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Regularity of Solutions to an Integral Equation on a Half-Space R_+^n*

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ABSTRACT

In this paper, we discuss the integral equation on a half space R_+^n

$$u(x)=\int_{R_+^n}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x^*-y|^{n-\alpha}}\right)u^p(y)\mathrm{d}y,\ u(x)>0,\ x\in R_+^n. \tag{0.1}$$

where $0<\alpha<n$, $x^*=(x_1,\cdots,x_{n-1},-x_n)$ is the reflection of the point x about the ∂R_+^n. We study the regularity for the positive solutions of (0.1). A regularity lifting method by contracting operators is used in proving the boundedness of solutions, and the Lipschitz continuity is derived by combinations of contracting and shrinking operators introduced by Ma-Chen-Li ([1]).

Keywords: Regularity Lifting; HLS Inequality; Contracting Operators; Shrinking Operators

1. Introduction

Let R_+^n be the upper half Euclidean space

$$R_+^n=\{x=(x_1,x_2,\cdots,x_n)\in R^n \mid x_n>0\}.$$

In this paper we consider the regularity of positive solution of the following integral equation in R_+^n

$$u(x)=\int_{R_+^n}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x^*-y|^{n-\alpha}}\right)u^p(y)\mathrm{d}y,\ x\in R_+^n. \tag{1.1}$$

where $p>1$. It relates closely to the higher-order PDEs with Navier boundary conditions in R_+^n:

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u=u^p, u\ge 0 & \text{in } R_+^n;\\ u=(-\Delta)u=\cdots=(-\Delta)^{\frac{\alpha}{2}-1}u=0, & \text{on } \partial R_+^n.\end{cases} \tag{1.2}$$

D. Li and R. Zhuo proved the following result:

Proposition 1.1. ([2]) *Let α be an even number and $p=\frac{n+\alpha}{n-\alpha}$. If $u(x)$ is the smooth solution of the integral Equation* (1.1), *then $u(x)$ satisfies the PDEs* (1.2).

In particular, when $\alpha=2$ and $p=\frac{n+2}{n-2}$, Chen and Li ([3]) showed the equivalence between the integral Equation (1.1) and partial differential Equation (1.2). For more results concerning integral equations, see [4-6].

Firstly, in this paper we have the boundedness for the positive solutions of (1.1) by using the contracting operators.

Theorem 1.1. *Let u be a solution of* (1.1). *If $p>\frac{n}{n-\alpha}$, and $u\in L^{\frac{n(p-1)}{\alpha}}(R_+^n)$, then u is in $L^r(R_+^n)\cap L^\infty(R_+^n)$ for any $1<r<\infty$.*

Remark 1. In [2], the authors proved that Theorem 1.1 is true for the critical case $p=\frac{n+\alpha}{n-\alpha}$. While our result also covers subcritical case $\frac{n}{n-\alpha}<p<\frac{n+\alpha}{n-\alpha}$ and super critical case $p>\frac{n+\alpha}{n-\alpha}$.

Then we employ the brand new method which is the combinations of contracting and shrinking operators introduced by Ma-Chen-Li ([1]) to derive the Lipschitz continuity of solutions.

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Theorem 1.2. *Under the same conditions of Theorem 1.1, u is Lipschitz continuous in* R_+^n.

2. L^∞ Estimate by Contracting Operators

In this section, we obtain L^∞ estimate for positive solutions to the equation (0.1) by using the contracting operators. To prove the Theorem 1.1, we need the following equivalent form of Hardy-Littlewood-Sobolev inequality.

Lemma 2.1. *Let* $g \in L^{\frac{nr}{n+\alpha r}}\left(R^n\right)$ *for* $\frac{n}{n-\alpha} < r < \infty$. *Define*

$$Tg(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} g(y)\,\mathrm{d}y. \tag{2.1}$$

Then

$$\|Tg\|_{L^r(R^n)} \le C(n,\alpha,r)\|g\|_{L^{\frac{nr}{n+\alpha r}}(R^n)}.$$

Proof of Theorem 1.1: The proof is divided into two steps.

