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Scattering of the Radial Focusing Mass-Supercritical and Energy-Subcritical Nonlinear Schrödinger Equation in 3D

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ABSTRACT

This paper studies the global behavior to 3D focusing nonlinear Schrödinger equation (NLS), the scaling index here is $(0 < s_c < 1)$, which is the mass-supercritical and energy-subcritical, and we prove under some condition the solution $u(t)$ is globally well-posed and scattered. We also show that the solution "blows-up in finite time" if the solution is not globally defined, as $t \to T$ we can provide a depiction of the behavior of the solution, where T is the "blow-up time".

Keywords: NLS; Blows-Up in Finite Time; Supremum; Precompactness

1. Introduction

Consider the Cauchy problem for the nonlinear Schrödinger equation (NLS) in dimensions d = 3:

$$\begin{cases} iu_t + \Delta u + |u|^2 u = 0 \\ u(x,0) = u_0(x) \in H^1(R^3) \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function in $\mathbb{R}^3 \times \mathbb{R}$. The initial-value problem $u_0 = u(x,0)$ is locally well-posed in H^1.

In this paper we will study the focusing (NLS) problem, which is the mass-supercritical and energy-subcritical, where $(0 < s_c < 1)$.

The Equation (1.1) has mass $M[u](t) = M[u_0]$ where

$$M[u](t) = \int |u(x,t)|^2 \,\mathrm{d}x,$$

Energy $E[u](t) = E[u_0]$ where

$$E[u](t) = \frac{1}{2}\int |\nabla u(x,t)|^2 - \frac{1}{4}\int |u(x,t)|^4 \,\mathrm{d}x,$$

and Momentum $P[u](t) = P[u_0]$ where

$$P[u](t) = \mathrm{Im}\int \bar{u}(x,t) \nabla u(x,t) \,\mathrm{d}x .$$

If $\|xu_0\|_{L^2} < \infty$, then u satisfies

$$\partial_t^2 \int |x|^2 |u(x,t)|^2 \,\mathrm{d}x = 24E[u] - 4\|\nabla u(t)\|_{L_x^2}^2 \tag{1.2}$$

Equation (1.2) is said to be the Virial identity.

The Equation (1.1) has the scaling:

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

and also this scaling is a solution if $u(x,t)$ is a solution.

Moreover, u_0 is a solution that is globally defined by u, if it is globally defined $(T = +\infty)$, and it does scatter (See [1,2]). We say the solution "blows-up in finite time". If the solution is not globally defined, as $t \to T$, we can provide a depiction of the behavior of the solution, where T is the "blow-up time". It follows from the H^1 local theory optimized by scaling, that if blow-up in finite-time $T > 0$ happens, (see [3] or [4]), then there is a lower-bound on the "blow-up rate":

$$\|\nabla u(t)\|_{L_x^2} \ge \frac{c}{(T-t)^{\frac{1}{4}}} \tag{1.3}$$

for some constant c. Thus, to prove global presence, it suffices to prove a global axiomatic bound on $\|\nabla u(t)\|_{L^2}$. From the Strichartz estimates, there is a constant $c_{ST} > 0$ such that if $\|u_0\|_{\dot{H}^{\frac{1}{2}}} < c_{ST}$, then the solution u is globally defined and scattered.

Note that the quantities $\|u_0\|_{L^2} \nabla u_{0_{L^2}}$ and $M[u_0]E[u_0]$ are also scale-invariant (See also [5]).

Let $u(x,t) = \mathrm{e}^{it}\psi(x)$ then u solves (1.1) as long as ψ solves the nonlinear elliptic equation

$$-\psi + \Delta\psi + |\psi|^2 \psi = 0 \tag{1.4}$$

Equation (1.4) has an infinite number of solutions in $H^1\left(\mathbb{R}^3\right)$. The solution of minimal mass is denoted by $\psi(x)$ and for the properties of ψ see [3,5,6].

Under the condition $M[u]E[u] < M[\psi]E[\psi]$, solutions to (1.1) globally exist if u_0 satisfies;

$$\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|\psi\|_{L^2}\|\nabla\psi\|_{L^2}, \tag{1.5}$$

and there exist $\phi_\pm \in H^1$ such that

$$\lim_{t\to\pm\infty}\left\|u(t) - \mathrm{e}^{it\nabla}\phi_\pm\right\|_{H^1} = 0.$$

Theorem 1.1. Let $u_0 \in H^1$, and let u be the corresponding solution to (1.1) in H^1. Suppose

$$M[u]E[u] < M[\psi]E[\psi] \tag{1.6}$$

If $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|\psi\|_{L^2}\|\nabla\psi\|_{L^2}$ then u scatters in H^1.

The argument of [6] in the radial case followed a strategy introduced by [7] for proving global well-posedness and scattering for the focusing energy-critical NLS. The beginning used a contradiction to the argument: suppose the sill for scattering is strictly below that claimed. This uniform localization enabled the use of a local Virial identity to be established, with the support of the sharp Gagliardo-Nirenberg inequality, an accurately positive lower bound on the convexity (in time) of the local mass of u_c Mass conservation is then violated at enough large time.

We show in this paper, that the above program carries over to the non-radial setting with the extension of two key components.

