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S^1-Equivariant CMC Surfaces in the Berger Sphere and the Corresponding Lagrangians

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ABSTRACT

The periodic S^1-equivariant hypersurfaces of constant mean curvature can be obtained by using the Lagrangians with suitable potential functions in the Berger spheres. In the corresponding Hamiltonian system, the conservation law is effectively applied to the construction of periodic S^1-equivariant surfaces of arbitrary positive constant mean curvature.

Keywords: S^1-Equivariant CMC Surfaces; Conservation Laws

1. Introduction

W.-Y. Hsiang [1] investigated the rotation hypersurfaces of constant mean curvature in the hyperbolic or spherical n-space. In [2], Eells and Ratto have constructed the rotation (S^1-equivariant) minimal hypersurfaces in the unit 3-sphere with standard metric by using a certain first integral, which is invariant with respect to the rotation angle of generating curves on the orbit space. In [3], a family of S^1-equivariant periodic CMC surfaces was constructed in the Berger spheres when the constant mean curvature (CMC) is a sufficiently small positive number, and it was cleared that the conserved quantity can be obtained by using the Lagrangian equipped with suitable potential function of the corresponding dynamical system with respect to the Hsiang-Lawson metric [1,4] on the orbit space via the Hamilton equation, where the rotation angle of generating curves can be regarded as "time". We should remark that the corresponding Lagrangian has the vanishing potential when we construct the S^1-equivariant minimal hypersurfaces. However, in case that we construct the S^1-equivariant non-minimal CMC-hypersurface in the Berger sphere, the potential of the Lagrangian is a nonvanishing function. In Theorem 4.3, we determine the potential function of the Lagrangian which corresponds to the S^1-equivariant CMC-surfaces immersed in the Berger sphere. As a result we can obtain a family of periodic S^1-equivariant CMC surfaces in the Berger spheres when the constant mean curvature is an arbitrary positive number (Theorem 5.2).

2. Preliminaries

In [3], a generalized inner product $g_{\alpha,\beta}$ on the unit 3-sphere $S^3 \subset \mathbf{C}\times\mathbf{C}$ was defined by

$$\left(g_{\alpha,\beta}\right)_z (v,w) = \alpha\langle v,w\rangle + \beta\langle v,iz\rangle\langle w,iz\rangle,$$

where $v=(v_1,v_2)$, $w=(w_1,w_2)\in T_zS^3$ and $\langle v,w\rangle = \Re\left(v_1\bar{w}_1 + v_2\bar{w}_2\right)$, α and β are positive and nonnegative parameters, respectively. The Cartan hypersurface $SO(3)/\mathbf{Z}_2\times\mathbf{Z}_2$ in the unit 4-sphere is covered by S^3 (via an 8-fold covering), whose metric is rescaled along the Hopf fibres and its metric on S^3 coincides with $g_{4,12}(\alpha=4,\beta=12)$ [5,6]. The family of metrics $g_{\alpha,\beta}$ defined on S^3 contains this one as a special case. In particular $g_{\alpha,\beta}$ is a left-invariant metric on $S^3 = SU(2)$ and $\left(S^3, g_\beta\right)$ is called the Berger sphere with metric $g_\beta := g_{1,\beta}$ in case that $\beta > -1$. The Berger metrics g_β are obtained from the canonical metric by multiplying the metric along the Hopf fiber by $1+\beta$ [7].

Throughout the paper we consider the Berger spheres $\left(S^3, g_\beta\right)(\beta > -1)$. Here we summarize the notations which are used in the paper.

X denotes the orbit space by g_β-isometric S^1-ction $r_t : S^3 \to S^3$ as follows.

$$r_t(z) = \left(z_1, \mathrm{e}^{it}z_2\right), z = (z_1, z_2).$$

As the parametrization of X we use the following map:

$$(\theta,\phi) \to \left(\mathrm{e}^{i\phi}\cos\theta, \sin\theta\right), 0 \le \phi \le 2\pi, 0 \le \theta \le \frac{\pi}{2}.$$

h_β stands for the orbital metric on $X^* = X \setminus \left(\partial X \cup \{\text{pole}\}\right)$:

$$h_\beta := \mathrm{d}\theta^2 + \frac{(1+\beta)\cos^2\theta}{1+\beta\sin^2\theta}\mathrm{d}\phi^2.$$

