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Oscillation Theorems for a Class of Nonlinear Second Order Differential Equations with Damping

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ABSTRACT

The oscillatory behavior of solutions of a class of second order nonlinear differential equations with damping is studied and some new sufficient conditions are obtained by using the refined integral averaging technique. Some well known results in the literature are extended. Moreover, two examples are given to illustrate the theoretical analysis.

Keywords: Nonlinear Differential Equations; Damping Equations; Second Order; Oscillation Solutions

1. Introduction

In this paper, we are concerned with the oscillatory behavior of solutions of the second-order nonlinear differential equations with damping

$$\left(r(t)\Psi(x(t))k(x'(t))\right)' + p(t)k(x'(t)) + q(t)f(x(t))g(x'(t)) = 0, t \ge t_0 \ge 0, \quad (1.1)$$

where $r(t), p(t), q(t) \in C([t_0,\infty), R)$ and $\Psi, k, f, g \in C(R,R)$.

In what follows with respect to Equation (1.1), we shall assume that there are positive constants c, c_1, c_2, γ_1 and γ_2 satisfying

(A1) $r(t) > 0$ and $xf(x) > 0$ for all $x \ne 0$;
(A2) $0 < c \le \Psi(x(t)) \le c_1$ for all x;
(A3) $\gamma_1 > 0$ and $k^2(y) \le \gamma_1 y k(y)$ for all $y \in R$;
(A4) $q(t) \ge 0$ and $0 < c_2 \le g(x'(t))$;
(A5) $\frac{f(x)}{x} \ge \gamma_2 > 0$ for all $x \ne 0$.

We shall consider only nontrivial solutions of Equation (1.1) which are defined for all large *t*. A solution of Equation (1.1) is said to be oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

The oscillation problem for various particular cases of Equation (1.1) such as the nonlinear differential equation

$$\left[r(t)x'(t)\right]' + q(t)f(x(t)) = 0, \quad (1.2)$$

the nonlinear damped differential equation

$$\left[r(t)(x'(t)\right]' + p(t)x'(t) + q(t)f(x(t)) = 0 \quad (1.3)$$

and

$$\left(r(t)\Psi(x(t))k(x'(t))\right)' + p(t)k(x'(t)) + q(t)f(x(t)) = 0, \quad (1.4)$$

have been studied extensively in recent years, see e.g. [1-21] and the references quoted therein. Moreover, in 2011, Wang [22] established some oscillation criteria for Equation (1.1) firstly, some new sharper results are obtained in the present paper.

An important method in the study of oscillatory behaviour for Equations (1.1)-(1.4) is the averaging technique which comes from the classical results of Wintner [19] and Hartman [18]. Using the generalized Riccati technique and the refined integral averaging technique introduced by Rogovchenko and Tuncay [20,21], several new oscillation criteria for Equation (1.1) are established in Section 2, we also show some examples to explain the application of our oscillation theorems in Section 2. Our results strengthen and improve the recent results of [1] and [21,22].

2. The Main Results

Following Philos [10], let us introduce now the class of functions Θ which will be extensively used in the sequel. Let

$$D_0 = \{(t,s): t > s \ge t_0\} \text{ and } D = \{(t,s): t \ge s \ge t_0\}.$$

The function $H \in C(D;R)$ is said to belong to the class Θ if

1) $H(t,t) = 0$ for $t \ge t_0$; $H(t,s) > 0$ on D_0;
2) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable;

3) There exists a function $h(t,s)\in C(D,R)$ such that

$$-\frac{\partial H(t,s)}{\partial s}=h(t,s)\sqrt{H(t,s)}.$$

In this section, several oscillation criteria for Equation (1.1) are established under the assumptions (A1)-(A5). The first result is the following theorem.