Theorem 1.2. Suppose the radial H^1 solution u to (1.1) blows-up at time $T < \infty$. Then either there is a non-absolute $c_1 \gg 1$ constant such that, as $t \to T$

$$\int_{|x|\le c_1^2\|\nabla u(t)\|_{L^2}^{-2}} |u(x,t)|^3\,\mathrm{d}x \ge c_1^{-1}, \tag{1.7}$$

or there exists a sequence of times $t_n \to T$ such that for an absolute constant c_2

$$\int_{|x|\le c_2\|u_0\|_{L^2}^{\frac{3}{2}}\|\nabla u(t)\|_{L^2}^{\frac{-1}{2}}} |u(x,t_n)|^3\,\mathrm{d}x \to \infty \tag{1.8}$$

From (1.3), we have that the concentration in (1.7) satisfies $\|\nabla u(t)\|_{L^2}^{-2} \le c(T-t)^{\frac{1}{2}}$, and the concentration in (1.8) satisfies $\|\nabla u(t)\|_{L^2}^{\frac{-1}{2}} \le c(T-t)^{\frac{1}{8}}$ (For more additional information see [8-10]).

Notation

Let $\mathrm{e}^{it\Delta}f$ be the free Schrödinger propagator, and let $u_t + \nabla u = 0$, with $u(0,x) = f(x)$ be linear equation, a solution in physical space, is given by:

$$\mathrm{e}^{it\Delta}f(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}}\int_{R^d} \mathrm{e}^{\frac{i|x-y|^2}{4t}} f(y)\,\mathrm{d}y,$$

and in frequency space

$$\widehat{\mathrm{e}^{it\Delta}f}(\xi) = \mathrm{e}^{-4\pi^2 it|\xi|^2}\hat{f}(\xi)$$

In particular, they save the Farewell homogeneous Sobolev norms and obey the dispersive inequality

$$\left\|\mathrm{e}^{it\Delta}f\right\|_{L_x^\infty(R^d)} \lesssim |t|^{-\frac{d}{2}} f_{L_x^1} \tag{1.9}$$

For all times $t \ne 0$.

Let $\phi(x) \in C_c^\infty\left(\mathbb{R}^3\right)$ be a radial function, so that, $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| \ge 2$, Define the inner and outer spatial localizations of $u(x,t)$ at radius $R(t) > 0$ as

$$u_1(x,t) = \phi(x/R(t))u(x,t),$$

$$u_2(x,t) = \left(1 - \phi(x/R(t))\right)u(x,t)$$

Let $\chi(x) \in C_c^\infty\left(\mathbb{R}^3\right)$ be a radial function so that, $\chi(x) = 1$ for $|x| \le \frac{1}{8\pi}$ and $\chi(x) = 0$ for $|x| \ge \frac{1}{2\pi}$ then $\hat{\chi}(0) = 1$, and define the inner and outer indecision localizations at radius $\rho(t)$ of u_1 as

$$\hat{u}_{1L}(\xi,t) = \hat{\chi}(\xi/\rho(t))\hat{u}_1(\xi,t),$$

and

$$\hat{u}_{1H}(\xi,t) = \left(1 - \hat{\chi}(\xi/\rho(t))\right)\hat{u}_1(\xi,t),$$

(the $\frac{1}{8\pi}$ and $\frac{1}{2\pi}$ radii are chosen to be consistent with the assumption $\hat{\chi}(0) = 1$, since $\hat{\chi}(0) = \int_{R^3}\chi(x)\,\mathrm{d}x$. In reality, this is for suitability only; the argument is easily proper to the case where $\hat{\chi}(0)$ is any number $\ne 0$). We note that the indecision localization of $u_1 = u_{1L} + u_{1H}$ is inaccurate, though decisively we have;

$$\left|1 - \hat{\chi}(\xi)\right| \le c\min\left(|\xi|, 1\right) \tag{1.10}$$

2. Proof of Theorem 1.2

In this section we discuss a proof of Theorem (1.2).

Proposition 2.1. Let u be an H^1 radial solution to (1.1) that blows-up in finite $T > 0$. Let

$$R(t) = c_1\|u_0\|_{L^2}^{\frac{3}{2}}\|\nabla u(t)\|_{L^2}^{-\frac{1}{2}}$$

and $\rho(t) = c_2\|\nabla u(t)\|_{L_x^2}^2$, (Where c_1 and c_2 are absolute constants), and $u = u_{1L} + u_{1H} + u_2$ as characterized in the paragraph above.

1) There exists an absolute constant $c > 0$ such that

$$\left\|u_{1L}(t)\right\|_{L_x^3} \geq c \text{ as } t \to T. \quad (2.1)$$

2) Let us assume that there exists a constant c^* such that $\left\|u_1(t)\right\|_{L^3} \leq c^*$. Then

$$\left\|u_1(t)\right\|_{L_x^3\left(\left|x-x_0(t)\right|\leq\rho(t)^{-1}\right)} \geq \frac{c}{\left(c^*\right)^3} \text{ as } t \to T \quad (2.2)$$

for some absolute constant $c > 0$, where $x_0(t)$ is a stance function such that

$$\left|x_0(t)\right| / \rho(t)^{-1} \leq c \cdot \left(c^*\right)^6.$$

We recall, an "exterior" estimate, usable to radially symmetric functions only, originally due to [11]:

$$\|\upsilon\|^4_{L^4_{\{|x|>R\}}} \leq \frac{c}{R^2} \upsilon^3_{L^2_{\{|x|>R\}}} \nabla \upsilon_{L^2_{\{|x|>R\}}} \quad (2.3)$$

where c is independent of $R > 0$. We recall the generally usable symmetric functions and for any function υ,

$$\|\upsilon\|^4_{L^4\left(R^3\right)} \leq c\|\upsilon\|^2_{L^3\left(R^3\right)} \|\nabla \upsilon\|^2_{L^2_{R^3}}. \quad (2.4)$$

(2.3), (2.4) are Gagliardo-Nirenberg estimates for functions on $\mathbb{R}^3$.

