{\alpha}(W\alpha). \tag{4.8}$$

Now, applying by ϕ in (2.11) and using (2.10), we find

$$\begin{aligned} &\phi(\nabla_X A)\xi \\ &= \nabla_X U - \mu w(AX)\xi - \phi A\phi AX - \alpha AX + \alpha\eta(AX)\xi. \end{aligned}$$

If we put $X = U$ in this and make use of (2.5), (3.5) and (4.3), then we obtain

$$\nabla_U U = \phi(\nabla_U A)\xi + c\left(\frac{\lambda}{\alpha} - 1\right)U. \tag{4.9}$$

Taking the inner product U to (2.11), we also obtain

$$\begin{aligned} &g(\nabla_X W, U) \\ &= \frac{1}{\mu}g\big((\nabla_U A)X, \xi\big) - \left(\frac{c}{\alpha} + \alpha\right)g(AW, X) - 2cw(X), \end{aligned}$$

where we have used (2.4), (2.5) and (4.3), which together with (4.7) implies that

$$\begin{aligned} &\mu(\nabla_U A)\xi - \mu^2\left\{\left(\alpha + \frac{c}{\alpha}\right)AW + 2cW\right\} \\ &= (U\mu)A\xi + \frac{c}{\alpha}\big((W\alpha)U - \lambda\mu^2 W - \mu\rho A\xi\big) \\ &\quad + c\alpha\mu\xi - \alpha\mu^2 AW. \end{aligned}$$

If we apply by ϕ to this and make use of (3.5) and (4.9), then we obtain

$$\nabla_U U = -\frac{c}{\alpha}(W\alpha)W + \delta U \tag{4.10}$$

for some function δ on Ω.

On the other hand, differentiating (4.3) covariantly, and using itself again, we find

$$-\frac{c}{\alpha}(X\alpha)U+\alpha(\nabla_X A)U+\alpha A\nabla_X U+c\nabla_X U=0,$$

which together with (2.4) and (2.5) gives

$$\begin{aligned}&\frac{c}{\alpha}\big((Y\alpha)u(X)-(X\alpha)u(Y)\big)\\&+c\alpha\mu\big(\eta(X)w(Y)-\eta(Y)w(X)\big)\\&+\alpha\big(g(A\nabla_X U,Y)-g(A\nabla_Y U,X)\big)+cdu(X,Y)=0,\end{aligned}\tag{4.11}$$

where du is the exterior derivative of a 1-form u is given by

$$du(X,Y)=X\big(u(Y)\big)-Y\big(u(X)\big)-u\big([X,Y]\big).$$

Putting $X=\xi$ in (4.11) and taking account of (2.7) and (2.10), we get

$$\begin{aligned}&\alpha\mu g(W,\nabla_Y U)\\&=-\frac{c}{\alpha}(\xi\alpha)u(Y)+c\alpha\mu w(Y)+\mu(\alpha^2+c)g(AW,Y)\\&+g(\alpha A\nabla_\xi U+c\nabla_\xi U,Y),\end{aligned}$$

or putting $Y=U$ and making use of (4.3),

$$\alpha^2 g(\nabla_U U,W)=-c\mu(\xi\alpha).$$

From this and (4.8) it follows that

$$\alpha(W\alpha)=\mu(\xi\alpha)\tag{4.12}$$

because U is orthogonal to W. Comparing this with (4.10), we have

$$\nabla_U U=-\frac{c}{\alpha^2}\mu(\xi\alpha)W+\delta U,$$

which together with (4.3) implies that

$$\alpha A\nabla_U U+c\nabla_U U=-\frac{c}{\alpha}\mu(\xi\alpha)\left(AW+\frac{c}{\alpha}W\right).$$

By virtue of (2.7), (3.5) and (4.2), we can write this as

$$\alpha A\nabla_U U+c\nabla_U U=-\frac{c}{\alpha^2}\mu^2(\xi\alpha)A\xi.$$

If we put $X=U$ in (4.11) and take account of the last equation, then we obtain

$$\mu^2(Y\alpha)-(U\alpha)u(Y)-\frac{1}{\alpha}g\big(\mu^2(\xi\alpha)A\xi,Y\big)=0.$$

Therefore we have

$$\alpha\nabla\alpha=\frac{\alpha(U\alpha)}{\mu^2}U+(\xi\alpha)A\xi.\tag{4.13}$$

Using (4.2) and (4.3), we can write (3.14) as

$$\frac{1}{2}(\alpha\nabla\rho-\rho\nabla\alpha)=c\left(2-\frac{2\rho}{\alpha}-\frac{3\rho}{\alpha^2}\right)U+(\xi\rho)A\xi-A\nabla\alpha,$$

which together with (4.4) implies that

$$A\nabla\alpha+3c\left(\frac{c}{\alpha^2}-1\right)U=(\xi\alpha)A\xi+(W\alpha)AW.$$

If we take the inner product U to this, and make use of (4.3), we deduce that

$$\alpha(U\alpha)=3(c-\alpha^2)\mu^2.$$

Thus, (4.13) reformed as

$$\alpha\nabla\alpha=3(c-\alpha^2)U+\theta A\xi,\tag{4.14}$$

where we have put $\theta=\xi\alpha$.

