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Common Fixed Point Result of Multivalued and Singlevalued Mappings in Partially Ordered Metric Space

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ABSTRACT

In recent times the fixed point resulting in partially ordered metric spaces has greatly developed. In this paper we prove common fixed point results for multivalued and singlevalued mappings in partially ordered metric space. Our theorems generalized the theorem in [1] and extended much more recent results in such spaces.

Keywords: Multi-Valued Mapping; Single-Valued Mapping; Partial Ordering; Control Function; Fixed Point Theorem

1. Introduction and Preliminaries

Throughout this paper, let (X,d) be a metric space unless mentioned otherwise and $B(X)$ is the set of all non-empty bounded subsets of X. Let $\delta(A,B)$ and $D(A,B)$ be the functions defined by

$$\delta(A,B)=\sup\{d(a,b):a\in A,b\in B\}$$

$$D(A,B)=\inf\{d(a,b):a\in A,b\in B\}$$

for all A, B in $B(X)$. If A is a singleton *i.e.* $A=\{a\}$, we write

$$\delta(A,B)=\delta(a,B)$$

and

$$D(A,B)=D(a,B)$$

If B is also a singleton *i.e.* $B=\{b\}$, we write

$$\delta(A,B)=\delta(A,b)$$

and

$$D(A,B)=D(A,b)$$

It is obvious that $D(A,B)=\delta(a,B)$. For all $A,B,C\in B(X)$. The definition of $d(A,B)$ yields the following:

$$\delta(A,B)=\delta(B,A)\le 0$$

$$\delta(A,B)\le\delta(A,C)+\delta(C,B)$$

$$\delta(A,B)=0 \text{ iff } A=B=\{a\}$$

and

$$\delta(A,A)=\operatorname{diam}\{a\}.$$

Several authors used these concepts of weakly contraction, compatibility, weak compatibility to prove some common fixed point theorems for set valued mappings (see [2-8]).

Definition 1.1. [9] A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if

1) Given $a\in A$, there is a sequence $\{a_n\}$ in X such that $a_n\in A_n$ for $n=1,2,\cdots$, and $\{a_n\}$ converges to a.

2) Given $\varepsilon>0$, there exists a positive integer N such that $A_n\subseteq A_\varepsilon$ for $n>N$ where A_e is the union of all open spheres with centers in A and radius ε.

Lemma 1.1. [9,10] If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.

Lemma 1.2. [9] Let $\{A_n\}$ be a sequence in $B(X)$ and y a point in X such that $\delta(A_n,y)\to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.

In [11], Jungck and Rhoades extended definition of compatibility to set valued mappings setting as follows:

Definition 1.2. The mapping $I:X\to X$ and $f:X\to B(X)$ are δ-compatible if $\lim_{n\to\infty}\delta(fIx_n,Ifx_n)=0$, whenever $\{x_n\}$ is a sequence in X such the $Ifx_n\in B(X), fx_n\to\{t\}$ and $Ix_n\to t$, for some $t\in X$.

Recently, the following definition is given by Jungck and Rhoades [12].

Definition 1.3. The mapping $I:X\to X$ and $f:X\to B(X)$ are weakly compatible if for each point u in X such that $fu=\{Iu\}$, we have $fIu=Ifu$.

It can be seen that any δ-compatible mappings are weakly compatible but the converse is not true as shown by an example in [13]. We will use the following relation between two nonempty subsets of a partially ordered set.

Definition 1.4. [3] Let A and B be two nonempty subsets of a partially ordered set $(X;\preccurlyeq)$. The relation between A and B is denoted and defined as follows: $A \preccurlyeq B$, if for every $a \in A$ there exists $b \in B$ such that $a \preccurlyeq b$.

We will utilize the following control function which is also referred to as altering distance function.

Definition 1.5. [14] A function $\psi : [0,\infty) \to [0,\infty)$ is called an Altering distance function if the following properties are satisfied:

1) ψ is monotone increasing and continuous,
2) $\psi(t) = 0$ if and only if $t = 0$.

For the use of control function in metric fixed point theory see some recent references ([15,16]).

2. Main Result

Recently fixed point theory in partially ordered metric spaces has greatly developed. Choudhury and Metiya [17] proved certain fixed point theorems for multi valued and single valued mappings in partially ordered metric spaces. They proved the following:

Theorem 2.1. Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T : X \to B(X)$ be a multi valued mappings such that the following conditions are satisfied:

There exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$,

1) For $x, y \in X, x \prec_1 y$ implies $Tx \prec_1 Ty$,

2) If $x_n \to x$ is a non decreasing sequence in X, then $x_n \preccurlyeq x$, for all n,

3) $$\psi\left(d\left(Tx,Ty\right)\right) \le \alpha\psi\left(\max\left\{d\left(x,y\right), D\left(x,Tx\right), D\left(y,Ty\right), \frac{D\left(x,Ty\right)+D\left(y,Tx\right)}{2}\right\}\right),$$

for all comparable $x, y \in X$, where $0 < \alpha < 1$ and ψ is an Altering distance function. Then T has a fixed point.