Proof of Prop 2.1: Since by (1.3), $\left\|\nabla u(t)\right\|_{L_x^2} \to +\infty$ as $t \to T$, by energy conservation, we have $u(t)^4_{L_x^4} / \nabla u(t)^2_{L_x^2} \to 2$. Thus, for t to be large enough to close to T

$$\|\nabla u\|^2_{L_x^2} \leq u^4_{L_x^4} \leq \|u_{1L}\|^4_{L_x^4} + \|u_{1H}\|^4_{L_x^4} + \|u_2\|^4_{L_x^4}. \quad (2.5)$$

By (2.3), the selection of $R(t)$ and mass conservation;

$$\|u_2\|^4_{L_x^4} \leq \frac{c}{R^2}\|u_0\|^3_{L_x^2} \|\nabla u\|_{L_x^2} \leq \frac{1}{4}\|\nabla u\|^2_{L_x^2} \quad (2.6)$$

where c_1 in the definition of $R(t)$ has been selected to obtain the factor $\frac{1}{4}$ here. By Sobolev embedding, (1.10), and the selected $\rho(t)$

$$\begin{aligned}\|u_{1H}\|^4_{L_x^4} &\leq c\|u_{1H}\|^4_{\dot{H}^{\frac{3}{4}}} = c\left\||\xi|^{\frac{3}{4}}\left(1-\hat{\chi}(\xi/\rho)\right)\hat{u}_1(\xi)\right\|^4_{L_\xi^2} \\ &\leq c\rho^{-1}\left\||\xi|\hat{u}_1(\xi)\right\|^4_{L_\xi^2} \leq c\rho^{-1}\|\nabla u_1\|^4_{L_x^2} \\ &\leq c\rho^{-1}\|\nabla u\|^4_{L_x^2} \leq \frac{1}{4}\nabla u^2_{L_x^2}\end{aligned} \quad (2.7)$$

where c_2 in the definition of $\rho(t)$ has been selected to obtain the factor $\frac{1}{4}$ here. Bring together (2.5), (2.6), and (2.7), to obtain

$$\|\nabla u\|^2_{L_x^2} \leq c\|u_{1L}\|^4_{L_x^4} \quad (2.8)$$

By (2.8) and (2.4), we obtain (2.1), completing the proof of part (1) of the proposition.

To prove part (2), we assume $u_1(t)_{L^3} \leq c^*$, by (2.8)

$$\begin{aligned}\|\nabla u\|^2_{L^2} &\leq c\|u_{1L}\|^3_{L_x^3}\|u_{1L}\|_{L_x^\infty} \leq c\cdot\left(c^*\right)^3\|u_{1L}\|_{L_x^\infty} \\ &\leq c\cdot\left(c^*\right)^3 \sup_{x\in R^3}\left|\int \rho^3\chi\rho(x-y)u_1(y)\,\mathrm{d}y\right|.\end{aligned}$$

There exists $x_0 = x_0(t) \in \mathbb{R}^3$ for which at least $\frac{1}{2}$ of this supremum is attained. Thus,

$$\begin{aligned}\|\nabla u\|^2_{L^2} &\leq c\cdot\left(c^*\right)^3\left|\int\rho^3\chi\rho(x_0-y)u_1(y)\,\mathrm{d}y\right| \\ &\cdot\left|\int\rho^3\chi\rho(x_0-y)u_1(y)\,\mathrm{d}y\right| \\ &\leq c\cdot\left(c^*\right)^3\rho^3\int_{|x_0(t)-y|\leq\rho^{-1}}\left|u_1(y)\right|\mathrm{d}y \\ &\leq c\cdot\left(c^*\right)^3\rho\left(\int_{|x_0(t)-y|\leq\rho^{-1}}\left|u_1(y)\right|^3\mathrm{d}y\right)^{\frac{1}{3}}\end{aligned}$$

where we used Hölder's inequality in the last step. By the selected ρ, we obtain (2.2). To complete the proof, it keeps to obtain the remind control on $x_0(t)$ which will be a consequence of the radial supposition and the supposed bound $\left\|u_1(t)\right\|_{L^3} \leq c^*$.

Assume $\dfrac{\left|x_0(t_n)\right|}{\rho(t_n)^{-1}} \gg \left(c^*\right)^6$ along a sequence of times $t_n \to T$. Assume the spherical annulus;

$$A = \left\{x \in \mathbb{R}^3 : |x_0| - \rho^{-1} \leq |x| \leq |x_0| + \rho^{-1}\right\}.$$

And inside A place $\sim \dfrac{4\pi|x_0|^2}{\pi\left(\rho^{-1}\right)^2}$ disjoint balls, at radius x_0, both the radius ρ^{-1}, centered on the sphere. By the radiality supposition, on all ball B, we have $\|u_1\|_{L_B^3} \geq \dfrac{c}{\left(c^*\right)^3}$, and hence on the annulus A,

$$\|u_1\|^3_{L_A^3} \geq \frac{c}{\left(c^*\right)^9}\frac{|x_0|^2}{\left(\rho^{-1}\right)^2} \gg \left(c^*\right)^3.$$

which contradicts the assumption $\|u_1\|_{L^3} \leq c^*$. □

We now point out how to obtain Theorem 1.2 as a consequence.