Now, we are going to prove that $\theta=0$ on Ω. For this, the last equation is rewritten as

$$\frac{1}{2}Y\alpha^2=3(c-\alpha^2)u(Y)+\theta\eta(AY).$$

Differentiating this with respect to a vector field X again, and taking the skew-symmetric parts with respect to X and Y, then we eventually have

$$\begin{aligned}&(X\theta)\eta(AY)-(Y\theta)\eta(AX)+3(c-\alpha^2)du(X,Y)\\&=\theta\{6\big(\eta(AX)u(Y)-\eta(AY)u(X)\big)\\&-2g(A\phi AX,Y)+2cg(\phi X,Y)\}.\end{aligned}\tag{4.15}$$

Putting $Y=\xi$ in this, we find

$$\begin{aligned}&\alpha(X\theta)-(\xi\theta)\eta(AX)+3(c-\alpha^2)du(X,\xi)\\&=2\theta\big(u(AX)-3\alpha u(X)\big).\end{aligned}\tag{4.16}$$

By the way, we see, using (2.10) and (2.13), that $-du(X,\xi)=g(\mu AW+3\phi AU+\alpha A\xi-\beta\xi+\phi\nabla\alpha,X)$, which together with (2.7), (2.8), (3.5) and (4.3) gives

$$-du(X,\xi)=g\left(\mu\left(\rho+\frac{3c}{\alpha}\right)W+\phi\nabla\alpha,X\right).$$

Thus it follows, using (2.5) and (4.14), that

$$du(X,\xi)=-g\left(\mu(\rho+3\alpha)W+\frac{\theta}{\alpha}U,X\right).\tag{4.17}$$

Substituting this into (4.16), we find

$$\begin{aligned}&\alpha\nabla\theta-(\xi\theta)A\xi\\&=3\mu(c-\alpha^2)(\rho+3\alpha)W+\theta\left(\frac{c}{\alpha}-9\alpha\right)U,\end{aligned}\tag{4.18}$$

where we have used (4.3). Comparing this to (4.15), we get

$$\begin{aligned}&(c-\alpha^2)\\&\{\mu(\rho+3\alpha)\big(w(X)\eta(Y)-w(Y)\eta(X)\big)+du(X,Y)\}\\&=\theta\left(\frac{c}{3\alpha^2}-1\right)\big(\eta(AX)u(Y)-\eta(AY)u(X)\big)\\&+\frac{2}{3}\theta\big(g(A\phi AX,Y)+cg(\phi X,Y)\big).\end{aligned}\tag{4.19}$$

If we take the inner product ξ to (4.4) and make use of (2.8), (4.2) and (4.12), then we obtain

$$\alpha^2(\xi\rho) = (\rho\alpha + 2c)\xi\alpha. \tag{4.20}$$

Using this, we can write (4.4) as

$$\frac{1}{2}\alpha^2(\alpha\nabla\rho - \rho\nabla\alpha) = -c\alpha(2\rho+\alpha)U + c(\xi\alpha)(2A\xi - \alpha\xi),$$

where we have used (2.7), (2.8), (3.5), (4.2) and (4.12). Combining this with (4.14), we obtain

$$\frac{1}{2}\alpha^3\nabla\rho - \left(\frac{1}{2}\rho\alpha^2 + 2c\alpha\right)\nabla\alpha = c\left(5\alpha^2 - 2\rho\alpha - 6c\right) - c\alpha(\xi\alpha)\xi, \tag{4.21}$$

which tells us that

$$\begin{aligned}&\frac{1}{2}\alpha^3\left((X\rho)(Y\alpha) - (Y\rho)(X\alpha)\right)\\ &= c\left(5\alpha^2 - 2\rho\alpha - 6c\right)\left((Y\alpha)u(X) - (X\alpha)u(Y)\right)\\ &\quad - c\alpha(\xi\alpha)\left(\eta(X)(Y\alpha) - \eta(Y)(X\alpha)\right).\end{aligned} \tag{4.22}$$

Using the quite same method as that used to (4.19) from (4.14), we can drive from (4.21) the following:

$$\begin{aligned}&\left(10\alpha - 6\rho - \frac{24}{\alpha}c\right)\left((X\alpha)u(Y) - (Y\alpha)u(X)\right)\\ &\quad + 3(\xi\alpha)\left(\eta(X)(Y\alpha) - \eta(Y)(X\alpha)\right)\\ &\quad + 2\alpha\left((X\rho)u(Y) - (Y\rho)u(X)\right)\\ &= \alpha\left(\eta(X)(Y\theta) - \eta(Y)(X\theta)\right)\\ &\quad - \alpha(\xi\alpha)g\left((\phi A - A\phi)X, Y\right)\\ &\quad + \left(5\alpha^2 - 2\rho\alpha - 6c\right)du(X,Y),\end{aligned} \tag{4.23}$$

where we have used (2.2) and (4.22). Putting $X = \xi$ in this and using (4.14), (4.17) and (4.20), we obtain

$$\begin{aligned}&\left(3\alpha + 2\rho + \frac{5}{\alpha}c\right)(\xi\alpha)U + \alpha\left(\nabla\theta - (\xi\theta)\xi\right)\\ &+ \mu\left\{(\rho+3\alpha)\left(5\alpha^2 - 2\rho\alpha - 6c\right) - \frac{3}{\alpha}(\xi\alpha)^2\right\}W = 0,\end{aligned}$$

which together with (2.7) and (4.18) implies that

$$\begin{aligned}&\left(6\alpha - 2\rho - \frac{6}{\alpha}c\right)(\xi\alpha)U\\ &= \mu\left\{\xi\theta + (\rho+3\alpha)\left(2\alpha^2 - 2\rho\alpha - 3c\right) - \frac{3}{\alpha}(\xi\alpha)^2\right\}W.\end{aligned}$$

Thus, it follows that

$$\left(6\alpha^2 - 2\rho\alpha - 6c\right)\xi\alpha = 0$$

because U and W are orthogonal to each other, and hence $6\alpha^2 - 2\rho\alpha - 6c = 0$ if $\xi\alpha \neq 0$. Differentiation gives $(6\alpha - \rho)\nabla\alpha = \alpha\nabla\rho$ on this subset, which together with (4.20) yields $c = 0$, a contradiction. Thus, $\xi\alpha = 0$ on Ω is proved. Consequently we prove that $W\alpha = 0$ by virtue of (4.12) and Remark 1. By (4.20) we also have $\xi\rho = 0$. Therefore (4.14) and (4.21) are reduced respectively to

$$\alpha\nabla\alpha = 3\left(c - \alpha^2\right)U, \tag{4.24}$$

$$\frac{1}{2}\alpha^3\nabla\rho - \left(\frac{1}{2}\rho\alpha^2 + 2c\alpha\right)\nabla\alpha = c\left(5\alpha^2 - 2\rho\alpha - 6c\right)U. \tag{4.25}$$