We prove the following theorem for four single-valued and multivalued mappings:

Theorem 2.2. Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $I,J : X \to X$ be single valued and $F,G : X \to CB(X)$ be multivalued mappings such that the following conditions are satisfied:

1) $\cup F(X) \prec_1 J(X)$ and $\cup G(X) \prec_1 I(X)$,

2) $\{F,I\}$ and $\{G,J\}$ are weakly compatible,

3) If $x_n \to x$ is a strictly decreasing sequence in X, then $x_n \preccurlyeq x$, for all n,

4) $$\psi\left(d\left(Fx,Gy\right)\right) \le \alpha\psi\left(\max\left\{d\left(Ix,Jy\right), D\left(Ix,Fx\right), D\left(Jy,Gy\right), \frac{D\left(Ix,Gy\right)+D\left(Iy,Fx\right)}{2}\right\}\right),$$

for all comparable $x, y \in X$, $\operatorname{diam}Fx = \operatorname{diam}Gy$, where $0 < \alpha < 1$ and ψ is an Altering distance function and suppose that one of $I(X)$ or $J(X)$ is complete. Then there exists a unique point $p \in X$ such that

$$Fp = Gp = \{Ip\} = \{Jp\} = \{p\}$$

Proof: Let x_0 be an arbitrary point of X. By 1) we choose a point $x_1 \in X$ such that $y_1 = Fx_0 \prec_1 Jx_1$. For this point x_1, there exists a point $x_2 \in X$ such that $y_2 = Gx_1 \prec_1 Ix_2$, and so on. Continuing in this manner we can define a sequence $\{y_n\}$ as follows

$$\begin{aligned} y_{2n+1} &= Fx_{2n} \prec_1 Jx_{2n+1}, \\ y_{2n+2} &= Fx_{2n+1} \prec_1 Jx_{2n+2}, \end{aligned} \tag{2.1}$$

We claim that $\{y_n\}$ is a Cauchy sequence. For which two cases arise, either $y_n = y_{n+1}$ for some n, or $y_n \ne y_{n+1}$, for each n.

Case I. If $y_n = y_{n+1}$ for some n then, $y_n = y_{n+k}$ for each $k \ge 1$. For instance suppose $y_{2m+1} = y_{2m+2}$. Then $y_{2m+2} = y_{2m+3}$. Otherwise using 3), we get

$$\begin{aligned} &\psi\left(d\left(y_{2m+2}, y_{2m+3}\right)\right) = \psi\left(d\left(Fx_{2m+1}, Gx_{2m+2}\right)\right) \\ &\le \alpha\psi\left(\max\left\{d\left(Ix_{2m+1}, Jx_{2m+2}\right), D\left(Ix_{2m+1}, Fx_{2m+1}\right), D\left(Jx_{2m+2}, Gx_{2m+2}\right), \frac{D\left(Ix_{2m+1}, Gx_{2m+2}\right)+D\left(Ix_{2m+2}, Fx_{2m+1}\right)}{2}\right\}\right) \\ &\le \alpha\psi\left(\max\left\{d\left(y_{2m+1}, y_{2m+2}\right), d\left(y_{2m+1}, y_{2m+2}\right), d\left(y_{2m+2}, y_{2m+3}\right), \frac{d\left(y_{2m+1}, y_{2m+3}\right)+d\left(y_{2m+2}, y_{2m+2}\right)}{2}\right\}\right) \end{aligned}$$

Since

$$\frac{d\left(y_{2m+1} + y_{2m+3}\right)}{2} \le \max\left\{d\left(y_{2m+1}, y_{2m+2}\right), d\left(y_{2m+2}, y_{2m+3}\right)\right\}$$

It follows that

$$\psi\left(d\left(y_{2m+2},y_{2m+3}\right)\right)\le\alpha\psi\left(\max\left\{d\left(y_{2m+1},y_{2m+2}\right),d\left(y_{2m+2},y_{2m+3}\right)\right\}\right) \tag{2.2}$$

Suppose that if $d\left(y_{2m+1},y_{2m+2}\right)\le d\left(y_{2m+2},y_{2m+3}\right)$, for some positive integer n, then from (2.2), we have

$$\psi\left(d\left(y_{2m+2},y_{2m+3}\right)\right)\le\alpha\psi\left(d\left(y_{2m+2},y_{2m+3}\right)\right)$$

which implies that $d\left(y_{2m+2},y_{2m+3}\right)=0$, or $y_{2m+2}=y_{2m+3}$.