Proof of Theorem 1.2. By part (1) of Prop. 2.1 and the standard convolution inequality:

$$c \leq \|u_{1L}\|_{L_x^3} = \left\|\rho^3\chi(\rho\,\cdot) * u_1\right\|_{L_x^3} \leq \|u_1\|_{L^3}.$$

If $\|u_1(t)\|_{L^3}$ is not bounded, then there exists a sequence of times $t_n \to T$ such that $\|u_1(t_n)\|_{L^3} \to \infty$. Since $\|u(t_n)\|_{L^3(|x|\le 2R)} \ge u_1(t_n)_{L^3}$, we have (1.8) in Theorem 1.2; on the other hand, if $\|u_1(t)\|_{L^3} \le c^*$, for some c^*, as $t \to T$, we have (2.2) of Prop. 2.1. Since $|x_0(t)| \le c(c^*)^6 \rho(t)^{-1}$, we have

$$\frac{c}{(c^*)^3} \le \|u_1(t)\|_{L^3(|x-x_0(t)|\le\rho(t)^{-1})} \le \|u_1(t_n)\|_{L^3(|x|\le c(c^*)^6\rho(t)^{-1})}$$

which gives (1.7) in Theorem 1.2. □

3. Strichartz Estimates

In this section we show local theory and Strichartz estimates.

Strichartz Type Estimates

We say the pair (q,r) is $\dot{H}^s$ Strichartz admissible if $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}-s$, with $2\le q$, $r\le\infty$ and $(q,r,d)\ne(2,\infty,2)$.

And the pair (q,r) is $\frac{d}{2}$-passable if $1\le q$, $r\le\infty$, $\frac{1}{q}<d\left(\frac{1}{2}-\frac{1}{r}\right)$ or $(q,r)=(\infty,2)$.

As habitual we denote by q',r' the Hölder conjugates of q and r consecutive (*i.e.* $\frac{1}{r}+\frac{1}{r'}=1$).

Let

$$\|u\|_{S(L^2)} = \sup_{(q,r)L^2\text{admissible } 2\le r\le 6, 2\le q\le\infty} \|u\|_{L_t^q L_x^r}.$$

We consider dual Strichartz norms. Let

$$u_{S'(L^2)} = \inf_{(q,r)L^2\text{admissible } 2\le q\le\infty, 2\le r\le 6,} \|u\|_{L_t^{q'} L_x^{r'}}.$$

where (q',r') is the Hölder dual to (q,r). Also define

$$\|u\|_{S'\left(\dot{H}^{-\frac{1}{2}}\right)} = \sup_{(q,r)\dot{H}^{-\frac{1}{2}}\text{ admissible } \frac{4^+}{3}\le q\le 6^-, 3^+\le r\le 6^-} u_{L_t^{q'} L_x^{r'}}$$

The Strichartz estimates are:

$$\left\|e^{it\Delta}\phi\right\|_{S(L^2)} \le c\|\phi\|_{L^2}$$

and

$$\left\|\int_0^t e^{i(t-t')\Delta} f(\cdot,t')\,dt'\right\|_{S(L^2)} \le c\|f\|_{S'(L^2)}.$$

By bring together Sobolev embedding with the Strichartz estimates, we obtain

$$\left\|e^{it\Delta}\phi\right\|_{S\left(\dot{H}^{\frac{1}{2}}\right)} \le c\|\phi\|_{\dot{H}^{\frac{1}{2}}}$$

and

$$\left\|\int_0^t e^{i(t-t')\Delta} f(\cdot,t')\,dt'\right\|_{S\left(\dot{H}^{\frac{1}{2}}\right)} \le c\left\|D^{\frac{1}{2}} f\right\|_{S'(L^2)}. \quad (3.1)$$

We must also need the Kato inhomogeneous Strichartz estimate [12].

$$\left\|\int_0^t e^{i(t-t')\Delta} f(\cdot,t')\,dt'\right\|_{S\left(\dot{H}^{\frac{1}{2}}\right)} \le c\|f\|_{S'\left(\dot{H}^{-\frac{1}{2}}\right)}. \quad (3.2)$$

To point out a restriction to a time subinterval $\subset(-\infty,+\infty)$, we will write $S(\dot{H}^s;I)$ or $S'(\dot{H}^s;I)$.

Proposition 3.1 Assume $\|u_0\|_{S\left(\dot{H}^{\frac{1}{2}}\right)} \le M$. There is $\delta_{sd}=\delta_{sd}(M)>0$ such that if $\left\|e^{it\Delta}u_0\right\|_{S\left(\dot{H}^{\frac{1}{2}}\right)} \le \delta_{sd}$, then u solving (1.1) is global (in $\dot{H}^{\frac{1}{2}}$) and

$$\|u\|_{S\left(\dot{H}^{\frac{1}{2}}\right)} \le 2\left\|e^{it\Delta}u_0\right\|_{S\left(\dot{H}^{\frac{1}{2}}\right)},$$

$$\left\|D^{\frac{1}{2}}u\right\|_{S(L^2)} \le 2c\|u_0\|_{\dot{H}^{\frac{1}{2}}}.$$

(Observe that, by the Strichartz estimates, the assumptions are satisfied if $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \le c\delta_{sd}$).