Theorem 2.1. Let assumption (A1)-(A5) be fulfilled and $H\in\Theta$. If there exists functions $R,\phi\in C([t_0,\infty),R)$ such that $(rR)\in C^1(t_0,\infty,R)$ and

$$0<\inf_{s\ge t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\}\le\infty, \tag{2.1}$$

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\phi_+^2(s)}{\rho(s)r(s)}\mathrm{d}s=\infty, \tag{2.2}$$

and for any $T\ge t_0,\beta>1$,

$$\phi(T)\le\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)Q(s)-\frac{\beta c_1\gamma_1}{4}\rho(s)r(s)h^2(t,s)\right]\mathrm{d}s. \tag{2.3}$$

where

$$Q(t)=\rho(t)\cdot \tag{2.4}$$

$$\left\{c_2\gamma_2q(t)-\left[R(t)r(t)\right]'-\frac{1}{c_1}p(t)R(t)+\frac{1}{c_1\gamma_1}r(t)R^2(t)-\frac{\gamma_1}{4}\left(\frac{1}{c}-\frac{1}{c_1}\right)\frac{p^2(t)}{r(t)}\right\}$$

$$\rho(t)=\exp\left(-\frac{2}{c_1}\int^t\left(\frac{R(s)}{\gamma_1}-\frac{p(s)}{2r(s)}\right)\mathrm{d}s\right), \tag{2.5}$$

and $\phi_+(s)=\max\{\phi(s),0\}$, then Equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (1.1). Then there exists a $T_0\ge t_0$ such that $x(t)\ne 0$ for all $t\ge T_0$. Without loss of generality, we may assume that $x(t)>0$ on interval $[T_0,\infty)$. A similar argument holds also for the case when $x(t)$ is eventually negative. As in [1], define a generalized Riccati transformation by

$$v(t)=\rho(t)\left[\frac{r(t)\Psi(x(t))k(x'(t))}{x(t)}+r(t)R(t)\right] \tag{2.6}$$

for all $t\ge T_0$, then differentiating Equation (2.6) and using Equation (1.1), we obtain

$$\begin{aligned}v'(t)=&\frac{\rho'(t)}{\rho(t)}v(t)-\frac{\rho(t)p(t)k(x'(t))}{x(t)}\\&-\frac{\rho(t)q(t)g(x'(t))f(x(t))}{x(t)}\\&-\frac{\rho(t)r(t)\Psi(x(t))k(x'(t))x'(t)}{x^2(t)}\\&+\rho(t)\left[R(t)r(t)\right]'\end{aligned} \tag{2.7}$$

In view of (A1)-(A5), we get

$$\begin{aligned}v'(t)&\le\frac{\rho'(t)}{\rho(t)}v(t)-\frac{\rho(t)p(t)k(x'(t))}{x(t)}-\gamma_2c_2\rho(t)q(t)-\frac{\rho(t)r(t)\Psi(x(t))k^2(x'(t))}{\gamma_1x^2(t)}+\rho(t)\left[R(t)r(t)\right]'\\
&=\rho(t)\left[R(t)r(t)\right]'-\gamma_2c_2\rho(t)q(t)+\frac{\rho'(t)}{\rho(t)}v(t)+\frac{\gamma_1\rho(t)p^2(t)}{4r(t)\Psi(x(t))}-\frac{\rho(t)}{\Psi(x(t))}\left[\sqrt{\frac{r(t)}{\gamma_1}}\frac{\Psi(x(t))k(x'(t))}{x(t)}+\frac{p(t)}{2}\sqrt{\frac{\gamma_1}{r(t)}}\right]^2\\
&\le\rho(t)\left[R(t)r(t)\right]'-\gamma_2c_2\rho(t)q(t)+\frac{\rho'(t)}{\rho(t)}v(t)+\frac{\gamma_1\rho(t)p^2(t)}{4cr(t)}-\frac{\rho(t)}{c_1}\left[\sqrt{\frac{r(t)}{\gamma_1}}\frac{\Psi(x(t))k(x'(t))}{x(t)}+\frac{p(t)}{2}\sqrt{\frac{\gamma_1}{r(t)}}\right]^2\\
&\le\rho(t)\left[R(t)r(t)\right]'-\gamma_2c_2\rho(t)q(t)+\frac{\rho'(t)}{\rho(t)}v(t)+\frac{\gamma_1\rho(t)p^2(t)}{4cr(t)}-\frac{\rho(t)}{c_1}\left[\sqrt{\frac{r(t)}{\gamma_1}}\frac{\Psi(x(t))k(x'(t))}{x(t)}+\frac{p(t)}{2}\sqrt{\frac{\gamma_1}{r(t)}}\right]^2\\
&=\rho(t)\left[R(t)r(t)\right]'-\gamma_2c_2\rho(t)q(t)+\frac{\rho'(t)}{\rho(t)}v(t)+\frac{\gamma_1\rho(t)p^2(t)}{4cr(t)}-\frac{\rho(t)}{c_1}\left[\sqrt{\frac{r(t)}{\gamma_1}}\left(\frac{v(t)}{\rho(t)r(t)}-R(t)\right)+\frac{p(t)}{2}\sqrt{\frac{\gamma_1}{r(t)}}\right]^2\\
&=-Q(t)+\left(\frac{\rho'(t)}{\rho(t)}+\frac{2R(t)}{c_1\gamma_1}-\frac{p(t)}{c_1r(t)}\right)v(t)-\frac{1}{c_1\gamma_1\rho(t)r(t)}v^2(t)=-Q(t)-\frac{1}{c_1\gamma_1\rho(t)r(t)}v^2(t)\end{aligned}$$