From (4.24) we have $\left(\alpha^2 - c\right)du(X,Y) = 0$ and hence $\left(\alpha^2 - c\right)du(X,\xi) = 0$, which together with (4.17) gives

$$\left(\alpha^2 - c\right)(\rho + 3\alpha) = 0.$$

If $\alpha^2 - c \neq 0$ on Ω, and then we restrict our arguments on such a place. Then we have $\rho + 3\alpha = 0$ and thus $U\rho = -3(U\alpha)$, which together with (4.5) yields $2\rho + \alpha = 0$, a contradiction because of Remark 1. Accordingly we have $\alpha^2 = c$ on Ω and hence α is constant. Thus, (4.25) becomes

$$\frac{1}{2}\nabla\rho = -(2\rho + \alpha)U.$$

On the other hand, using $\xi\alpha = 0$ and the last equation, we can write (4.23) as
$\left(5\alpha^2 - 2\rho\alpha - 6c\right)du(X,\xi) = 0$ and consequently
$(2\rho + \alpha)(\rho + 3\alpha) = 0$ by virtue of (4.17). Therefore we have $2\rho + \alpha = 0$, which together with (4.2) and $\alpha^2 = c$ implies that $2\beta = \alpha^2$, a contradiction. Thus, we deduce that $\Omega = \varnothing$. Accordingly we have

Lemma 1. $\Omega = \varnothing$ *if it satisfies* $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ *and* $R_\xi A = AR_\xi$.

From this we conclude that

Theorem 1. *Let* M *be a real hypersurface in* $M_n(c), c \neq 0$. *If it satisfies* $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ *and at the same time* $R_\xi A = AR_\xi$, *then* M *is a Hopf hypersurface in* $M_n(c)$.

5. Real Hypersurfaces with $\mathrm{Tr}R_\xi = \mathrm{const}$

In this section, we will continue our arguments under the same hypotheses as those stated in section 3, namely $R_\xi S = SR_\xi$ and $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ hold on M. Then we have

$$h = \rho. \tag{5.1}$$

Indeed, if not, then we have $\beta = \rho\alpha + c$ because of (3.3). Thus, (3.4) becomes $A^2\xi = \rho A\xi + c\xi$. So we have $R_\xi A = AR_\xi$ by virtue of the first equation of section 4. By Lemma 1, we verify that $\Omega = \varnothing$, a contradiction. Thus, $h = \rho$ is established on the whole space.

Furthermore, we assume that if $\mathrm{Tr}R_\xi = \mathrm{const}$. Then we obtain from (2.14)

$$\beta - h\alpha = \text{const.} \tag{5.2}$$

This is equivalent to $g(S\xi,\xi) = \mathrm{const}$. by virtue of (2.6). Because of (3.5), we can write (3.18) as

$$\lambda(\alpha AU + cU) = 0. \tag{5.3}$$

Frist of all, we prove

Lemma 2. $\alpha AU + cU = 0$ *on* Ω *if* $\mathrm{Tr}R_\xi = \mathrm{const}$.

Proof. If not, then we have $\lambda = 0$, that is $h - \alpha = 0$ with the aid of (5.1) and (5.3). And we restrict arguments on such a place. Then (3.5) becomes

$$AW = \mu\xi. \tag{5.4}$$

Because of (2.8), (3.12) and (5.1), we can write (3.16) as

$$\alpha A^2U + 2(\mu^2 + c)AU = \mu A\nabla\mu - \mu(W\alpha)A\xi. \tag{5.5}$$

Using (2.8) and (5.4), the Equation (3.21) reformed as

$$\mu\nabla\mu = 2cU - \alpha AU + \mu(W\alpha)\xi.$$

Since $h = \alpha$, we see, using (5.2), that $\nabla\mu = 0$. Hence, taking the inner product ξ to the above equation, we have $W\alpha = 0$. Thus, we verify from the last equation

$$\alpha AU = 2cU. \tag{5.6}$$

We also have from (5.5) $\alpha A^2U + 2(\mu^2 + c)AU = 0$. Combining the last two equations, we obtain $(\mu^2 + 2c)AU = 0$. Thus, we see, using (5.6), that

$$\mu^2 + 2c = 0. \tag{5.7}$$

On the other hand, if we take the inner product U to (2.11) and make use of (2.4), (5.4), (5.6) and (5.7), then we obtain

$$g(\nabla_X W, U) = \frac{1}{\mu} g((\nabla_U A)\xi, X) - 2cw(X) + \frac{2c}{\mu}\left(\alpha - \frac{2c}{\alpha}\right)\eta(X). \tag{5.8}$$

Applying (2.11) by ϕ and taking account of (2.10), we also deduce that

$$\phi(\nabla_X A)\xi = \nabla_X U + \mu w(AX)\xi - \phi A\phi AX - \alpha AX + \alpha g(A\xi, X)\xi.$$

If we put $X = U$ in this and use (5.4) and (5.6), then we get

$$\nabla_U U = \phi(\nabla_U A)\xi + 2cU. \tag{5.9}$$

Now, differentiating (5.4) covariantly and using (2.2) and (5.7), we find

$$(\nabla_X A)W + A\nabla_X W = \mu\phi AX.$$

If we take the inner product U to this and make use of (2.4), (5.4), (5.6) and (5.7), then we obtain

$$\frac{2c}{\alpha} g(\nabla_X W, U) = -c\mu\eta(X) - g((\nabla_U A)W, X). \tag{5.10}$$