Proof. Define

$$\Psi_{u_0}(v) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-t')\Delta}|v|^2 v(t')\,dt'.$$

Applying the Strichartz estimates, we obtained

$$\left\|D^{\frac{1}{2}}\Psi_{u_0}(v)\right\|_{S(L^2)} \le c\|u_0\|_{\dot{H}^{\frac{1}{2}}} + c\left\|D^{\frac{1}{2}}\left(|v|^2 v\right)\right\|_{L_t^{\frac{5}{2}} L_x^{\frac{10}{9}}}$$

and

$$\left\|\Psi_{u_0}(v)\right\|_{S\left(\dot{H}^{\frac{1}{2}}\right)} \le c\left\|e^{it\Delta}u_0\right\|_{S\left(\dot{H}^{\frac{1}{2}}\right)} + c\left\|D^{\frac{1}{2}}\left(|v|^2 v\right)\right\|_{L_t^{\frac{5}{2}} L_x^{\frac{10}{9}}}$$

We apply the Hölder inequalities and fractional Leibnitz [13] to get

$$\left\|D^{\frac{1}{2}}\left(|v|^2 v\right)\right\|_{L_t^{\frac{5}{2}} L_x^{\frac{10}{9}}} \le \|v\|_{L_t^5 L_x^5}^2 \left\|D^{\frac{1}{2}}v\right\|_{L^\infty L^2} \le \|v\|_{S\left(\dot{H}^{\frac{1}{2}}\right)}^2 \left\|D^{\frac{1}{2}}v\right\|_{S(L^2)}$$

Let

$$\delta_{sd} \le \min\left(\frac{1}{\sqrt{24c}}, \frac{1}{24cM}\right).$$

Then $\Psi_{u_0}: N \to N$, where

$$N=\left\{v\left|\|v\|_{S\left(\dot{H}^{\frac{1}{2}}\right)}\le\left\|2\mathrm{e}^{it\Delta}u_0\right\|_{S\left(\dot{H}^{\frac{1}{2}}\right)},\left\|D^{\frac{1}{2}}v\right\|_{S\left(\dot{L}^2\right)}\le 2c\|u_0\|_{\dot{H}^{\frac{1}{2}}}\right.\right\}$$

and Ψ_{u_0} is a contraction on *N*. □

Proposition 3.2. If $u_0\in H^1, u(t)$ is global with globally finite $\dot{H}^{\frac{1}{2}}$ Strichartz norm $\|u\|_{S\left(\dot{H}^{\frac{1}{2}}\right)}<+\infty$ and a uniformly bounded H^1 norm $\sup_{t\in[0,+\infty)}\|u(t)\|_{H^1}\le N$, then $u(t)$ scatters in H^1 as $t\to+\infty$.

Meaning that there exist $\phi^+\in H^1$ such that

$$\lim_{t\to+\infty}\left\|u(t)-\mathrm{e}^{it\Delta}\phi^+\right\|_{H^1}=0.$$

Proof. Since $u(t)$ resolves the integral equation

$$u(t)=\mathrm{e}^{it\Delta}u_0+i\int_0^t\mathrm{e}^{i(t-t')\Delta}\left(|u|^2u\right)(t')\,\mathrm{d}t',$$

we have

$$u(t)-\mathrm{e}^{it\Delta}\phi^+=-i\int_t^{+\infty}\mathrm{e}^{i(t-t')\Delta}\left(|u|^2u\right)(t')\,\mathrm{d}t' \tag{3.3}$$

where

$$\phi^+=u_0+i\int_0^{+\infty}\mathrm{e}^{-it'\Delta}\left(|u|^2u\right)(t')\,\mathrm{d}t'.$$

Apply the Strichartz estimates to (3.3), to get

$$\begin{aligned}&\left\|u(t)-\mathrm{e}^{it\Delta}\phi^+\right\|_{H^1}\\&\le c\left\||u|^2\left(1+|\nabla|\right)u\right\|_{L^{\frac{5}{2}}_{[t,+\infty)}L_x^{\frac{10}{9}}}\\&\le c\|u\|^2_{L^5_{[t,+\infty)}L^5_x}\|u\|_{L_t^\infty H_x^1}\le cN\|u\|^2_{L^5_{[t,+\infty)}L^5_x}\end{aligned}$$

As $t\to\infty$ above inequality get the claim. □

4. Some Lemma

4.1. Here We Discuss the Precompactness of the Flow Implies Regular Localization

Let *u* be a solution to (1.1) such that

$$K=\left\{u\left(\cdot-\eta(t),t\right)\middle|t\in[0,+\infty)\right\} \tag{4.1}$$

is precompact in H^1. Then for each $\varepsilon>0$ there exist $R>0$ so that π for all $0\le t<+\infty$.

We proof (4.2) by contradiction, there exists $\varepsilon>0$ and a sequence of times t_n and by changing the variables,

$$\begin{aligned}&\int_{|\eta|>n}\left|\nabla u\left(\eta-\eta(t_n),t_n\right)\right|^2+\left|u\left(\eta-\eta(t_n),t_n\right)\right|^2\\&+\left|u\left(\eta-\eta(t_n),t_n\right)\right|^4\varepsilon\ge\varepsilon\end{aligned} \tag{4.3}$$

Since *K* is precompact, there exists $\varphi\in H^1$, such that $u\left(\cdot-\eta(t_n),t_n\right)\to\varphi$ in H^1, by (4.3),

$$\forall R>0,\int_{|\eta|>R}\left|\nabla\varphi(\eta)\right|^2+\left|\varphi(\eta)\right|^2+\left|\varphi(\eta)\right|^4\mathrm{d}\eta\ge\varepsilon.$$

Which is a contradiction with the fact that $\varphi\in H^1$. The proof is complete.