for all $t \geq T_0$ with $Q(t)$ defined as above. Then we obtain

$$Q(t) \leq -v'(t) - \frac{1}{c_1\gamma_1\rho(t)r(t)}v^2(t). \qquad (2.8)$$

On multiplying Equation (2.8) (with t replaced by s) by $H(t,s)$, integrating with respect to s from T to t for $t \geq T \geq T_0$, using integration by parts and property 3), we get

$$\begin{aligned}\int_T^t H(t,s)Q(s)\mathrm{d}s &\leq -\int_T^t H(t,s)v'(s)\mathrm{d}s - \int_T^t H(t,s)\frac{1}{c_1\gamma_1\rho(s)r(s)}v^2(s)\mathrm{d}s \\ &= -H(t,s)v(s)\Big|_T^t + \int_T^t v(s)\mathrm{d}H(t,s) - \int_T^t H(t,s)\frac{1}{c_1\gamma_1\rho(s)r(s)}v^2(s)\mathrm{d}s \\ &= H(t,T)v(T) - \int_T^t\left[v(s)h(t,s)\sqrt{H(t,s)} + H(t,s)\frac{v^2(s)}{\gamma_1 c_1\rho(s)r(s)}\right]\mathrm{d}s.\end{aligned}$$

Then, for any $\beta > 1$

$$\begin{aligned}\int_T^t H(t,s)Q(s)\mathrm{d}s \leq H(t,T)v(T) &- \int_T^t\left[\sqrt{\frac{H(t,s)}{\beta\gamma_1 c_1\rho(s)r(s)}}v(s) + \frac{1}{2}\sqrt{\beta\gamma_1 c_1\rho(s)r(s)}h(t,s)\right]^2\mathrm{d}s \\ &+ \frac{\beta\gamma_1 c_1}{4}\int_T^t \rho(s)r(s)h^2(t,s)\mathrm{d}s - \int_T^t\frac{(\beta-1)H(t,s)}{\beta\gamma_1 c_1\rho(s)r(s)}v^2(s)\mathrm{d}s,\end{aligned}$$

and, for all $t \geq T \geq T_0$,

$$\begin{aligned}&\int_T^t\left[H(t,s)Q(s) - \frac{\beta\gamma_1 c_1}{4}\rho(s)r(s)h^2(t,s)\right]\mathrm{d}s \\ &\leq H(t,T)v(T) - \int_T^t\left[\sqrt{\frac{H(t,s)}{\beta\gamma_1 c_1\rho(s)r(s)}}v(s) + \frac{1}{2}\sqrt{\beta\gamma_1 c_1\rho(s)r(s)}h(t,s)\right]^2\mathrm{d}s - \int_T^t\frac{(\beta-1)H(t,s)}{\beta\gamma_1 c_1\rho(s)r(s)}v^2(s)\mathrm{d}s\end{aligned} \qquad (2.9)$$

Furthermore,

$$\frac{1}{H(t,T)}\int_T^t\left[H(t,s)Q(s) - \frac{\beta\gamma_1 c_1}{4}\rho(s)r(s)h^2(t,s)\right]\mathrm{d}s \leq v(T) - \frac{1}{H(t,T)}\int_T^t\frac{(\beta-1)H(t,s)}{\beta\gamma_1 c_1\rho(s)r(s)}v^2(s)\mathrm{d}s.$$