Step 1. We first show that $u(x) \in L^r\left(R_+^n\right)$, $1 < r < \infty$, $\forall x \in R_+^n$. Define

$$a(x) = u(x)^{p-1}.$$

Then

$$u(x) = \int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) a(y)u(y)\,\mathrm{d}y$$

For a positive number *A*, define

$$a_A(x) = \begin{cases} a(x), & \text{if } a(x) \ge A, \text{or } |x| \ge A \\ 0, & \text{elsewhere.} \end{cases}$$

Let

$$a_B(x) = a(x) - a_A(x).$$

Obviously, $|a_B(x)| \le A$, and $a_B(x)$ vanishes outside the ball $B_A(0)$.

Define

$$(T_A v)(x) = \int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) a_A(y)v(y)\,\mathrm{d}y.$$

$$F_A(x) = \int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) a_B(y)u(y)\,\mathrm{d}y.$$

The Equation (0.1) can be rewritten as

$$u(x) = (T_A u)(x) + F_A(x).$$

We will show that, for any $1 < r < \infty$,

1) T_A is a contracting map from $L^r\left(R_+^n\right)$ to $L^r\left(R_+^n\right)$ for *A* large, and

2) $F_A(x)$ is in $L^r\left(R_+^n\right)$.

1) Assume $v \in L^r\left(R_+^n\right)$, then

$$|T_A v| \le \int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) |a_A(y)v(y)|\,\mathrm{d}y.$$

For any $r > \frac{n}{n-\alpha}$, we apply Hardy-Littlewood-Sobolev inequality and Hölder inequality to obtain

$$\|T_A v\|_{L^r(R_+^n)} \le C\|a_A v\|_{L^{\frac{nr}{n+\alpha r}}(R_+^n)} \le C\|a_A\|_{L^{\frac{n}{\alpha}}(R_+^n)} \|v\|_{L^r(R_+^n)}.$$

Since $a(x) \in L^{\frac{n}{\alpha}}\left(R_+^n\right)$, by the definition of $a_A(x)$, one can choose a large number *A*, such that

$$C\|a_A\|_{L^{\frac{n}{\alpha}}(R_+^n)} \le \frac{1}{2}.$$

and hence arrives at

$$\|T_A v\|_{L^r(R_+^n)} \le \frac{1}{2}\|v\|_{L^r(R_+^n)}.$$

That is $T_A : L^r\left(R_+^n\right) \to L^r\left(R_+^n\right)$ is a contracting operator.

2) Consider

$$F_A(x) = \int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) a_B(y)u(y)\,\mathrm{d}y.$$

For any $r > \frac{n}{n-\alpha}$, we apply Hardy-Littlewood-Sobolev inequality and Hölder inequality to obtain

$$\|F_A\|_{L^r(R_+^n)} \le C\|a_B u\|_{L^{\frac{nr}{n+\alpha r}}(R_+^n)} \le C\|a_B\|_{L^s(R_+^n)} \|u\|_{L^t(R_+^n)}.$$

We require

$$\frac{n+\alpha r}{nr} = \frac{1}{s} + \frac{1}{t},\ s,t > 1.$$

By the bounded-ness of a_B, we see that *s* can be arbitrary. Since $u \in L^{\frac{n(p-1)}{\alpha}}\left(R_+^n\right)$, we take $t = \frac{n(p-1)}{\alpha}$, and hence

$$r = \frac{n(p-1)}{\frac{n(p-1)}{s} + \alpha(2-p)} \to \frac{n(p-1)}{\alpha(2-p)}, \text{as } s \to \infty,$$

we see $F_A \in L^{\frac{n(p-1)}{\alpha(2-p)} - \varepsilon}$, for any small $\varepsilon > 0$. Obviously, $\frac{n(p-1)}{\alpha(2-p)} > \frac{n(p-1)}{\alpha}$ since $p > 1$.

If $p > 2$, we are done. If $p < 2$, repeat the above process and after a few steps, we arrive at

$$u(x) \in L^r(R_+^n), \frac{n}{n-\alpha} < r < \infty.$$

Step 2. In this step we will show that $u(x) \in L^\infty(R_+^n)$.