Lemma 4.1. Let *u* be a solution of (1.1) defined on $[0,+\infty)$, such that $P[u]=0$ and *K* such as in (4.1) is precompact in H^1, for some continuous function $\eta(\cdot)$ then;

$$\frac{\eta(t)}{t}\to 0\text{ as }t\to+\infty \tag{4.4}$$

Proof. Suppose that (4.4) does not hold. Then there exists a sequence $t_n\to+\infty$, such that $\frac{\eta(t_n)}{t_n}\ge\varepsilon_0$ for some $\varepsilon_0>0$. Retaining generality, we assume $\eta(0)=0$. For $R>0$, let

$$t_0(R)=\inf\left\{t\ge 0:\left|\eta(t)\right|\ge R\right\}$$

i.e. $t_0(R)$ is the first time when $\eta(t)$ arrives at the boundary of the ball of radius *R*. By continuity of $\eta(t)$, the value $t_0(R)$ is well-defined. Furthermore, the following hold:

1) $t_0(R)>0$;
2) $|\eta(t)|<R$, for $0\le t\le t_0(R)$;
3) $|\eta(t_0(R))|=R$.

Let $R_n=|\eta(t_n)|$ and $\tilde{t}_n=t_0(R_n)$. We note that $t_n\ge\tilde{t}_n$, which combined with $\frac{|\eta(t_n)|}{t_n}\ge\varepsilon_0$, gives $\frac{R_n}{\tilde{t}_n}\ge\varepsilon_0$. Since $t_n\to+\infty$ and $\frac{|\eta(t_n)|}{t_n}\ge\varepsilon_0$, we have $R_n=|\eta(t_n)|\to+\infty$. Thus $\tilde{t}_n=t_0(R_n)\to+\infty$. We can disregard t_n. We will concentrate our work on the time interval $[0,\tilde{t}_n]$, and we will use in the proof:

1) for $0\le t\le\tilde{t}_n$ we have $|\eta(t)|<R_n$;
2) $|\eta(\tilde{t}_n)|=R_n$;
3) $\frac{R_n}{t_n}\ge\varepsilon_0$ and $\tilde{t}_n\to+\infty$.

By the precompactness of *K* and (4.2) it follows that for any $\varepsilon>0$, there exists $R_0(\varepsilon)\ge 0$, such that for any $t\ge 0$

$$\int_{|\eta+\eta(t)|\ge R_0(\varepsilon)}\left(|u|^2+|\nabla u|^2\right)\mathrm{d}\eta\le\varepsilon. \tag{4.5}$$

We will select *ε* later; for $\eta\in\mathbb{R}$ let $\gamma(\eta)\in C_0^\infty(\mathbb{R})$

be such that $\gamma(\eta)=\eta$ for $-1\le\eta\le 1$, $\gamma(\eta)=0$ for $|\eta|\ge 2^{\frac{1}{3}}$, $|\gamma(\eta)|\le|\eta|$, $\gamma'_\infty\le 4$ and $\gamma_\infty\le 2$ for $\eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{R}^3$. Let $\phi(\eta)=(\gamma(\eta_1),\gamma(\eta_2),\gamma(\eta_3))$. Then $\phi(\eta)=\eta$ for $|\eta|\le 1$ and $\|\phi\|_\infty\le 2$. For $R>0$, set $\phi_R(\eta)=R\phi(\eta/R)$. Let $z_R:\mathbb{R}\to\mathbb{R}^3$ be the truncation center of mass given by

$$z_R(t)=\int\phi_R(\eta)|u(\eta,t)|^2\,\mathrm{d}\eta.$$

Then $z'_R(t)=\left([z'_R(t)]_1,[z'_R(t)]_2,[z'_R(t)]_3\right)$, where

$$[z'_R(t)]_j=2\operatorname{Im}\int\gamma'(\eta_j/R)\partial_j u\bar{u}\,\mathrm{d}\eta.$$

Observe that $\gamma'\left(\frac{\eta_j}{R}\right)=1$ for $|\eta_j|\le 1$. By the zero momentum property

$$\operatorname{Im}\int_{|\eta_j|\le R}\partial_j u\bar{u}=-\operatorname{Im}\int_{|\eta_j|>R}\partial_j u\bar{u}.$$

Thus,

$$[z'_R(t)]_j=-2\operatorname{Im}\int_{|\eta_j|\ge R}\partial_j u\bar{u}\,\mathrm{d}\eta+2\operatorname{Im}\int_{|\eta_j|\ge R}\gamma'(\eta_j/R)\partial_j u\bar{u}\,\mathrm{d}\eta.$$

By Cauchy-Schwarz, we obtain;

$$|z'_R(t)|\le 5\int_{|\eta|\ge R}\left(|u|^2+|\nabla u|^2\right). \tag{4.6}$$

Set $\tilde{R}_n=R_n+R_0(\varepsilon)$. Observe that for $0\le t\le\tilde{t}_n$ and $|\eta|\ge\tilde{R}_n$, we have $|\eta+\eta(t)|\ge\tilde{R}_n-R_n=R_0(\varepsilon)$, and thus (4.6), (4.5) give

$$\left|z'_{\tilde{R}_n}(t)\right|\le 5\varepsilon. \tag{4.7}$$

We now obtain an upper bound for $z'_{\tilde{R}_n}(0)$ and a lower bound for $z'_{\tilde{R}_n}(t)$

$$z'_{\tilde{R}_n}(0)=\int_{|\eta|<R_0(\varepsilon)}\phi_{\tilde{R}_n}(\eta)|u_0|^2\,\mathrm{d}\eta+\int_{|\eta+\eta(0)|\ge R_0(\varepsilon)}\phi_{\tilde{R}_n}(\eta)|u_0|^2\,\mathrm{d}\eta$$