By the way, we have from (3.20)

$$\mu\alpha(\nabla_W A)X = -4c(g(A\xi, X)U + u(X)A\xi),$$

where we have used $W\alpha = 0, \lambda = 0$, (5.6) and (5.7), which together with (2.4) gives

$$(\nabla_U A)W = 2c\mu\xi - \frac{4c\mu}{\alpha}A\xi.$$

Substituting this into (5.10) and using (5.7), we find

$$g(\nabla_X W, U) = \frac{3c\alpha}{\mu}\eta(X) + 2\mu g(A\xi, X),$$

which together with (5.8) implies that

$$(\nabla_U A)\xi = -2c\mu W + c\left(-3\alpha + \frac{4c}{\alpha}\right)\xi.$$

Thus, it follows that $\phi(\nabla_U A)\xi = -2cU$. From this and (5.9) we verify that

$$\nabla_U U = 0. \tag{5.11}$$

Differentiating (5.6) covariantly, and using itself again, we get

$$\frac{2c}{\alpha}(X\alpha)U + \alpha(\nabla_X A)U + \alpha A\nabla_X U = 2c\nabla_X U,$$

or, using (2.4) and (2.5)

$$\frac{2c}{\alpha}(X\alpha)u(Y) + \alpha g((\nabla_U A)X, Y) - c\alpha\mu\eta(X)w(Y) - 2c\alpha\mu w(X)\eta(Y) + \alpha g(A\nabla_X U, Y) - 2cg(\nabla_X U, Y) = 0.$$

Taking skew-symmetric part with respect to X and Y, we find

$$\frac{2c}{\alpha}((X\alpha)u(Y) - (Y\alpha)u(X)) + c\alpha\mu(\eta(X)w(Y) - \eta(Y)w(X)) + \alpha(g(A\nabla_X U, Y) - g(A\nabla_Y U, X)) - 2cdu(X, Y) = 0.$$

If we put $Y = U$ in this and make use of (5.6) and (5.11), then we obtain

$$\mu^2\nabla\alpha = (U\alpha)U, \tag{5.12}$$

which enables us to obtain $\xi\alpha = 0$. Accordingly (3.14) turns out to be

$$3A^2U - 2\alpha AU - (\mu^2 + c)U + A\nabla\alpha = 0,$$

where we have used (2.8) and $\rho = \alpha$, which together with (5.6) and (5.7) gives $3c\left(4c-\alpha^2\right)U+\alpha^2 A\nabla\alpha=0$. Thus, it is seen that $\alpha\left(U\alpha\right)=3c\left(4c-\alpha^2\right)$. So we can write (5.12) as $\nabla\alpha^2=3\left(\alpha^2-4c\right)U$. Differentiating this covariantly and taking the skew-symmtric part, we eventually have $\left(\alpha^2-4c\right)du\left(X,Y\right)=0$, which implies that $\left(\alpha^2-4c\right)du\left(X,\xi\right)=0$. This together with (5.4), (5.6) and (5.12) yields $\alpha\left(\alpha^2-4c\right)=0$, Thus we have $\alpha^2-4c=0$ because of Remark 1. It is contradictory by virtue of (5.7). Consequently Lemma 2 is proved.

Using (5.1) and Lemma 2, we can write respectively (3.20) and (3.21) as

$$\begin{aligned}&\alpha\left(\nabla_W A\right)X\\&=-\left(W\alpha\right)AX-\frac{c}{\alpha}\lambda\left(w\left(X\right)U+u\left(X\right)W\right)\\&+g\left(\nabla\mu-\frac{ch}{\alpha\mu}U,X\right)A\xi+g\left(A\xi,X\right)\left(\nabla\mu-\frac{ch}{\alpha\mu}U\right),\end{aligned}\tag{5.13}$$

$$\mu\nabla\mu-\alpha\nabla\lambda=\left(W\alpha\right)AW-\left(\xi\lambda\right)A\xi+c\left(3+\frac{2\lambda}{\alpha}\right)U.\tag{5.14}$$

Replacing X by U in (5.13) and remembering Lemma 2, we find

$$\alpha\left(\nabla_W A\right)U=\left(U\mu\right)A\xi+\frac{c}{\alpha}\left(\left(W\alpha\right)U-\lambda\mu^2W-\mu hA\xi\right).$$

On the other hand, if we take the inner product (3.7) with U, and make use of (2.4), (2.10) and Lemma 2, then we get

$$\begin{aligned}&\left(\alpha\lambda+c\right)g\left(\nabla_X W,U\right)\\&=\alpha g\left(\left(\nabla_W A\right)U,X\right)+c\alpha\mu\eta\left(X\right)-\alpha\mu^2g\left(AW,X\right),\end{aligned}$$

which connected to the last equation implies that

$$\begin{aligned}&\left(\alpha\lambda+c\right)g\left(\nabla_X W,U\right)\\&=g\left(X,\left(U\mu\right)A\xi+\frac{c}{\alpha}\left(\left(W\alpha\right)U-\lambda\mu^2W-\mu hA\xi\right)\right)\\&+c\alpha\mu\eta\left(X\right)-\alpha\mu^2g\left(AW,X\right).\end{aligned}\tag{5.15}$$

Thus, it follows that

$$\left(\alpha\lambda+c\right)g\left(\nabla_U W,U\right)=\frac{c}{\alpha}\mu^2\left(W\alpha\right).\tag{5.16}$$

Using the quite same method as that used to derive (4.10) from (3.7), (4.6) and (2.11), we can derive from (3.7), (5.15) and (2.11) the following:

$$\left(\alpha\lambda+c\right)\nabla_U U=-\frac{c}{\alpha}\mu^2\left(W\alpha\right)W+\delta U.\tag{5.17}$$

where the function δ is given by

$$\begin{aligned}&\delta=\mu\left(U\mu\right)+\left(\alpha\lambda+c\right)\\&\left\{2c+\lambda\left(\frac{c}{\alpha}+\alpha\right)\right\}-\left\{\frac{c}{\alpha}\left(2\lambda+\alpha\right)+\alpha\lambda\right\}\mu^2.\end{aligned}\tag{5.18}$$

We notice here that the following:

Remark 2. $\alpha\lambda+c\neq0$ *on* Ω *if* $TrR_\xi=\text{const}$.