For any point $x \in R_+^n$, we divide the integral into two parts

$$\begin{aligned} u(x) &= \int_{B_1(x)} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u(y)^p \, \mathrm{d}y \\ &\quad + \int_{R_+^n \setminus B_1(x)} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u(y)^p \, \mathrm{d}y \\ &\le \int_{B_1(x)} \frac{1}{|x-y|^{n-\alpha}} u(y)^p \, \mathrm{d}y \\ &\quad + \int_{R_+^n \setminus B_1(x)} \frac{1}{|x-y|^{n-\alpha}} u(y)^p \, \mathrm{d}y \\ &= I_1 + I_2 \end{aligned}$$

Consider I_2. Since $\frac{1}{|x-y|^{n-\alpha}}$, and by the result in *Step* 1, $u(x) \in L^r(R_+^n)$, for $\frac{n}{n-\alpha} < r < \infty$, we have $I_2 < C_1$.

For I_1, we apply Hölder inequality

$$I_1 \le \left(\int_{B_1(x)} \frac{1}{|x-y|^{(n-\alpha)q}} \mathrm{d}y \right)^{\frac{1}{q}} \left(\int_{B_1(x)} \left| u(y)^p \right|^{\frac{q}{q-1}} \mathrm{d}y \right)^{\frac{q-1}{q}}$$

Choose appropriate *q*, so that $(n-\alpha)q < n$, and hence

$$\left(\int_{B_1(x)} \frac{1}{|x-y|^{(n-\alpha)q}} \mathrm{d}y \right)^{\frac{1}{q}} \le C_2$$

Since $u(x) \in L^r(R_+^n), \forall r > \frac{n}{n-\alpha}$,

$$\left(\int_{B_1(x)} \left| u(y)^p \right|^{\frac{q}{q-1}} \mathrm{d}y \right)^{\frac{q-1}{q}} \le C_3$$

We conclude that

$$u(x) \in L^\infty(R_+^n).$$

3. Lipschitz Continuity by Combinations of Contracting Operators and Shrinking Operators

In the previous section we showed that the solution $u(x)$ of (0.1) is in $L^\infty(R_+^n)$. In this section, we will use the regularity lifting by combinations of contracting and shrinking operators to prove $u(x) \in C^{0,1}(R_+^n)$, the space of Lipschitz continuous functions with norm

$$\|v\|_{C^{0,1}(R_+^n)} = \|v\|_{L^\infty(R_+^n)} + \sup_{x \ne y} \left\{ \frac{|v(x)-v(y)|}{|x-y|} \right\} \tag{3.1}$$

To prove the Theorem 1.2, we need introduce the following definition, property and a more general Regularity Lifting Theorem on the combined use of contracting and shrinking operators.

Let *V* be a Hausdorff topological vector space. Suppose there are two extented norms (*i.e.* the norm of an element in *V* might be infinity) defined on *V*,

$$X := \{ v \in V : \|v\|_X < \infty \} \text{ and } Y := \{ v \in V : \|v\|_Y < \infty \}.$$

Definition. ("XY-pair") Suppose *X*, *Y* are two normed subspaces described above, *X* and *Y* are called "XY-pair", if whenever the sequence $\{u_n\} \subset X$ with $u_n \to u$ in *X* and $\|u_n\|_Y \le C$ will imply $u \in Y$.

Remark 2. The "XY-pair" are quite common, here we choose $X = L^r(R_+^n)$ for $1 \le r \le \infty$, and $Y = C^{0,1}(R_+^n)$ with the norm defined in (3.1).

Theorem 3.1. (*Regularity Lifting Theorem*) *Suppose Banach spaces X, Y are an "XY-pair", and let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be closed subsets of X and Y respectively. Suppose* $T : \mathfrak{X} \to X$ *is a contraction*:

$$\|Tf - Tg\|_X \le \eta \|f - g\|_X, \ \forall f, g \in \mathfrak{X} \text{ for some } 0 < \eta < 1;$$

and $T : \mathfrak{Y} \to Y$ is shrinking:

$$\|Tg\|_Y \le \theta \|g\|_Y, \ \forall g \in \mathfrak{Y}, \text{ for some } 0 < \theta < 1.$$

Define

$$Sf = Tf + F \text{ for some } F \in \mathfrak{X} \cap \mathfrak{Y}.$$

Moreover, assume that

$$S : \mathfrak{X} \cap \mathfrak{Y} \to \mathfrak{X} \cap \mathfrak{Y}.$$

Then there exists a solution *u* of equation

$$u = Tu + F \text{ in } \mathfrak{X}$$

and more importantly,

$$u \in Y.$$

The proof and some applications of Theorem 3.1 can be found in [1,7,8].