$V = 2\pi \sin\theta\sqrt{1+\beta\sin^2\theta}$ is the volume function of orbits and $\hat{h}_\beta = V^2 h_\beta$ is the Hsiang-Lawson metric on X^*:

$$\hat{h}_\beta := \left(\hat{h}_\beta\right)_1 \mathrm{d}\theta^2 + \left(\hat{h}_\beta\right)_2 \mathrm{d}\phi^2,$$

where

$$\left(\hat{h}_\beta\right)_1 = 4\pi^2 \sin^2\theta\left(1+\beta\sin^2\theta\right),$$
$$\left(\hat{h}_\beta\right)_2 = 4\pi^2\left(1+\beta\right)\sin^2\theta\cos^2\theta.$$

$\gamma : J \subset \boldsymbol{R} \to \left(X^*, h_\beta\right)$ denotes a curve parametrized by arclength s. And also $\tau(\gamma) := \nabla_{\dot\gamma}\dot\gamma$ and $\hat\tau(\gamma) := \hat\nabla_{\dot\gamma}\dot\gamma$ stand for the tension fields of γ with respect to the metrics h_β and $\hat{h}_\beta$, respectively. The geodesic curvature $\kappa_\beta(\gamma)$ at $\gamma(s)$ is defined by $\kappa_\beta(\gamma) := h_\beta\left(\tau(\gamma), \eta\right)$ where η denotes the unit normal vector field to γ.

3. S^1-Equivariant CMC-Immersion

For a curve $\gamma : J \to X^*$, we consider an S^1-equivariant map $\mu : M = \gamma^{-1}\left(\left(S^3, g_\beta\right)\right) \to \left(S^3, g_\beta\right)$ such that $\gamma\circ\pi = \sigma\circ\mu$, where $\pi : M \to J$ and $\sigma : \left(S^3, g_\beta\right) \to X^*$ are Riemannian submersions. Throughout the paper, we assume that μ is an S^1-equivariant constant mean curvature H immersion. Then we have

$$\kappa_\beta(\gamma) - \eta(\log V) = 2H, \tag{1}$$

since

$$h_\beta\left(\tau(\gamma), \eta\right) - \eta(\log V) = h_\beta\left(\hat\tau(\gamma), \eta\right).$$

On the orbit space $\left(X^*, h_\beta\right)$, the velocity vector field of a curve $\gamma(s) = \left(\theta(s), \phi(s)\right)$ is given by the following component functions.

$$\theta'(s) = \cos\lambda(s), \phi'(s) = \frac{\sqrt{1+\beta\sin^2\theta(s)}\sin\lambda(s)}{\sqrt{1+\beta}\cos\theta(s)}.$$

Lemma 3.1. The following formulas hold on $\left(X^*, h_\beta\right)$.

$$\eta(s) = -\sin\lambda(s)\frac{\partial}{\partial\theta} + \frac{\sqrt{1+\beta\sin^2\theta(s)}\cos\lambda(s)}{\sqrt{1+\beta}\cos\theta(s)}\frac{\partial}{\partial\phi}. \tag{2}$$

$$\tau(\gamma) = \tau(\gamma)_1\frac{\partial}{\partial\theta} + \tau(\gamma)_2\frac{\partial}{\partial\phi}, \tag{3}$$

where

$$\tau(\gamma)_1 = -\left(\sin\lambda(s)\right)\lambda'(s) + \frac{(1+\beta)\tan\theta(s)\sin^2\lambda(s)}{1+\beta\sin^2\theta(s)},$$

and

$$\tau(\gamma)_2 = -\frac{\cos\lambda(s)}{\sqrt{1+\beta}}\left\{\frac{(1+\beta)\sin\theta(s)\sin\lambda(s)}{\sqrt{1+\beta\sin^2\theta(s)}\cos^2\theta(s)} - \frac{\sqrt{1+\beta\sin^2\theta(s)}\lambda'(s)}{\cos\theta(s)}\right\}.$$

Then using the formula (1) we have the following differential Equation (4) of generating curves which corresponds to the CMC-rotation hypersurfaces immersed in $\left(S^3, g_\beta\right)$, since using Lemma 3.1 the geodesic curvature $\kappa_\beta(\gamma)$ is given by

$$\begin{aligned}&\kappa_\beta(\gamma) = \lambda'(s) - \frac{(1+\beta)\tan\theta(s)\sin\lambda(s)}{1+\beta\sin^2\theta(s)},\\ &\lambda'(s) + \left(\cot\theta(s) - \tan\theta(s)\right)\sin\lambda(s) - 2H = 0.\end{aligned} \tag{4}$$