In fact, if not, then $\alpha\lambda+c=0$. So we have from (5.16) $W\alpha=0$ and hence $\delta=0$ because of (5.17). Thus, (5.18) implies that $\alpha\left(U\mu\right)=2c\lambda\mu$ on this subset. However, by putting $X=\xi$ in (5.15) we obtain $\alpha\left(U\mu\right)=\left(\alpha\mu^2+c\lambda\right)\mu$. Combining the last two equations, we verify that $\alpha\mu^2=c\lambda$, which together with $\alpha\lambda+c=0$ gives $\mu^2+\lambda^2=0$, which will produce a contradiction. Therefore $\alpha\lambda+c\neq0$ on Ω is proved.

Because of (5.17) and Lemma 2, it is seen that

$$\alpha A\nabla_U U+c\nabla_U U=-\frac{c}{\alpha\lambda+c}\mu^2\left(W\alpha\right)\left(AW+\frac{c}{\alpha}W\right)\tag{5.19}$$

by virtue of Remark 2.

Now, differentiating $\alpha AU+cU=0$ covariantly and using itself, we find

$$-\frac{c}{\alpha}\left(X\alpha\right)U+\alpha\left(\nabla_X A\right)U+\alpha A\nabla_X U+c\nabla_X U=0,$$

which together with (2.4) and (2.5) implies that

$$\begin{aligned}&\frac{c}{\alpha}\left(\left(Y\alpha\right)u\left(X\right)-\left(X\alpha\right)u\left(Y\right)\right)\\&+c\alpha\mu\left(\eta\left(X\right)w\left(Y\right)-\eta\left(Y\right)w\left(X\right)\right)\\&+\alpha\left(g\left(A\nabla_X U,Y\right)-g\left(A\nabla_Y U,X\right)\right)+cdu\left(X,Y\right)=0.\end{aligned}\tag{5.20}$$

If we put $X=\xi$ in this, and make use of (2.7) and (2.10), then we get

$$\begin{aligned}&\alpha\mu g\left(W,\nabla_Y U\right)\\&=-\frac{c}{\alpha}\left(\xi\alpha\right)u\left(Y\right)+c\alpha\mu w\left(Y\right)\\&+\mu\left(\alpha^2+c\right)g\left(AW,Y\right)+g\left(\alpha A\nabla_\xi U+c\nabla_\xi U,Y\right).\end{aligned}\tag{5.21}$$

By putting $Y=U$ in this and using Lemma 2, we have $\alpha^2g\left(W,\nabla_U U\right)=-c\mu\left(\xi\alpha\right)$ and hence $\alpha^2g\left(\nabla_U W,U\right)=c\mu\left(\xi\alpha\right)$ because U and W are mutually orthogonal. From this and (5.16) we verify that

$$\alpha\mu\left(W\alpha\right)=\left(\alpha\lambda+c\right)\left(\xi\alpha\right).\tag{5.22}$$

If we replace X by U in (5.20) and make use of (5.19), (5.22) and Lemma 2, we obtain

$$\alpha\nabla\alpha=\frac{\alpha\left(U\alpha\right)}{\mu^2}U+\frac{\xi\alpha}{\mu}\left(\alpha AW+cW\right).\tag{5.23}$$

We are now going to prove the following:

Lemma 3. *If* TrR_ξ *is constant, then we have*

$$\left(\alpha\lambda+c\right)\alpha\nabla\alpha=\alpha\left(W\alpha\right)\left(\alpha AW+cW\right)+2fU\tag{5.24}$$

on Ω, *where the function* f *is given by*

$$2f = 2\alpha^3\lambda + 4c\alpha\lambda + 2c\alpha^2 + 3c^2 - \alpha^2\mu^2. \quad (5.25)$$

Proof. Using (2.7), (2.8) and Lemma 2, we can write (2.13) as

$$\alpha\nabla_\xi U = \left(\alpha^2 + 3c\right)\mu W - \alpha\mu^2\xi + \alpha\phi\nabla\alpha,$$

which together with (3.5), (5.22) and (5.23) implies that

$$\nabla_\xi U = \left(\alpha + \frac{3c}{\alpha} - \frac{U\alpha}{\mu^2}\right)\mu W - \mu^2\xi + \frac{W\alpha}{\mu}U.$$

From this and Lemma 2, we obtain

$$\begin{aligned}&\alpha A\nabla_\xi U + c\nabla_\xi U \\ &= \left(\alpha + \frac{3c}{\alpha} - \frac{U\alpha}{\mu^2}\right)\mu\left(\alpha AW + cW\right) - \mu^2\left(\alpha A\xi + c\xi\right).\end{aligned}$$

Using (5.12), Lemma 2 and the last equation, we have

$$\begin{aligned}&\alpha g\left(\nabla_X W, U\right) \\ &= \frac{c}{\alpha\mu}(\xi\alpha)u(X) + \frac{U\alpha}{\mu^2}g\left(\alpha AW + cW, X\right) \\ &\quad - 2\left(\alpha^2 + 2c\right)g\left(AW, X\right) \\ &\quad - c\left(2\alpha + \frac{3c}{\alpha}\right)w(X) + \mu g\left(\alpha A\xi + c\xi, X\right).\end{aligned}$$

By virtue of this and (5.15), it is verified that

$$\begin{aligned}&\frac{\alpha\lambda + c}{\mu^2}(U\alpha)\left(\alpha AW + cW\right) - \alpha\left(U\mu\right)A\xi \\ &= -c\lambda\mu^2 W - c\left(\lambda + \alpha\right)\mu A\xi + c\alpha^2\mu\xi - \alpha^2\mu^2 AW \\ &\quad + \left(\alpha\lambda + c\right) \\ &\quad \left\{2\left(\alpha^2 + 2c\right)AW + c\left(2\alpha + \frac{3c}{\alpha}\right)W - \mu\left(\alpha A\xi + c\xi\right)\right\}.\end{aligned}$$