Proof of Theorem 1.2: For any $x \in R_+^n$, by elementary calculus one can verify that

$$\begin{aligned} \int_0^\infty \int_{B_t(x)} u^p(y) \mathrm{d}y \frac{\mathrm{d}t}{t^{n-\alpha+1}} &= \int_{R_+^n} \int_{|x-y|}^\infty \frac{\mathrm{d}t}{t^{n-\alpha+1}} u^p(y) \mathrm{d}y \\ &= \frac{1}{n-\alpha} \int_{R_+^n} \frac{1}{|x-y|^{n-\alpha}} u^p(y) \mathrm{d}y. \end{aligned}$$

It follows that the solution of (0.1) only differs by a constant multiple from the solution of the following equation

$$u(x)=\int_0^{\infty}\left(\int_{B_t(x)}u^p(y)\mathrm{d}y-\int_{B_t(x^*)}u^p(y)\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}. \quad (3.2)$$

Hence, for convenience of argument, we prove that every positive solution u of (3.2) is Lipschitz continuous.

Let

$$\mathfrak{X}=\left\{v\in X\equiv L^{\infty}\left(R_+^n\right)\Big|\|v\|_{L^\infty}\le 2\|u\|_{L^\infty}\right\}$$

and

$$\mathfrak{Y}=\left\{v\in Y\equiv C^{0,1}\left(R_+^n\right)\Big|\|v\|_{L^\infty}\le 2\|u\|_{L^\infty}\right\}.$$

For every $\epsilon>0$, define

$$T_\epsilon v(x)=\int_0^{\epsilon}\left(\int_{B_t(x)}v^p(y)\mathrm{d}y-\int_{B_t(x^*)}v^p(y)\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}.$$

and

$$F(x)=\int_{\epsilon}^{\infty}\left(\int_{B_t(x)}u^p(y)\mathrm{d}y-\int_{B_t(x^*)}u^p(y)\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}.$$

Then obviously, u is a solution of the equation

$$v=T_\epsilon v+F.$$

Write $S_\epsilon v=T_\epsilon v+F$. We will show that for ϵ sufficiently small,

1) T_ϵ is a contracting operator from $\mathfrak{X}$ to X.
2) T_ϵ is a shrinking operator from $\mathfrak{Y}$ to Y.
3) $F\in\mathfrak{X}\cap\mathfrak{Y}$ and $S_\epsilon:\mathfrak{X}\cap\mathfrak{Y}\to\mathfrak{X}\cap\mathfrak{Y}$.

1) For any $f,g\in\mathfrak{X}$ and for any $x\in R_+^n$, we have

$$T_\epsilon f(x)-T_\epsilon g(x)=\int_0^{\epsilon}\left\{\int_{B_t(x)}\left(f^p(y)-g^p(y)\right)\mathrm{d}y-\int_{B_t(x^*)}\left(f^p(y)-g^p(y)\right)\mathrm{d}y\right\}\frac{\mathrm{d}t}{t^{n-\alpha+1}}$$