Now, it follows that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)Q(s) - \frac{\beta c_1\gamma_1}{4}\rho(s)r(s)h_1^2(t,s)\right]\mathrm{d}s \\ &\leq v(T) - \liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\frac{(\beta-1)H(t,s)}{\beta\gamma_1 c_1\rho(s)r(s)}v^2(s)\mathrm{d}s.\end{aligned} \qquad (2.10)$$

From (2.3) and (2.10), we have

$$v(T) \geq \phi(T) + \liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\frac{(\beta-1)H(t,s)}{\beta\gamma_1 c_1\rho(s)r(s)}v^2(s)\mathrm{d}s$$

for all $T \geq T_0$ and $\beta > 1$. Obviously,

$$v(T) \geq \phi(T) \quad \text{for all } T \geq T_0 \qquad (2.11)$$

and

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^t\frac{H(t,s)}{\rho(s)r(s)}v^2(s)\mathrm{d}s \\ &\leq \frac{\beta\gamma_1 c_1}{(\beta-1)}\big(v(T_0) - \phi(T_0)\big) < \infty\end{aligned} \qquad (2.12)$$

Now, we can claim that

$$\int_{T_0}^{\infty}\frac{v^2(s)}{\rho(s)r(s)}\mathrm{d}s < \infty, \qquad (2.13)$$

Otherwise,

$$\int_{T_0}^{\infty}\frac{v^2(s)}{\rho(s)r(s)}\mathrm{d}s = \infty. \qquad (2.14)$$

By (2.1), there exists a positive constant η such that

$$\inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\} > \eta > 0,$$

and there exists a $T_2 \geq T_1$ satisfying

$$H(t,T_1)/H(t,t_0)\ge\eta \quad \text{for all} \quad t\ge T_2 .$$

On the other hand, by (2.14) for any $\xi>0$, there exists a $T_1>T_0$ such that

$$\int_{T_0}^{t}\frac{v^2(s)}{\rho(s)r(s)}\mathrm{d}s\ge\frac{\xi}{\eta} \quad \text{for all} \quad t\ge T_1 .$$

Using integration by parts, we obtain

$$\begin{aligned}&\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{H(t,s)}{\rho(s)r(s)}v^2(s)\mathrm{d}s\\&=\frac{1}{H(t,T_0)}\int_{T_0}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]\left[\int_{T_0}^{s}\frac{v^2(\tau)}{\rho(\tau)r(\tau)}\mathrm{d}\tau\right]\mathrm{d}s\\&\ge\frac{\xi}{\eta}\frac{1}{H(t,T_0)}\int_{T_1}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]\mathrm{d}s\\&=\frac{\xi}{\eta}\frac{H(t,T_1)}{H(t,T_0)}.\end{aligned}$$

This implies that

$$\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{H(t,s)}{\rho(s)r(s)}v^2(s)\mathrm{d}s\ge\xi \quad \text{for all} \quad t\ge T_2 .$$

Since ξ is an arbitrary positive constant, we get

$$\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{H(t,s)}{\rho(s)r(s)}v^2(s)\mathrm{d}s=+\infty ,$$

which contradicts (2.12), so (2.13) holds, and from (2.11)

$$\int_{T_0}^{\infty}\frac{\phi_+^2(s)}{\rho(s)r(s)}\mathrm{d}s\le\int_{T_0}^{\infty}\frac{v^2(s)}{\rho(s)r(s)}\mathrm{d}s<+\infty ,$$

which contradicts (2.2), then Equation (1.1) is oscillatory.

Now, we define $H(t,s)=(t-s)^{n-1}$, $(t,s\in D)$, here $n>2$. Evidently, $H\in\Theta$ and

$$h(t,s)=(n-1)(t-s)^{n-3/2},\ (t,s\in D).$$

Thus, by Theorem 2.1, we obtain the following result.

Corollary 2.1. Let assumption (A1)-(A5) be fulfilled. Suppose that (2.2) holds. If there exist functions $R,\phi\in C([t_0,\infty),R)$ such that $(rR)\in C^1([t_0,\infty),R)$,

$$\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{T}^{t}\left[(t-s)^{n-1}Q(s)-\frac{\beta c_1\gamma_1(n-1)^2}{4}\rho(s)r(s)(t-s)^{n-3}\right]\mathrm{d}s\ge\phi(T),$$

where $Q(t)$ and $\rho(t)$ are defined as in Theorem 2.1, then Equation (1.1) is oscillatory.