Since ξ and W are mutually orthogonal, we see from the last equation the following:

$$\begin{aligned}&\frac{\alpha\lambda + c}{\mu^2}U\alpha = -\left(c\lambda + \alpha\mu^2\right) + \frac{\alpha\left(U\mu\right)}{\mu} + \left(\alpha\lambda + c\right)\left(\alpha + \frac{3c}{\alpha}\right), \\ &\frac{\left(\alpha\lambda + c\right)^2}{\mu^2}U\alpha = \alpha\mu\left(U\mu\right) - \left(2c\lambda + c\alpha + \lambda\alpha^2\right)\mu^2 \\ &+ \left(\alpha\lambda + c\right)\left\{2\left(\alpha^2 + 2c\right)\lambda + c\left(2\alpha + \frac{3c}{\alpha}\right) - \alpha\mu^2\right\}.\end{aligned} \quad (5.26)$$

Eliminating $U\alpha$ from above two equations, we eventually have

$$\left\{\alpha\left(U\mu\right) - \left(\lambda\alpha^2 + 2c\lambda + c\alpha\right)\right\}\mu\left(\alpha\lambda + c - \mu^2\right) = 0. \quad (5.27)$$

Now, suppose that $\alpha\left(U\mu\right) - \left(\lambda\alpha^2 + 2c\lambda + c\alpha\right)\mu \neq 0$ on Ω. Then we have $\alpha\lambda + c - \mu^2 = 0$, which together with (2.8) and (5.1) yields $\beta - \rho\alpha - c = 0$ on this subset. So (3.4) becomes $A^2\xi = \rho A\xi + c\xi$ and consequently $R_\xi A = AR_\xi$ on this set. By Lemma 1 we see that $\Omega = \varnothing$, a contradiction. Therefore we have

$$\alpha\left(U\mu\right) = \left(\lambda\alpha^2 + 2c\lambda + c\alpha\right)\mu \quad (5.28)$$

with the aid of (5.27). Comparing this with (5.26), it follows that

$$\begin{aligned}&\alpha\left(\alpha\lambda + c\right)U\alpha \\ &= \left(2\alpha^3\lambda + 4c\alpha\lambda + 2c\alpha^2 + 3c^2 - \alpha^2\mu^2\right)\mu^2.\end{aligned} \quad (5.29)$$

Therefore, (5.24) and (5.25) are established on Ω because of (5.22), (5.23) and (5.29). This completes the proof.

Now, differentiating (2.7) covariantly and using (2.2), we find

$$\begin{aligned}&\left(\nabla_X A\right)\xi + A\phi AX \\ &= \left(X\alpha\right)\xi + \alpha\phi AX + \left(X\mu\right)W + \mu\nabla_X W.\end{aligned} \quad (5.30)$$

Putting $X = \xi$ in this and making use of (2.12) and (3.23), we find

$$3AU - \alpha U - \left(\xi\alpha\right)\xi - \left(W\alpha\right)W + \nabla\alpha = \mu\nabla_\xi W,$$

which together with (5.22), (5.24), (5.25) and Lemma 2 gives

$$\mu\nabla_\xi W = \frac{1}{\alpha\lambda + c}\left(\alpha^2\lambda + c\lambda + c\alpha - \alpha\mu^2\right)U.$$

By the way, if we replace X by ξ in (3.7) and take account of (3.11), (3.23) and Lemma 1, then we obtain

$$\begin{aligned}&\mu\left(A\nabla_\xi W - \lambda\nabla_\xi W\right) \\ &= \left(\mu^2 + \frac{c}{\alpha}\lambda\right)U - \mu\nabla\mu + \mu\left(\xi\lambda\right)W + \mu\left(W\alpha\right)\xi,\end{aligned}$$

which connected to the last equation and Lemma 1 implies that

$$\mu\nabla\mu = \mu\left(W\alpha\right)\xi + \mu\left(\xi\lambda\right)W + \left(\alpha\lambda + \frac{2c\lambda}{\alpha} + c\right)U. \quad (5.31)$$

Using this and (3.24), we can write (5.14) as

$$\alpha\nabla\lambda = \alpha\left(\xi\lambda\right)\xi + \alpha\left(W\lambda\right)W + \left(\alpha\lambda - 2c\right)U. \quad (5.32)$$

From (5.2) we have

$$\nabla\beta = h\left(\nabla\alpha\right) + \alpha\left(\nabla h\right), \quad (5.33)$$

which shows that $\xi\beta = h\left(\xi\alpha\right) + \alpha\left(\xi h\right)$. Since we have

$$2\mu\left(W\alpha\right) = \xi\beta - 2\alpha\left(\xi\alpha\right)$$

because of (2.8) and (3.23), we verify, using the last equation, that $2\mu\left(W\alpha\right) = \alpha\left(\xi h\right) + \left(h - 2\alpha\right)\xi\alpha$. By virtue of this and (5.22), it is seen that $2\left(\alpha\lambda + c\right)\xi\alpha = \alpha^2\left(\xi h\right) + \alpha\left(\lambda - \alpha\right)\xi\alpha$. Thus, it follows that

$$\alpha^2(\xi\lambda) = (\alpha\lambda + 2c)\xi\alpha. \quad (5.34)$$

If we take the inner product U to (5.32), then we have $\alpha(U\lambda) = (\alpha\lambda - 2c)\mu^2$, which together with (5.33) gives

$$U\beta = (\alpha\lambda - 2c)\mu^2 + (\alpha + h)U\alpha.$$

On the other hand, we have from (2.8)

$$U\beta = 2\mu(U\mu) + 2\alpha(U\alpha).$$

Combining to the last two equations, we verify that

$$2\mu(U\mu) = (\alpha\lambda - 2c)\mu^2 + \lambda(U\alpha),$$

which together with (5.28) and (5.29) implies that

$$\alpha^3\lambda^2 - 3c\alpha^2\lambda - c^2\lambda - 4c^2\alpha = \alpha^2\lambda\mu^2. \quad (5.35)$$