Thus

$$\begin{aligned}\left|T_\epsilon f(x)-T_\epsilon g(x)\right|&\le\int_0^{\epsilon}\left\{\int_{B_t(x)}\left|f^p(y)-g^p(y)\right|\mathrm{d}y+\int_{B_t(x^*)}\left|f^p(y)-g^p(y)\right|\mathrm{d}y\right\}\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&=p\int_0^{\epsilon}\left\{\int_{B_t(x)}\left|\xi_1^{p-1}(y)\right|\left|f(y)-g(y)\right|\mathrm{d}y+\int_{B_t(x^*)}\left|\xi_2^{p-1}(y)\right|\left|f(y)-g(y)\right|\mathrm{d}y\right\}\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&\le C\|u\|_{L^\infty}^{p-1}\|f-g\|_{L^\infty}\int_0^{\epsilon}\left(\left|B_t(x)\right|+\left|B_t(x^*)\right|\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&=C_1\|u\|_{L^\infty}^{p-1}\|f-g\|_{L^\infty}\int_0^{\epsilon}\frac{\mathrm{d}t}{t^{1-\alpha}}=C_2\|u\|_{L^\infty}^{p-1}\varepsilon^{\alpha}\|f-g\|_{L^\infty}.\end{aligned}$$

Here we applied the Mean Value Theorem with both $\xi_1(y)$ and $\xi_2(y)$ valued between $f(y)$ and $g(y)$, and $\left|B_t(\cdot)\right|$ denotes the volume of the ball $B_t(\cdot)$.

Choose ϵ sufficiently small such that

$$C_2\|u\|_{L^\infty}^{p-1}\epsilon^{\alpha}\le\frac{1}{4},$$

and consequently

$$\|T_\epsilon f-T_\epsilon g\|_{L^\infty}\le\frac{1}{4}\|f-g\|_{L^\infty}.$$

Therefore T_ϵ is a contracting operator from $\mathfrak{X}$ to X for such a small ϵ.

2) Assume $v\in\mathfrak{Y}$, then for any $x,z\in R_+^n$,

$$\begin{aligned}T_\epsilon v(x)-T_\epsilon v(z)&=\int_0^{\epsilon}\left\{\left[\int_{B_t(x)}v^p(y)\mathrm{d}y-\int_{B_t(z)}v^p(y)\mathrm{d}y\right]-\left[\int_{B_t(x^*)}v^p(y)\mathrm{d}y-\int_{B_t(z^*)}v^p(y)\mathrm{d}y\right]\right\}\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&=\int_0^{\epsilon}\left\{\left[\int_{B_t(x)}\left(v^p(y)-v^p(y+z-x)\right)\mathrm{d}y\right]-\left[\int_{B_t(x^*)}\left(v^p(y)-v^p(y+z^*-x^*)\right)\mathrm{d}y\right]\right\}\frac{\mathrm{d}t}{t^{n-\alpha+1}}\end{aligned}$$

the last equality above is from the fact

$$\int_{B_t(z)}v^p(y)\mathrm{d}y=\int_{B_t(x)}v^p(y'+z-x)\mathrm{d}y'=\int_{B_t(x)}v^p(y+z-x)\mathrm{d}y$$

Therefore

$$\begin{aligned}\left|T_\epsilon v(x)-T_\epsilon v(z)\right|&\le\int_0^{\epsilon}\int_{B_t(x)}\left|\left(v^p(y)-v^p(y+z-x)\right)\right|\mathrm{d}y\frac{\mathrm{d}t}{t^{n-\alpha+1}}+\int_0^{\epsilon}\int_{B_t(x^*)}\left|\left(v^p(y)-v^p(y+z^*-x^*)\right)\right|\mathrm{d}y\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&\le\int_0^{\epsilon}\int_{B_t(x)}p\left|v^{p-1}(\xi_1)\right|\left|v(y)-v(y+z-x)\right|\mathrm{d}y\frac{\mathrm{d}t}{t^{n-\alpha+1}}+\int_0^{\epsilon}\int_{B_t(x^*)}p\left|v^{p-1}(\xi_2)\right|\left|v(y)-v(y+z^*-x^*)\right|\mathrm{d}y\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&\le C\|u\|_{L^\infty}^{p-1}\|v\|_{C^{0,1}}|z-x|\int_0^{\epsilon}\left(\left|B_t(x)\right|+\left|B_t(x^*)\right|\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}=C_1\|u\|_{L^\infty}^{p-1}\|v\|_{C^{0,1}}|z-x|\varepsilon^{\alpha}.\end{aligned}$$

Again by choosing ϵ sufficiently small, we derive

$$\sup_{x\neq z}\frac{\left|T_\epsilon v(x)-T_\epsilon v(z)\right|}{|z-x|}\leq\frac{1}{4}\|v\|_{C^{0,1}}.$$

Combining this with the estimate in 1), we arrive at

$$\|T_\epsilon v\|_{C^{0,1}}\leq\frac{1}{2}\|v\|_{C^{0,1}},\ \forall v\in\mathfrak{Y}.$$

Hence T_ϵ is a shrinking operator from $\mathfrak{Y}$ to Y.