Example 2.1. Consider the nonlinear damped differential equation

$$\left(t^2\left(\frac{1}{2}+\frac{\mathrm{e}^{-|x|}}{2}\right)\frac{x'(t)}{1+x'^2(t)}\right)'+2t^3\frac{x'(t)}{1+x'^2(t)}+\left(2+2t^4+6t^2-6t^2\sin^2 t\right)x(t)\left(1+x^2(t)\right)\left(1+x'^2(t)\right)=0 .$$

where $x\in(-\infty,+\infty)$ and $t\ge1$, $c=\frac{1}{2}$, $c_1=c_2=1$,

$$\frac{f(x)}{x}=1+x^2(t)\ge1=\gamma_2=\gamma_1 .$$

The assumptions (A1)-(A5) hold. If we take $\beta=2$, $n=3$ and $R(t)=t$, then $\rho(t)=1$, and

$$Q(t)=2+3t^2-6t^2\sin^2 t .$$

A direct computation yields that the conditions of Corallary 2.1 are satisfied, Equation (1.1) is oscillatory.

As a direct consequence of Theorem 2.1, we get the following result.

Corollary 2.2. In Theorem 2.1, if condition (2.3) is replaced by

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)Q(s)-\frac{\beta c_1\gamma_1}{4}\rho(s)r(s)h_1^2(t,s)\right]\mathrm{d}s\ge\phi(T),$$

where $Q(t),\rho(t)$ and $\phi_+(t)$ are the same as in Theorem 2.1, then Equation (1.1) is oscillatory.

Theorem 2.2. Let assumption (A1)-(A5) be fulfilled. For some $\beta\ge1$, if there exist functions $R,\phi\in C([t_0,\infty),R)$ such that

$$(rR)\in C^1([t_0,\infty),R)$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)Q(s)-H(t,s)\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)}-\frac{\beta c_1\gamma_1}{2}\rho(s)r(s)h^2(t,s)\right]\mathrm{d}s=\infty,$$

where $Q(t)$ is the same as in Theorem 2.1, $H \in \Theta$ and

$$\rho(t) = \exp\left(-\frac{2}{c_1\gamma_1}\int^t R(s)\,\mathrm{d}s\right) \tag{2.16}$$

Then Equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (1.1). Then there exists a $T_0 \ge t_0$ such that $x(t) \ne 0$ for all $t \ge T_0$. Without loss of generality, we may assume that $x(t) > 0$ on interval $[T_0, \infty)$. A similar argument holds also for the case when $x(t)$ is eventually negative.

Define the function $v(t)$ as in (2.6). Using (A1)-(A5) and (2.7), we have

$$\begin{aligned} v'(t) &\le \frac{\rho'(t)}{\rho(t)}v(t) - \frac{\rho(t)p(t)k(x'(t))}{x(t)} - \gamma_2 c_2\rho(t)q(t) - \frac{\rho(t)r(t)\Psi(x(t))k^2(x'(t))}{\gamma_1 x^2(t)} + \rho(t)\left[R(t)r(t)\right]' \\ &\le -Q(t) + \left(\frac{\rho'(t)}{\rho(t)} + \frac{2R(t)}{c_1\gamma_1} - \frac{p(t)}{c_1 r(t)}\right)v(t) - \frac{1}{c_1\gamma_1\rho(t)r(t)}v^2(t) \\ &= -Q(t) - \frac{p(t)}{c_1 r(t)}v(t) - \frac{1}{c_1\gamma_1\rho(t)r(t)}v^2(t) \end{aligned} \tag{2.17}$$

where $Q(t)$ is the same as in Theorem 2.1. On the other hand, since the inequality

$$ml - nl^2 \le \frac{m^2}{2n} - \frac{n}{2}l^2$$

holds for all $n > 0$ and $m, l \in R$. Let

$$m = -\frac{\rho(t)}{c_1 r(t)},\ n = \frac{1}{\gamma_1 c_1 r(t)\rho(t)},\ l = v(t),$$

we get from (2.17) that

$$Q(t) - \frac{\gamma_1 p^2(t)\rho(t)}{2c_1 r(t)} \le -v'(t) - \frac{v^2(t)}{2c_1\gamma_1\rho(t)r(t)}, \quad t > T_0. \tag{2.18}$$