4. Conservation Laws

We consider a generating curve $\gamma(s) = \left(\theta(s), \phi(s)\right)$ on X^* such that $\theta = \theta(\phi)$ and $\phi'(s) > 0$. Then we can consider the space $\Xi\left(\theta, \theta^\#\right)$ of motion with $\theta^\# = \frac{\mathrm{d}\theta}{\mathrm{d}\phi}$ and time ϕ. Let $\mathcal{L} = \mathcal{L}\left(\theta, \theta^\#\right)$ be a Lagrangian on $\Xi\left(\theta, \theta^\#\right)$. Via the Legendre transformation we have the Hamiltonian $\mathcal{H}$ on the phase space $\Xi^*(\theta, p)$:

$$\mathcal{H} = \theta^\# p - \mathcal{L},\ p = \frac{\partial\mathcal{L}}{\partial\theta^\#}.$$

The conservation laws of our system imply the following

Proposition 4.1. Let the Lagrangian $\mathcal{L}$ on $\Xi\left(\theta, \theta^\#\right)$ be the following form:

$$\mathcal{L} = \sqrt{\left(\hat{h}_\beta\right)_1\left(\theta^\#\right)^2 + \left(\hat{h}_\beta\right)_2} + G(\theta),$$

where $\hat{h}_\beta$ is the Hsiang-Lawson metric on X^* and $G(\theta)$ is a potential function on the configuration space.

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\left\{\frac{\left(\hat{h}_\beta\right)_2}{\sqrt{\left(\hat{h}_\beta\right)_1\left(\theta^\#\right)^2 + \left(\hat{h}_\beta\right)_2}} + G(\theta)\right\} = 0, \tag{5}$$

where the conserved quantity in the formula represents the Hamiltonian of our system.

By means of the Hamilton Equation (5), we shall determine the potential $G(\theta)$ which corresponds to the S^1-equivariant CMC surfaces immersed in $\left(S^3, g_\beta\right)$ via the differential Equation (4) of generating curves on the orbit space X^*.

The direct computation yields the following

Lemma 4.2. Assume that θ and λ are functions of

ϕ and $\frac{\mathrm{d}\lambda}{\mathrm{d}\phi}=\frac{\lambda'(s)}{\phi'(s)}$. Then we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\left(\hat{h}_\beta\right)_2}{\sqrt{\left(\hat{h}_\beta\right)_1\left(\theta^\#\right)^2+\left(\hat{h}_\beta\right)_2}}\\&=\Psi\left(\lambda'(s)+2\cot 2\theta(s)\sin\lambda(s)\right),\end{aligned}\tag{6}$$

where

$$\Psi=\frac{2(1+\beta)\pi\sin\theta(s)\cos^2\theta(s)\cot\lambda(s)}{\sqrt{1+\beta\sin^2\theta(s)}}.$$

As a consequence, we have the following

Theorem 4.3. On our system, the Lagrangian $\mathcal{L}$ and the Hamiltonian $\mathcal{H}$ which correspond to the S^1-equivariant CMC-H hypersurface immersed in $\left(S^3,g_\beta\right)$ can be determined as follows:

$$\mathcal{L}=\sqrt{\left(\hat{h}_\beta\right)_1\left(\theta^\#\right)^2+\left(\hat{h}_\beta\right)_2}+\sqrt{1+\beta}\pi H\cos 2\theta$$

$$\mathcal{H}=-\left\{\frac{\left(\hat{h}_\beta\right)_2}{\sqrt{\left(\hat{h}_\beta\right)_1\left(\theta^\#\right)^2+\left(\hat{h}_\beta\right)_2}}+\sqrt{1+\beta}\pi H\cos 2\theta\right\}.$$

Proof. Using Lemma 4.2 and the differential equation of generating curves (4) we have

$$\frac{\mathrm{d}}{\mathrm{d}\phi}G(\theta)=-2H\Psi,$$

from which we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\theta}G(\theta)\\&=-\frac{4H\pi(1+\beta)\sin\theta(s)\cos^2\theta(s)\cot\lambda(s)\phi'(s)}{\sqrt{1+\beta\sin^2\theta(s)}\theta'(s)}.\end{aligned}$$

Since H is a constant mean curvature and

$$\frac{\phi'(s)}{\theta'(s)}=\frac{\sqrt{1+\beta\sin^2\theta(s)}\tan\lambda(s)}{\sqrt{1+\beta}\cos\theta(s)},$$

we can choose such as $G(\theta)=\sqrt{1+\beta}\pi H\cos 2\theta$. Q.E.D.