Differentiating this with respect to ξ and using (3.23) and (5.22), we find

$$\left(\alpha^2\lambda^2 - 8c\alpha\lambda - 2\alpha\lambda\mu^2 - 4c^2\right)\xi\alpha$$
$$+\left(2\alpha^3\lambda - 3c\alpha^2 - c^2 - \alpha^2\mu^2\right)\xi\lambda = 0,$$

which together with (5.34) and (5.35) yields

$$\left(\lambda^2 + 4\alpha\lambda + 4c\right)\alpha(\xi\alpha) = 0.$$

From this and (5.34), we easily see that $\xi\alpha = 0$. So it is, using (5.22) and (5.34), seen that $W\alpha = 0$ and $\xi\lambda = 0$. Hence (3.24) becomes $W\lambda = 0$. Using these facts, (5.24), (5.31) and (5.32) turn out respectively to

$$(\alpha\lambda + c)\alpha\nabla\alpha = \left(2\alpha^3\lambda + 4c\alpha\lambda + 2c\alpha^2 + 3c^2 - \alpha^2\mu^2\right)U, \quad (5.36)$$

$$\alpha\mu\nabla\mu = \left(\alpha^2\lambda + 2c\lambda + c\alpha\right)U, \quad (5.37)$$

and

$$\alpha\nabla\lambda = (\alpha\lambda - 2c)U. \quad (5.38)$$

Now, we prove

Lemma 4. *$\alpha\lambda = 2c$ and $2\lambda + 3\alpha = 0$ on Ω if $TrR_\xi = \text{const}$.*

Proof. From (5.38) we have

$$(X\alpha)(Y\lambda) + \alpha X(Y\lambda)$$
$$= \left(\lambda(X\alpha) + \alpha(X\lambda)\right)u(Y) + (\alpha\lambda - 2c)X\left(u(Y)\right).$$

If we take the skew-symmtric parts of this and make use of (5.36) and (5.38), then we obtain $(\alpha\lambda - 2c)du(X,Y) = 0$ and hence

$$(\alpha\lambda - 2c)du(X,\xi) = 0.$$

In the same way we see from (5.36) that

$$\left(\alpha^2\lambda + 2c\lambda + c\alpha\right)du(X,\xi) = 0. \quad (5.39)$$

Now, we assume $\alpha\lambda - 2c \neq 0$ on Ω and that restrict our arguments on such a place. Then we have $du(\xi, X) = 0$, that is $g(\nabla_X\xi, U) + g(\nabla_\xi U, X) = 0$, which together with (2.10) and (2.13) gives

$$3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha + \mu AW = 0.$$

Thus, it is, using (2.7), (2.8), (3.5) and Lemma 2, seen that

$$\alpha\nabla\alpha = \left(\alpha\lambda + \alpha^2 + 3c\right)U, \quad (5.40)$$

where we have used the fact that $\xi\alpha = 0$, which connected to (5.36) implies that $\mu^2 = \alpha\lambda - \lambda^2 + c$. Differentiation gives $2\mu\nabla\mu = (\alpha - 2\lambda)\nabla\lambda + \lambda\nabla\alpha$, or using (5.37), (5.38) and (5.40)

$$2\left(\alpha^2\lambda + 2c\lambda + c\alpha\right)$$
$$= (\alpha - 2\lambda)(\alpha\lambda - 2c) + \lambda\left(\alpha\lambda + \alpha^2 + 3c\right).$$

Therefore, we obtain

$$\alpha\lambda^2 + 4c\alpha - 3c\lambda = 0. \quad (5.41)$$

If we differentiate this, then we have $\left(\lambda^2 + 4c\right)\nabla\alpha + (2\alpha\lambda - 3c)\nabla\lambda = 0$, which together with (5.38) and (5.40) gives

$$\alpha\lambda^3 + 3\lambda^2\alpha^2 + 3c\lambda^2 - 3c\alpha\lambda + 4c\alpha^2 + 18c^2 = 0.$$

Combining this to (5.41), we obtain $\lambda^6 + 12c\lambda^4 + 32c^2\lambda^2 + 48c^3 = 0$. This means that λ is constant and hence $\alpha\lambda = 2c$ because of (5.38), a contradiction. Accordingly $\alpha\lambda = 2c$ is valid on Ω.

In the same way we verify from (5.39) that $\alpha^2\lambda + 2c\lambda + c\alpha = 0$, which together with $\alpha\lambda = 2c$ implies that $2\lambda + 3\alpha = 0$ on Ω. This completes the proof of Lemma 4.

Putting $X = W$ in (5.30) and using (3.10), Lemma 2 and Lemma 4, we get

$$\nabla_W W = 0. \quad (5.42)$$

If we put $X = \mu W$ in (2.11) and make use of (3.5), (3.10), (5.36), (5.37) and Lemma 4, then we obtain $\phi\nabla_W U = 0$, which together with (2.10) and (3.5) yields

$$\nabla_W U = -\mu\lambda\xi. \quad (5.43)$$

Finally we prove

Theorem 2. *Let M be a real hypersurface with $TrR_\xi = \text{const}$. in $M_n(c), c \neq 0$. If it satisfies $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ and at the same time $R_\xi S = SR_\xi$, then M is a Hopf hypersurface, where S denotes the Ricci tensor of M.*

Proof. From Lemma 4, we see that

$$\alpha h = \alpha^2 + 2c, 2h + \alpha = 0. \quad (5.44)$$

Because of Lemma 4, we also verify that α, μ and λ are constant on Ω by virtue of (5.36)-(5.38). Using these and (5.44) we can write (5.13) as

$$\left(\nabla_W A\right)X = u\left(X\right)\left(\frac{c}{2\mu}\xi + \frac{2c}{\alpha}W\right) + \left(\frac{c}{2\mu}\eta\left(X\right) + \frac{2c}{\alpha}w\left(X\right)\right)U, \quad (5.45)$$

which together with the Codazzi Equation (2.4) implies that

$$\left(\nabla_X A\right)W = u\left(X\right)\left(\frac{5c}{2\mu}\xi + \frac{2c}{\alpha}W\right) + \left(\frac{3c}{2\mu}\eta\left(X\right) + \frac{2c}{\alpha}w\left(X\right)\right)U.$$