Hence, by (4.5) we have

$$\left|z'_{\tilde{R}_n}(0)\right|\le R_0(\varepsilon)M[u]+2\tilde{R}_n\varepsilon. \tag{4.8}$$

For $0\le t\le\tilde{t}_n$, we divide $z'_{\tilde{R}_n}(t)$ as

$$z'_{\tilde{R}_n}(t)=\int_{|\eta+\eta(t)|\ge R_0(\varepsilon)}\phi_{\tilde{R}_n}(\eta)|u_0|^2\,\mathrm{d}\eta+\int_{|\eta+\eta(t)|\ge R_0(\varepsilon)}\phi_{\tilde{R}_n}(\eta)|u_0|^2\,\mathrm{d}\eta=I+II$$

To deduce the expression for I, we observed that $\left|\phi_{\tilde{R}_n}(\eta)\right|\le 2\tilde{R}_n$.

And use (4.5) to obtain $|I|\le 2\tilde{R}_n\varepsilon$.

For II we first observe that,

$$|\eta|\le|\eta+\eta(t)|+|\eta(t)|\le R_0(\varepsilon)+R_n=\tilde{R}_n$$

and thus $\phi_{\tilde{R}_n}(\eta)=\eta$.

We rewrite II as

$$\begin{aligned}II&=\int_{|\eta+\eta(t)|\le R_0(\varepsilon)}(\eta+\eta(t))|u(\eta,t)|^2\,\mathrm{d}\eta-\eta(t)\int_{|\eta+\eta(t)|\le R_0(\varepsilon)}|u(\eta,t)|^2\,\mathrm{d}\eta\\&=\int_{|\eta+\eta(t)|\le R_0(\varepsilon)}(\eta+\eta(t))|u(\eta,t)|^2\,\mathrm{d}\eta-\eta(t)M[u]+\eta(t)\int_{|\eta+\eta(t)|\le R_0(\varepsilon)}|u(\eta,t)|^2\,\mathrm{d}\eta\\&=IIA+IIB+IIC\end{aligned}$$

Trivially, $|IIA|\le R_0(\varepsilon)M[u]$, and by (4.5)

$$|IIC|\le|\eta(t)|\varepsilon\le\tilde{R}_n\varepsilon.$$

Thus,

$$\begin{aligned}\left|z'_{\tilde{R}_n}(t)\right|&\ge|IIB|-|I|-|IIA|-|IIC|\\&\ge|\eta(t)|M[u]-R_0(\varepsilon)M[u]-3\tilde{R}_n\varepsilon\end{aligned}$$

Taking $t=\tilde{t}_n$, we can get

$$\left|z'_{\tilde{R}_n}(\tilde{t}_n)\right|\ge\tilde{R}_n(M[u]-3\varepsilon)-R_0(\varepsilon)M[u] \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we have

$$\begin{aligned}5\varepsilon\tilde{t}_n\int_0^{\tilde{t}_n}\left|z'_{\tilde{R}_n}(t)\right|\mathrm{d}t&\ge\left|\int_0^{\tilde{t}_n}z'_{\tilde{R}_n}(t)\,\mathrm{d}t\right|\ge\left|z'_{\tilde{R}_n}(\tilde{t}_n)-z'_{\tilde{R}_n}(0)\right|\\&\ge\tilde{R}_n(M[u]-5\varepsilon)-2R_0(\varepsilon)M[u].\end{aligned}$$

Suppose $\varepsilon\le\frac{1}{5}M[u]$ and use $\tilde{R}_n\ge R_n$ to obtain

$$5\varepsilon\ge\frac{\tilde{R}_n}{\tilde{t}_n}(M[u]-5\varepsilon)-\frac{2R_0(\varepsilon)M[u]}{\tilde{t}_n}$$

Since $\frac{\tilde{R}_n}{\tilde{t}_n}\ge\varepsilon_0$ we have

$$5\varepsilon\ge\varepsilon_0(M[u]-5\varepsilon)-\frac{2R_0(\varepsilon)M[u]}{\tilde{t}_n}$$

(Assume $\varepsilon_0\le 1$) take $\varepsilon=\frac{M[u]\varepsilon_0}{16}$, as $n\to+\infty$, since $\tilde{t}_n\to+\infty$ we get a contradiction. □

4.2. We Now Prove the Following Rigidity Theorem

Lemma 4.2. If (1.5) and (1.6) hold, then for all t

$$\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}<\alpha\|\psi\|_{L^2}\|\nabla\psi\|_{L^2}. \tag{4.10}$$

where $\alpha=\left(\frac{M[u]E[u]}{M[\psi]E[\psi]}\right)^{\frac{1}{2}}$. We have also the bound for all t;

$$\begin{aligned}&8\|\nabla u(t)\|^2-6\|u(t)\|_{L^4}^4\\&\geq 8(1-\alpha)\|\nabla u(t)\|_{L^2}^2\geq 16(1-\alpha)E[u]\end{aligned} \tag{4.11}$$

The hypothesis here is $E[u]>0$ except if $u\equiv 0.$ In fact, $E[u]\geq\frac{1}{6}\|\nabla u_0\|_{L^2}^2$.

Theorem 4.3. Assume $u_0\in H^1$ satisfies $P[u_0]=0$,

$$M[u_0]E[u_0]<M[\psi]E[\psi] \tag{4.12}$$

and

$$\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}<\|\psi\|_{L^2}\|\nabla\psi\|_{L^2}. \tag{4.13}$$

Let u be the global H^1 solution of (1.1) with initial data u_0 and assume that $K=\{u(\cdot-\eta(t),t)\,|\,t\in[0,+\infty)\}$ is precompact in H^1. Then $u_0=0$.