5. Generating Curves for S^1-Equivariant CMC Surfaces

Let $\gamma(s)=\left(\theta(s),\phi(s)\right)$ be a generating curve on X^* such that $\theta=\theta(\phi)$ and $\phi'(s)>0$ with the arc length s. Then we set the following initial conditions:

$$\theta_0:=\theta(0),\phi(0)=0,\theta'(0)=0,\lambda(0)=\frac{\pi}{2}.$$

The Hamilton equation $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\phi}=0$ (Theorem 4.3) implies that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\sqrt{1+\beta}\pi\sin 2\theta(s)\sin\lambda(s)+\sqrt{1+\beta}\pi H\cos 2\theta(s)\right)\\&=0,\end{aligned}$$

from which we have

$$\sqrt{1+\beta}\pi\sin 2\theta(s)\sin\lambda(s)+\sqrt{1+\beta}\pi H\cos 2\theta(s)=K,$$

where

$$K=\sqrt{1+\beta}\pi\sin 2\theta_0+\sqrt{1+\beta}\pi H\cos 2\theta_0.$$

On the other hand, using the formulas

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2=\left\{\frac{\left(\hat{h}_\beta\right)_2}{\left(K-G(\theta)\right)^2}-1\right\}\frac{\left(\hat{h}_\beta\right)_2}{\left(\hat{h}_\beta\right)_1}$$

and

$$\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2}\right)_{s=0}=\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)_{s=0}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2,$$

$$\left(K-G(\theta_0)\right)^2=\left(\hat{h}_\beta\right)_2(\theta_0),$$

we have

$$\begin{aligned}\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2}\right)_{s=0}&=\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)_{s=0}\left\{\frac{\left(\hat{h}_\beta\right)_2(\theta)}{\left(K-G(\theta)\right)^2}\right\}\frac{\left(\hat{h}_\beta\right)_2(\theta_0)}{\left(\hat{h}_\beta\right)_1(\theta_0)}\\&=\frac{1}{\left(\hat{h}_\beta\right)_1(\theta_0)}\left\{\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)_{s=0}G\left(\theta(s)\right)\left(K-G(\theta_0)\right)\right.\\&\quad\left.+\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)_{s=0}\left(\hat{h}_\beta\right)_2\left(\theta(s)\right)\right\}.\end{aligned}$$

Consequently we have the following

Lemma 5.1. Under the initial conditions for generating curves which correspond to the CMC-H rotation hypersurfaces, we have

$$\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2}\right)_{s=0}=\frac{(1+\beta)\sin^2 2\theta_0\left(\cot 2\theta_0-H\right)}{2\sin^2\theta_0\left(1+\beta\sin^2\theta_0\right)},$$

and

$\theta_0\ge\theta_H$ (resp., $\le\theta_H$) if and only if,

$$\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2}\right)_{s=0}\le 0\left(\text{resp.},\ge 0\right),$$

where

$$\theta_H:=\arctan\left(-H+\sqrt{H^2+1}\right).$$

Assume that H is an arbitrary positive number. In

Lemma 5.1 we now choose θ_0 such that

$$\theta_H = \operatorname{arccot}\left(H + \sqrt{H^2+1}\right) < \theta_0 < 2\theta_H = \operatorname{arccot} H < \frac{\pi}{2}.$$

From Lemma 5.1, $\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2}\right)_{s=0} < 0$ and there exists the value ϕ_1 of ϕ such that $\theta(\phi) = \theta(\phi(s))$ decreases strictly until $\phi_1 = \phi(s_1)$, where the value of $\frac{\mathrm{d}\theta}{\mathrm{d}\phi}$ equals to zero at $\phi = \phi_1$, and $\theta(\phi) = \theta(\phi(s))$ takes a local minimum at $\phi = \phi_1$. In fact, if $\theta(\phi)$ does not take a local minimum, then we may assume that there exists a such that $0 \le a < \theta_0 < \frac{\pi}{2}$ and

$$\lim_{s\to+\infty} \theta(s) = a, \lim_{s\to+\infty} \theta'(s) = 0, \lim_{s\to+\infty} \lambda(s) = \frac{\pi}{2}.$$