3) To show F is Lipschitz continuous, we split it into two parts:

$$\begin{aligned}F(x)&=\int_\epsilon^1\left(\int_{B_t(x)}u^p(y)\mathrm{d}y-\int_{B_t(x^*)}u^p(y)\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&+\int_1^\infty\left(\int_{B_t(x)}u^p(y)\mathrm{d}y-\int_{B_t(x^*)}u^p(y)\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&\equiv I_1(x)+I_2(x)\end{aligned}$$

For the first part, we have

$$\begin{aligned}\left|I_1(x)-I_1(z)\right|&=\left|\int_\epsilon^1\left[\left(\int_{B_t(x)}u^p(y)\mathrm{d}y-\int_{B_t(z)}u^p(y)\mathrm{d}y\right)-\left(\int_{B_t(x^*)}u^p(y)\mathrm{d}y-\int_{B_t(z^*)}u^p(y)\mathrm{d}y\right)\right]\frac{\mathrm{d}t}{t^{n-\alpha+1}}\right|\\&\leq C\int_\epsilon^1\|u\|_{L^\infty}^p t^{n-1}\left(|x-z|+\left|x^*-z^*\right|\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}=C_1\|u\|_{L^\infty}^p|x-z|\left|1-\epsilon^{\alpha-1}\right|.\end{aligned}$$

Here we used the fact that

$$\text{the volume of }\left[\left(B_t(x)\setminus B_t(z)\right)\cup\left(B_t(z)\setminus B_t(x)\right)\right]\leq Ct^{n-1}|x-z|.$$

It follows that

$$\sup_{x\neq z}\frac{\left|I_1(x)-I_1(z)\right|}{|x-z|}\leq C(\epsilon).\tag{3.3}$$

For the second part, we use a different approach. Write $\delta=|x-z|$, then

$$\begin{aligned}I_1(x)-I_1(z)&=\int_1^\infty\left\{\left[\int_{B_t(x)}u^p(y)\mathrm{d}y-\int_{B_t(z)}u^p(y)\mathrm{d}y\right]+\left[\int_{B_t(z^*)}u^p(y)\mathrm{d}y-\int_{B_t(x^*)}u^p(y)\mathrm{d}y\right]\right\}\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&\equiv I_3(x,z)+I_3\left(z^*,x^*\right)\end{aligned}$$

$$\begin{aligned}I_3(x,z)&=\int_1^\infty\left(\int_{B_t(x)}u^p(y)\mathrm{d}y-\int_{B_t(z)}u^p(y)\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}\leq\int_1^\infty\left(\int_{B_{t(1+\delta)}(z)}u^p(y)\mathrm{d}y-\int_{B_t(z)}u^p(y)\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&=\int_{1+\delta}^\infty\int_{B_t(z)}u^p(y)\mathrm{d}y(1+\delta)^{n-\alpha}\frac{\mathrm{d}t}{t^{n-\alpha+1}}-\int_1^\infty\int_{B_t(z)}u^p(y)\mathrm{d}y\frac{\mathrm{d}t}{t^{n-\alpha+1}}\leq\left(\int_1^\infty\int_{B_t(z)}u^p(y)\mathrm{d}y\frac{\mathrm{d}t}{t^{n-\alpha+1}}\right)\left[(1+\delta)^{n-\alpha}-1\right]\\&=\left(\int_{|y-z|\geq1}\frac{1}{|y-z|^{n-\alpha}}u^p(y)\mathrm{d}y\right)(1+\xi_1)^{n-\alpha-1}\delta\leq C_1\left(\int_{R_+^n}u^p(y)\mathrm{d}y\right)\delta\leq C_2\delta.\end{aligned}\tag{3.4}$$