Proof. Let $\in C_0^\infty$ be redial with

$$\varphi(\eta)=\begin{cases}|\eta|^2 & \text{for } |\eta|\leq 1\\ 0 & \text{for } |\eta|\geq 2\end{cases}.$$

For $R>0$, we define

$$z_R(t)=\int R^2\varphi\left(\frac{\eta}{R}\right)|u(\eta,t)|^2\,\mathrm{d}\eta.$$

Then

$$z_R'(t)=2\operatorname{Im}\int R\nabla\varphi\left(\frac{\eta}{R}\right)\cdot\nabla u(t)\bar{u}(t)\,\mathrm{d}\eta.$$

By the Hölder inequality:

$$\begin{aligned}|z_R'(t)|&\leq cR\int_{|\eta|\leq 2R}|\nabla u(t)||u(t)|\,\mathrm{d}\eta\\&\leq cR\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}\end{aligned} \tag{4.14}$$

By calculation, we have the local Virial identity

$$\begin{aligned}z_R''(t)&=4\sum_{j,k}\int\frac{\partial^2\varphi}{\partial\eta_j\partial\eta_k}\left(\frac{\eta}{R}\right)\frac{\partial u}{\partial\eta_j}\frac{\partial\bar{u}}{\partial\eta_k}\\&\quad-\frac{1}{R^2}\int(\Delta^2\varphi)\left(\frac{\eta}{R}\right)|u|^2-\int(\Delta\varphi)\left(\frac{\eta}{R}\right)|u|^4.\end{aligned}$$

Since φ is radial we have

$$z_R''(t)=\left(8\int|\nabla u|^2-6\int|u|^4\right)+A_R(u(t)). \tag{4.15}$$

where

$$\begin{aligned}A_R(u(t))&=4\sum_j\left(\int\left(\partial_{\eta_j}^2\varphi\right)\left(\frac{\eta}{R}\right)-2\right)|\partial\eta_j u|^2\\&\quad+4\sum_{j\neq k}\int_{R\leq|\eta|\leq 2R}\frac{\partial^2\varphi}{\partial\eta_j\partial\eta_k}\left(\frac{\eta}{R}\right)\frac{\partial u}{\partial\eta_j}\frac{\partial\bar{u}}{\partial\eta_k}\\&\quad-\frac{1}{R^2}\int\left(\Delta^2\varphi\right)\left(\frac{\eta}{R}\right)|u|^2-\int\left(\Delta\varphi\left(\frac{\eta}{R}\right)-6\right)|u|^4.\end{aligned}$$

Thus, we obtain

$$|A_R(u(t))|\leq c\int_{|\eta|\geq R}\left(|\nabla u(t)|^2+\frac{1}{R^2}|u(t)|^2+|u(t)|^4\right)\mathrm{d}\eta \tag{4.16}$$

Now discuss $z_R(t)$ for R chosen appropriate large and selection time interval $[t_0,t_1]$ where $1\ll t_0\ll t_1<\infty$. By (4.15) and (4.11) we have

$$|z_R''(t)|\geq 16(1-\alpha)E[u]-|A_R(u(t))|. \tag{4.17}$$

Set $\varepsilon=\frac{1-\alpha}{c}E[u]$ in (4.2), $R_0\geq 0$, such that $\forall t$

$$\int_{|\eta+\eta(t)|\geq R}\left(|\nabla u|^2+|u|^2+|u|^4\right)\leq\frac{1-\alpha}{c}E[u]. \tag{4.18}$$

Choosing $R\geq R_0+\sup_{t_0\leq t\leq t_1}|\eta|$. Then (4.16), (4.17) and (4.18) imply that for all $t_0\leq t\leq t_1$,

$$|z_R''(t)|\geq 8(1-\alpha)E[u]. \tag{4.19}$$

By Lemma 4.1, there exists $t_0\geq 0$ such that for all $t\geq t_0$ we have $|x(t)|\leq\mu t$ with $\mu>0.$ By taking $R=R_0+\mu t_1$, we obtain that (4.18) holds for all $t_0\leq t\leq t_1$. Integrating (4.19) over $[t_0,t_1]$ we obtain

$$|z_R'(t_1)-z_R'(t_0)|\geq 8(1-\alpha)E[u](t_1-t_0). \tag{4.20}$$

On the other hand, for all $t_0\leq t\leq t_1$, by (4.10) and (4.14), we have

$$\begin{aligned}|z_R'(t_1)|&\leq cRu(t)_{L^2}\|\nabla u(t)\|_{L^2}\leq cR\|\psi\|_{L^2}\|\nabla\psi\|_{L^2}\\&\leq c\|\psi\|_{L^2}\|\nabla\psi\|_{L^2}(R_0+\mu t_1)\end{aligned} \tag{4.21}$$

Combining (4.20) and (4. 21), we obtained

$$8(1-\alpha)E[u](t_1-t_0)\leq 2c\|\psi\|_{L^2}\|\nabla\psi\|_{L^2}(R_0+\mu t_1).$$

It is important to mention that α and R_0 are constant depending only on $\frac{M[u]E[u]}{M[\psi]E[\psi]}$, and $t_0=t_0(\mu)$.

Putting $\mu=\frac{(1-\alpha)E[u]}{c\|\psi\|_{L^2}\|\nabla\psi\|_{L^2}}$ and setting $t_1\to+\infty$, we obtain a contradiction except if $E[u]=0$, which implies $u\equiv 0$. □

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