Hence $y_{2m+2}=y_{2m+3}$. Similarly $y_{2m+3}=y_{2m+4}$ implie $y_{2m+4}=y_{2m+5}$. Proceeding in this manner, it follows that $y_{2m+1}=y_{2m+k}$ for each $k>1$, so that $y_n=y_{n+k}$ for each $k\ge1$, for some n, and $\{y_n\}$ is a Cauchy sequence.

Case II. When $y_n\ne y_{n+1}$ for each n. In this case, using 3), we obtain

$$\begin{aligned}\psi\left(d\left(y_{2n+1},y_{2n+2}\right)\right)&=\psi\left(d\left(Fx_{2n},Gx_{2n+1}\right)\right)\\&\le\alpha\psi\left(\max\left\{d\left(Ix_{2n},Jx_{2n+1}\right),D\left(Ix_{2n},Fx_{2n}\right),D\left(Jx_{2n+1},Gx_{2n+1}\right),\frac{D\left(Ix_{2n},Gx_{2n+1}\right)+D\left(Ix_{2n+1},Fx_{2n}\right)}{2}\right\}\right)\\&\le\alpha\psi\left(\max\left\{d\left(y_{2n},y_{2n+1}\right),d\left(y_{2n},y_{2n+1}\right),d\left(y_{2n+1},y_{2n+2}\right),\frac{d\left(y_{2n+1},y_{2n+2}\right)+d\left(y_{2n+1},y_{2n+1}\right)}{2}\right\}\right),\end{aligned}$$

Since

$$\frac{d\left(y_{2n+1},y_{2n+2}\right)}{2}\le\max\left\{d\left(y_{2n},y_{2n+1}\right),d\left(y_{2n+1},y_{2n+2}\right)\right\},$$

It follows that

$$\psi\left(d\left(y_{2n+1},y_{2n+2}\right)\right)\le\alpha\psi\left(\max\left\{d\left(y_{2n},y_{2n+1}\right),d\left(y_{2n+1},y_{2n+2}\right)\right\}\right) \tag{2.3}$$

Now if $d\left(y_{2n},y_{2n+1}\right)\le d\left(y_{2n+1},y_{2n+2}\right)$, for each positive integer n, then from (2.3), we have

$$\psi\left(d\left(y_{2n+1},y_{2n+2}\right)\right)\le\alpha\psi d\left(y_{2n+1},y_{2n+2}\right)$$

which implies that $d\left(y_{2n+1},y_{2n+2}\right)=0$, or $y_{2n+1}=y_{2n+2}$, contradicting our assumption that $y_n\ne y_{n+1}$, for each n. Therefore $d\left(y_{2n+1},y_{2n+2}\right)\le d\left(y_{2n},y_{2n+1}\right)$, for all $n\ge0$ and $\left\{d\left(y_n,y_{n+1}\right)\right\}$ is strictly decreasing sequence of positive numbers and therefore tends to a limit $r\ge0$. If possible suppose $r>0$. Then for given $\eta>0$, there exists a positive integer N such that for each $n\in N$, we have

$$r\le d\left(y_n,y_{n+1}\right)<r+\eta, \tag{2.4}$$

Taking the limit $n\to\infty$, in (2.3) and using the continuity of ψ, we have or

$$\psi(r)\le\alpha\psi(r+\eta)\le\alpha\psi(r),$$

which is a contradiction unless $r=0$. Hence

$$\lim_{n\to\infty}d\left(y_n,y_{n+1}\right)=0. \tag{2.5}$$

Next we show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not, then there exists an $\varepsilon>0$ and since $\lim_{n\to\infty}d\left(y_n,y_{n+1}\right)=0$, there exists two sequences of positive numbers $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ such that for all positive integers k, $n(k)>m(k)>k$ and $d\left(y_{m(k)},y_{n(k)}\right)\ge\varepsilon$. Assuming that $n(k)$ is the smallest positive integer, we get $n(k)>m(k)>k$,

$$d\left(y_{m(k)},y_{n(k)}\right)\ge\varepsilon \text{ and } d\left(y_{m(k)},y_{n(k)-1}\right)<\varepsilon.$$