On multiplying (2.18) (with t replaced by s) by $H(t,s)$, integrating with respect to s from T to t for $t \ge T \ge T_0$ and $\beta \ge 1$, using integration by parts and property 3), we get

$$\begin{aligned} &\int_T^t H(t,s)\left(Q(s) - \frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)}\right)\mathrm{d}s \\ &\le H(t,T)v(T) - \int_T^t\left(\sqrt{\frac{H(t,s)}{2\beta\gamma_1 c_1 r(s)\rho(s)}}v(s) + \frac{1}{2}\sqrt{2\beta\gamma_1 c_1 r(s)\rho(s)}h(t,s)\right)^2\mathrm{d}s \\ &\quad + \frac{\beta\gamma_1 c_1}{2}\int_T^t r(s)\rho(s)h^2(t,s)\,\mathrm{d}s - \int_T^t \frac{(\beta-1)H(t,s)}{2\beta\gamma_1 c_1 r(s)\rho(s)}v^2(s)\,\mathrm{d}s, \end{aligned}$$

This implies that

$$\begin{aligned} &\int_T^t\left(H(t,s)Q(s) - H(t,s)\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)} - \frac{\beta\gamma_1 c_1}{2}r(s)\rho(s)h^2(t,s)\right)\mathrm{d}s \\ &\le H(t,T)v(T) - \int_T^t \frac{(\beta-1)H(t,s)}{2\beta\gamma_1 c_1 r(s)\rho(s)}v^2(s)\,\mathrm{d}s \\ &\quad - \int_T^t\left(\sqrt{\frac{H(t,s)}{2\beta\gamma_1 c_1 r(s)\rho(s)}}v(s) + \frac{1}{2}\sqrt{2\beta\gamma_1 c_1 r(s)\rho(s)}h(t,s)\right)^2\mathrm{d}s \end{aligned}$$

Using the properties of $H(t,s)$, we have

$$\begin{aligned} &\int_{T_0}^t\left(H(t,s)Q(s) - H(t,s)\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)} - \frac{\beta\gamma_1 c_1}{2}r(s)\rho(s)h^2(t,s)\right)\mathrm{d}s \\ &\le H(t,T_0)v(T_0) \le H(t,T_0)\left|v(T_0)\right| \le H(t,t_0)\left|v(T_0)\right|. \end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{t_0}^{t}\left(H(t,s)Q(s)-H(t,s)\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)}-\frac{\beta\gamma_1 c_1}{2}r(s)\rho(s)h^2(t,s)\right)\mathrm{d}s\\&=\int_{t_0}^{T_0}\left(H(t,s)Q(s)-H(t,s)\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)}-\frac{\beta\gamma_1 c_1}{2}r(s)\rho(s)h^2(t,s)\right)\mathrm{d}s\\&\quad+\int_{T_0}^{t}\left(H(t,s)Q(s)-H(t,s)\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)}-\frac{\beta\gamma_1 c_1}{2}r(s)\rho(s)h^2(t,s)\right)\mathrm{d}s\\&\le H(t,t_0)\left[\int_{t_0}^{T_0}\left|Q(s)\right|\mathrm{d}s+\left|v(T_0)\right|\right]\end{aligned}$$

for all $t \ge T_0$, and so

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)Q(s)-H(t,s)\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)}-\frac{\beta\gamma_1 c_1}{2}r(s)\rho(s)h^2(t,s)\right)\mathrm{d}s\\&\le\int_{t_0}^{T_0}\left|Q(s)\right|\mathrm{d}s+\left|v(T_0)\right|<+\infty,\end{aligned}$$

which contradicts with the assumption (2.15). This completes the proof of Theorem 2.2.

Let $H(t,s)=(t-s)^{n-1},(t,s\in D)$, from Theorem 2.2, we obtain the next result.