Similarly,

$$\begin{aligned}I_3\left(z^*,x^*\right)&=\int_1^\infty\left(\int_{B_t(z^*)}u^p(y)\mathrm{d}y-\int_{B_t(x^*)}u^p(y)\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}\leq\int_1^\infty\left(\int_{B_{t(1+\delta)}(x^*)}u^p(y)\mathrm{d}y-\int_{B_t(x^*)}u^p(y)\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&=\int_{1+\delta}^\infty\int_{B_t(x^*)}u^p(y)\mathrm{d}y(1+\delta)^{n-\alpha}\frac{\mathrm{d}t}{t^{n-\alpha+1}}-\int_1^\infty\int_{B_t(x^*)}u^p(y)\mathrm{d}y\frac{\mathrm{d}t}{t^{n-\alpha+1}}\\&\leq\left(\int_1^\infty\int_{B_t(x^*)}u^p(y)\mathrm{d}y\frac{\mathrm{d}t}{t^{n-\alpha+1}}\right)\left[(1+\delta)^{n-\alpha}-1\right]\leq\left(\int_1^\infty\int_{B_t(x)}u^p(y)\mathrm{d}y\frac{\mathrm{d}t}{t^{n-\alpha+1}}\right)(1+\xi_2)^{n-\alpha-1}\delta\leq C_3\delta\end{aligned}\tag{3.5}$$

Note that here we applied the Mean Value Theorem with both ξ_1 and ξ_2 valued between 0 and δ.

Combining (3.4) and (3.5), we have

$$I_2(x)-I_2(z)=I_3(x,z)+I_3\left(z^*,x^*\right)\leq C\delta$$

The same inequality holds for $I_2(z)-I_2(x)$. Therefore,

$$\sup_{x\neq z}\frac{\left|I_2(x)-I_2(z)\right|}{|x-z|}\leq C.\tag{3.6}$$

Also from the definition of $F(x)$, we immediately have

$$\|F\|_{L^\infty} \le \|u\|_{L^\infty}. \tag{3.7}$$

Obviously (3.3), (3.6) and (3.7) imply that $F(x)$ is Lipschitz continuous, and this together with (3.7) imply

$$F \in \mathfrak{X} \cap \mathfrak{Y}.$$

Finally, to see that S_ε maps $\mathfrak{X} \cap \mathfrak{Y}$ into itself, we only need to verify that

$$\text{if } \|v\|_{L^\infty} \le 2\|u\|_{L^\infty}, \text{ then } \|T_\varepsilon v\|_{L^\infty} \le \|u\|_{L^\infty}. \tag{3.8}$$

In fact,

$$\begin{aligned} |T_\varepsilon v(x)| &= \left| \int_0^\varepsilon \left(\int_{B_t(x)} v^p(y)\,\mathrm{d}y - \int_{B_t(x^*)} v^p(y)\,\mathrm{d}y \right) \frac{\mathrm{d}t}{t^{n-\alpha+1}} \right| \\ &\le \|v\|_{L^\infty}^p \int_0^\varepsilon \left(|B_t(x)| + |B_t(x^*)| \right) \frac{\mathrm{d}t}{t^{n-\alpha+1}} \\ &\le C \|u\|_{L^\infty}^p \varepsilon^\alpha. \end{aligned}$$

Choosing ε sufficiently small (but independent of v), we can guarantee (3.8).

So far we have verified 1), 2) and 3), by the Theorem 3.1 and Remark 2, we conclude that the solution u of (0.1) is Lipschitz continuous. This completes the proof of the Theorem 1.2.

Usually, contracting operators are used to lift regularities. For a linear operator, if it is "shrinking", then it is contracting. While for nonlinear problems, as were seen in Section 3, sometimes it is very difficult or even impossible to prove that it is contracting in a given function space. However, one can show that it is "shrinking", and can still lift the regularity of solutions in many cases. The general Regularity Lifting Theorem is applied for integral equations and system of integral equations associated with Bessel potentials and Wolff potentials (see [1] and [7]), and therefore arrive at higher regularity as Lipschitz continuity of solutions.

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