Now,

$$\varepsilon\le d\left(y_{m(k)},y_{n(k)}\right)\le d\left(y_{m(k)},y_{n(k)-1}\right)+d\left(y_{n(k)-1},y_{n(k)}\right)$$

i.e.

$$\varepsilon\le d\left(y_{m(k)},y_{n(k)}\right)\le\varepsilon+d\left(y_{n(k)-1},y_{n(k)}\right) \tag{2.6}$$

Taking the limit as $k\to\infty$ in (2.6) and using (2.5), we have

$$\lim_{k\to\infty}d\left(y_{m(k)},y_{n(k)}\right)=\varepsilon. \tag{2.7}$$

Again

$$d\left(y_{m(k)},y_{n(k)}\right)\le d\left(y_{mk},y_{m(k)+1}\right)+d\left(y_{m(k)+1},y_{n(k)+1}\right)+d\left(y_{n(k)+1},y_{n(k)}\right)$$

and

$$d\left(y_{m(k)+1},y_{n(k)+1}\right)\le d\left(y_{m(k)+1},y_{m(k)}\right)+d\left(y_{m(k)},y_{n(k)}\right)+d\left(y_{n(k)},y_{n(k)+1}\right)$$

Taking the limit as $k\to\infty$ and using (2.6) and (2.7), we have

$$\lim_{k\to\infty}d\left(y_{m(k)+1},y_{n(k)+1}\right)=\varepsilon. \tag{2.8}$$

Again we have

$$d\left(y_{m(k)}, y_{n(k)}\right) \le d\left(y_{m(k)}, y_{n(k)+1}\right) + d\left(y_{n(k)+1}, y_{n(k)}\right)$$

and

$$d\left(y_{m(k)}, y_{n(k)+1}\right) \le d\left(y_{m(k)}, y_{n(k)}\right) + d\left(y_{n(k)}, y_{n(k)+1}\right).$$

Letting $k \to \infty$ and using (2.6) and (2.7), we have

$$\lim_{k\to\infty} d\left(y_{m(k)}, y_{n(k)+1}\right) = \varepsilon \tag{2.9}$$

Similarly, we have $\lim_{k\to\infty} d\left(y_{n(k)}, y_{m(k)+1}\right) = \varepsilon$.

For each positive integer k, $y_{m(k)}$ and $y_{n(k)}$ are comparable. Now using the monotone property of ψ in 4), we have

$$\psi\left(d\left(y_{m(k)+1}, y_{n(k)+1}\right)\right) = \psi\left(d\left(Fy_{m(k)}, Gy_{n(k)}\right)\right)$$
$$\le \alpha\psi\left(\max\left\{d\left(Iy_{m(k)}, Jy_{n(k)}\right), D\left(Iy_{m(k)}, Fy_{m(k)}\right), D\left(Jy_{n(k)}, Gy_{n(k)}\right), \frac{D\left(Iy_{m(k)}, Gy_{n(k)}\right) + D\left(Iy_{n(k)}, Fy_{m(k)}\right)}{2}\right\}\right)$$
$$\le \alpha\psi\left(\max\left\{d\left(y_{m(k)}, y_{n(k)}\right), d\left(y_{m(k)}, y_{m(k)+1}\right), d\left(y_{n(k)}, y_{n(k)+1}\right) \frac{d\left(y_{m(k)}, y_{n(k)+1}\right) + d\left(y_{n(k)}, y_{m(k)+1}\right)}{2}\right\}\right)$$

Letting $k \to \infty$ and using (2.6)-(2.9), and the continuity of ψ, we have $\psi(r) \le \alpha\psi(r)$, which is a contradiction by virtue of property of ψ. Therefore $\{y_n\}$ and hence any subsequence thereof, is a Cauchy sequence.

Suppose $J(X)$ is complete. Since $\{y_{2n+1}\} = \{Jx_{2n+1}\}$ is a subsequence of $\{y_n\}$, by the above $\{Jx_{2n+1}\}$ is Cauchy and $Jx_{2n+1} \to p = Jv$, for some $v \in X$.

We now show $Ix_{2n+2} \to p$. For suppose $Ix_{2n+2} \to q$.