Corollary 2.3. Let assumption (A1)-(A5) be fulfilled. If there exist functions $R,\phi\in C([t_0,\infty),R)$ such that

$$(rR)\in C^1([t_0,\infty),R),$$

and

$$\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{T}^{t}\left[(t-s)^{n-1}\left(Q(s)-\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)}\right)-\frac{\beta c_1\gamma_1(n-1)^2}{2}\rho(s)r(s)(t-s)^{n-3}\right]\mathrm{d}s=\infty$$

holds for some integer $n>2$ and $\beta\ge1$, where $Q(t)$ and $\rho(t)$ are defined as in Theorem 2.2, then Equation (1.1) is oscillatory.

Example 2.2. Consider the nonlinear damped differential equation

$$\begin{aligned}&\left(\left(1+t^2\right)\frac{2+x^2(t)}{1+x^2(t)}\frac{x'(t)}{1+x'^2(t)}\right)'+t\sqrt{1+t^2}\,\frac{x'(t)}{1+x'^2(t)}\\&+\left(2+\frac{3}{8}t^2\right)x(t)\left(1+\frac{1}{2+x^2(t)}\right)\left(1+x'^2(t)\right)=0.\end{aligned}\tag{2.19}$$

Evidently, for all $x\in(-\infty,+\infty)$, $\beta\ge1$ and $t\ge1$, we have

$$c=c_2=1\le\psi(x)t\le2=c_1,$$

and

$$\frac{f(x)}{x}=1+\frac{1}{2+x^2(t)}\ge1=\gamma_2=\gamma_1.$$

Let $R(t)=0,n=3$, then

$$\rho(t)=1\ \text{ and }\ Q(t)=2+\frac{1}{4}t^2.$$

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{T}^{t}\left[(t-s)^{n-1}\left(Q(s)-\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)}\right)-\frac{\beta c_1\gamma_1(n-1)^2}{2}\rho(s)r(s)(t-s)^{n-3}\right]\mathrm{d}s\\&=\mathrm{im}\sup_{t\to\infty}\frac{1}{t^2}\int_{1}^{t}\left[2(t-s)^2-4\beta\left(1+s^2\right)\right]\mathrm{d}s=\infty,\end{aligned}$$

Therefore, Equation (2.19) is oscillatory by Corallary 2.3.

Theorem 2.3. Let assumption (A1)-(A5) be fulfilled and $H\in\Theta$. If there exist functions $R,\phi\in C([t_0,\infty),R)$ such that (2.1) holds and $(rR)\in C^1([t_0,\infty),R)$, and for all $t\ge t_0$, any $T\ge t_0$, and for some $\beta>1$,

$$\phi(T)\le\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)Q(s)-H(t,s)\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)}-\frac{\beta c_1\gamma_1}{2}\rho(s)r(s)h^2(t,s)\right]\mathrm{d}s,\tag{2.20}$$

where $Q(t)$ and $\rho(t)$ are the same as in Theorem 2.2 and $\phi_+(s)=\max\{\phi(s),0\}$. If (2.2) is satisfied, then Equation (1.1) is oscillatory.

Proof. The proof of this theorem is similar to that of Theorem 2.1 and hence is omitted.

Theorem 2.4. Let all assumptions of Theorem 2.3 be fulfilled except the condition (2.20) be replaced by

$$\phi(T) \le \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)Q(s) - H(t,s)\frac{\gamma_1 p^2(s)\rho(s)}{2c_1 r(s)} - \frac{\beta c_1 \gamma_1}{2}\rho(s)r(s)h^2(t,s)\right] \mathrm{d}s,$$

then Equation (1.1) is oscillatory.

Remark 2.1. If we take $f(x) = x$, then the condition $q(t) \ge 0$ is not necessary.

Remark 2.2. If we take $g(x'(t)) \equiv 1, k(x') = x'$, then Theorem 2.3 and 2.4 reduce to Theorem 9 and 10 of [21] with $\gamma_1 = 1$, respectively.

Remark 2.3. If replace (A5) and (2.6) by $f'(x)$ exists, $f'(x) \ge \gamma_2 > 0$ for $x \ne 0$ and define

$$v(t) = \rho(t)r(t)\left[\frac{\Psi(x(t))k(x'(t))}{f(x(t))} + R(t)\right]$$

respectively, we can obtain similar oscillation results that are derived in the present paper.

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