Then from the differential Equation (4) of generating curves it follows that $a = \theta_H$. On the other hand we obtain the following formula:

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2 = \frac{(A(\theta)-1)(1+\beta)\cos^2\theta}{1+\beta\sin^2\theta}, \tag{7}$$

where

$$A(\theta) = \frac{\sin^2 2\theta}{\left\{\sin 2\theta_0 + H\left(\cos 2\theta_0 - \cos 2\theta\right)\right\}^2}.$$

The formula (7) implies that

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2 = \frac{4(1+\beta)\sin(\theta(s)+\theta_0)\sin(\theta(s)-\theta_0)\cos^2\theta(s)\Phi\Omega}{\left(1+\beta\sin^2\theta(s)\right)\left\{\sin 2\theta_0 + H\left(\cos 2\theta_0 - \cos 2\theta(s)\right)\right\}^2}, \tag{8}$$

where

$$\Phi = \cos(\theta(s)-\theta_0) + H\sin(\theta(s)-\theta_0),$$
$$\Omega = \cos(\theta(s)+\theta_0) - H\sin(\theta(s)+\theta_0).$$

The formula $\theta_0 < 2\theta_H = a + \theta_H$ implies that

$$\theta_0 - a < \theta_H = \arctan\left(\frac{1}{H+\sqrt{H^2+1}}\right) < \arctan\left(\frac{1}{H}\right),$$

from which we have

$$\cos(a-\theta_0) + H\sin(a-\theta_0) > 0,$$

since $0 < \theta_0 - a < \frac{\pi}{2}$.

Hence we see that $\lim_{s\to+\infty}\Phi$ is a positive number. Now if $\lim_{s\to+\infty}\Omega = 0$, then $\cos(a+\theta_0) = H\sin(a+\theta_0)$, which implies that $0 < a + \theta_0 < \frac{\pi}{2}$ and $a + \theta_0 = \operatorname{arccot} H = 2\theta_H = 2a$, hence $\theta_0 = a$, which is a contradiction. Therefore, the value $\lim_{s\to+\infty}\Omega$ is not zero. Consequently, since $0 \le a < \theta_0 < \frac{\pi}{2}$, from the formula (8) we see that $\lim_{s\to+\infty}\left(\mathrm{d}\theta/\mathrm{d}\phi\right)^2$ is not zero, which contradicts the assumption $\lim_{s\to+\infty}\theta'(s) = 0$. Hence $\theta(\phi) = \theta(\phi(s))$ takes a local minimum.

Thus we can continue $\theta = \theta(\phi(s))$ as the curve satisfying the differential Equation (4) by the reflection. Let $F_{\beta,H}$ be the right hand side of (7). We can define $\Lambda_{\beta,H}$ by $F_{\beta,H}$ as follows:

$$\Lambda_{\beta,H} = -\int_{\theta_0}^{\theta(s_1)} \frac{1}{\sqrt{F_{\beta,H}}}\mathrm{d}\theta.$$

Consequently we have the following

Theorem 5.2. Let H be an arbitrary positive number and choose θ_0 such that $\theta_H < \theta_0 < 2\theta_H$. If $\pi/\Lambda_{\beta,H}$ is a rational number, then the corresponding S^1-equivariant hypersurface is an immersed CMC-H torus in the Berger sphere (S^3, g_β). In particular, if $\pi/\Lambda_{\beta,H}$ is an integer, then this CMC-H torus is embedded.

Theorem 5.3. In the case $\theta_0 = \theta_H$, Then the corresponding S^1-equivariant CMC-H hypersurface in the Berger sphere (S^3, g_β) is an extended Clifford torus

$$S^1(r(\theta_0)) \times S^1(R(\theta_0)),$$
$$r(\theta_0) = \cos\theta_H\sqrt{1+\beta\cos^2\theta_H},$$
$$R(\theta_0) = \sin\theta_H\sqrt{1+\beta\sin^2\theta_H},$$

where

$$\theta_H = \arctan\left(-H + \sqrt{H^2+1}\right).$$

Corollary 5.4. There exists an embedded minimal torus in the Berger sphere (S^3, g_β)

$$S^1\left(\frac{\sqrt{2+\beta}}{2}\right) \times S^1\left(\frac{\sqrt{2+\beta}}{2}\right).$$

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