Differentiating this covariantly and using (2.2), we find

$$\begin{aligned}&\left(\nabla_Y\nabla_X A\right)W + \left(\nabla_X A\right)\left(\nabla_Y W\right)\\ &= \left(\frac{2c}{\alpha}\nabla_Y W + \frac{c}{2\mu}\phi AY\right)u\left(X\right) + \left\{\frac{2c}{\alpha}Y\left(w\left(X\right)\right) + \frac{3c}{2\mu}\left(g\left(\phi AY, X\right) + g\left(\xi, \nabla_X Y\right)\right)\right\}U\\ &\quad + Y\left(u\left(X\right)\right)\left(\frac{2c}{\alpha}W + \frac{5c}{2\mu}\xi\right) + \left(\nabla_Y U\right)\left(\frac{2c}{\alpha}w\left(X\right) + \frac{3c}{2\mu}\eta\left(X\right)\right).\end{aligned}$$

If we take the skew-symmetric part with respect to X and Y, and using the Ricci identity, we obtain

$$\begin{aligned}&R\left(Y,X\right)AW - A\left(R\left(Y,X\right)W\right) + \left(\nabla_X A\right)\left(\nabla_Y W\right) - \left(\nabla_Y A\right)\left(\nabla_X W\right)\\ &= \left(\frac{2c}{\alpha}\nabla_Y W + \frac{c}{2\mu}\phi AY\right)u\left(X\right) - \left(\frac{2c}{\alpha}\nabla_X W + \frac{c}{2\mu}\phi AX\right)u\left(Y\right)\\ &\quad + \frac{2c}{\alpha}dw\left(Y,X\right) + \frac{3c}{2\mu}\left(g\left(\phi AY, X\right) - g\left(\phi AX, Y\right) + g\left(\xi, \nabla_X Y\right) - g\left(\xi, \nabla_Y X\right)\right)U\\ &\quad + \left(\frac{5c}{2\mu}\xi + \frac{2c}{\alpha}W\right)du\left(Y,X\right) + \left(\nabla_Y U\right)\left(\frac{3c}{2\mu}\eta\left(X\right) + \frac{2c}{\alpha}w\left(X\right)\right) - \left(\nabla_X U\right)\left(\frac{2c}{\alpha}w\left(Y\right) + \frac{3c}{2\mu}\eta\left(Y\right)\right).\end{aligned}$$

By putting $Y = W$ and using (2.9), (3.5), (5.42), (5.43) and (5.45), we find

$$\begin{aligned}&\mu R\left(W,X\right)\xi + \lambda R\left(W,X\right)W - A\left(R\left(W,X\right)W\right) = u\left(\nabla_X W\right)\left(\frac{c}{2\mu}\xi + \frac{2c}{\alpha}W\right) + \frac{c}{2\mu}\eta\left(\nabla_X W\right)U\\ &- \frac{4c}{\mu^2}\lambda u\left(X\right)U\,\frac{3c}{2\mu^2}u\left(X\right)AU + \left(\frac{5c}{2\mu}\xi + \frac{2c}{\alpha}W\right)du\left(Y,X\right) - \frac{2c}{\alpha}\nabla_X U - \mu\lambda\left(\frac{2c}{\alpha}w\left(X\right) + \frac{3c}{2\mu}\eta\left(X\right)\right)\xi.\end{aligned} \quad (5.46)$$

On the other hand, using (2.7), (3.5) and Lemma 4, we can write (5.15) as

$$3cg\left(\nabla_X W, U\right) = -\mu\left(c\lambda + \alpha\mu^2\right)\eta\left(X\right).$$

From (2.3), we have

$$\begin{aligned}R\left(W,X\right)\xi &= c\eta\left(X\right)W + \alpha\eta\left(X\right)AW + \mu w\left(X\right)AW - \mu AX,\\ R\left(W,X\right)W &= c\left(w\left(X\right)W - X\right) - \frac{3c}{\mu^2}u\left(X\right)U + g\left(X, AW\right)AW - \lambda AX,\end{aligned}$$

which together with (2.8), (3.6) and (5.44) implies that

$$A\left(R\left(W,X\right)W\right) = c\left(w\left(X\right)AW - AX\right) - \frac{3c}{\mu^2}u\left(X\right)AU + g\left(X, AW\right)A^2W - \lambda A^2X. \quad (5.47)$$

Substituting above four equations into (5.46), we find

$$\begin{aligned}&-\lambda A^2X + \left(\mu^2 + \lambda^2 - c\right)AX + c\lambda X - \frac{2c}{\alpha}\nabla_X U + \frac{15c^2}{\mu^2}u\left(X\right)U\\ &+ c\mu\left(\eta\left(X\right)W - w\left(X\right)\xi\right) + \mu\eta\left(AX\right)AW - \mu w\left(AX\right)A\xi + \frac{\mu}{3c}\left(c\lambda + \alpha\mu^2\right)\eta\left(X\right)\left(\frac{c}{2\mu}\xi + \frac{2c}{\alpha}W\right)\\ &= -\left(\frac{5c}{2\mu}\xi - \frac{2c}{\alpha}W\right)du\left(W,X\right) - \mu\lambda\left(\frac{2c}{\alpha}w\left(X\right) + \frac{3c}{2\mu}\eta\left(X\right)\right)\xi.\end{aligned}$$

Since we have $g(U,U)=\mu^2$ from (2.5), if we take the inner product αU to this, then we obtain

$$-\alpha\lambda AU^2+\alpha\left(\mu^2+\lambda^2-c\right)AU+17c^2U=0,$$

which together with Lemma 2 and Lemma 4 implies that

$$\alpha^2\left(\mu^2+\lambda^2\right)+26c^2=0,$$

which will produce a contradiction. This completes the proof.

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