Since $y_{2n+1} = Jx_{2n+1} \to p$ and $y_{2n+2} = Ix_{2n+2} \to q$ therefore, $d\left(y_{2n+1}, y_{2n+2}\right) \to d(p,q)$. But $\left\{d\left(y_{2n+1}, y_{2n+2}\right)\right\}$ is a subsequence of the strictly decreasing sequence $\left\{d\left(y_n, y_{n+1}\right)\right\}$ which tends to the lim $r = 0$. Therefore $\left\{d\left(y_{2n+1}, y_{2n+2}\right)\right\}$ tends to limit $r = 0$ and hence $d(q,p) = 0$ implying $q = p$. Thus $Ix_{2n+2} \to p$. Now using (d), we have

$$\psi\left(d\left(Fx_{2n}, p\right)\right) = \psi\left(d\left(Fx_{2n}, Ix_{2n+2}\right)\right) = \psi\left(d\left(Fx_{2n}, Gx_{2n+1}\right)\right)$$
$$\le \alpha\psi\left(\max\left\{d\left(Ix_{2n}, Jx_{2n+1}\right), D\left(Ix_{2n}, Fx_{2n}\right), D\left(Jx_{2n+1}, Gx_{2n+1}\right), \frac{D\left(Ix_{2n}, Gx_{2n+1}\right) + D\left(Ix_{2n+1}, Fx_{2n}\right)}{2}\right\}\right)$$
$$\le \alpha\psi\left(\max\left\{d(q,p), d\left(p, Fx_{2n}\right), d(p,q), \frac{d(p,q) + d\left(p, Fx_{2n}\right)}{2}\right\}\right),$$

or

$$\psi\left(\delta\left(Fx_{2n}, p\right)\right) \le \psi\left(\delta\left(Fx_{2n}, p\right)\right),$$

which is a contradiction. Consequently $d\left(Fx_{2n}, p\right) \to 0$ as $n \to \infty$.

In the same manner, it follows that $d\left(Gx_{2n+1}, p\right) \to 0$ as $n \to \infty$. We now show $Gv = \{p\}$. For this, in view of $(d)'$, we have

$$\psi\left(\delta\left(Fx_{2n}, Gv\right)\right) \le \alpha\psi\left(\max\left\{d\left(Ix_{2n}, Jv\right), D\left(Ix_{2n}, Fx_{2n}\right), D(Jv, Gv), \frac{D\left(Ix_{2n}, Gv\right) + D\left(Iv, Fx_{2n}\right)}{2}\right\}\right)$$
$$\le \alpha\psi\left(\max\left\{d(Gv, p), d(p,p), d(p, Gv), \frac{d(p, Gv) + d(p,p)}{2}\right\}\right),$$

implies

$$\psi\left(d(p, Gv)\right) \le \psi d(Gv, p),$$

or

$$\psi\left(d(p, Gv)\right) \le \psi d(Gv, p),$$

which is a contradiction. Consequently, $d(Gv, p) \to 0$ as $n \to \infty$. Hence $Gv = \{p\} = \{Jv\}$. Since $G(X) \subset (X)$ there exists some $u \in X$ such that $Gv = \{Iu\}$. Hence $Gv = \{Jv\} = \{Iu\}$. We now show $Fu = \{Iu\}$. For this, first we prove $Iu \in Fu$. Suppose $Iu \notin Fu$ then $D(Iu, Fu) > 0$. Then in accordance with (d) such that

$$\psi\left(d\left(Fu,Iu\right)\right)=\psi\left(d\left(Fu,Gv\right)\right)\le\alpha\psi\left(\max\left\{d\left(Iu,Jv\right),D\left(Iu,Fu\right),D\left(Jv,Gv\right),\frac{D\left(Iu,Gv\right)+D\left(Iv,Fu\right)}{2}\right\}\right)$$
$$=\alpha\psi\left(\max\left\{0,D\left(Iu,Fu\right),0,\frac{0+D\left(Iv,Fu\right)}{2}\right\}\right).$$

implies $\psi\left(d\left(Fu,Iu\right)\right)\le\alpha\psi D\left(Iu,Fu\right)$, while $D\left(Fu,Iu\right)\le d\left(Fu,Iu\right)$. Therefore a contradiction arises. Hence $Iu\in Fu$. But then $M\left(u,v\right)=0$, which, by $\left(d\right)'$, implies $\operatorname{diam}\left(Fu\right)=\operatorname{diam}\left(Gv\right)=\operatorname{diam}\left\{Iu\right\}=0.$

Therefore Fu is a singleton. Since $Iu\in Fu$ and Fu is a singleton, $Fu=\left\{Iu\right\}$. Hence

$$Gv=\left\{Jv\right\}=Fu=\left\{Iu\right\}=\left\{p\right\}$$

Since the pair $\left\{F,I\right\}$ and $\left\{G,J\right\}$ are weakly compatible,

$$Fp=FIu=\left\{IFu\right\}=\left\{Ip\right\}$$

and

$$Gp=GJu=\left\{JGu\right\}=\left\{Jp\right\}.$$

From the above, it is clear that Fp and Gp are singletons and $d\left(Fp,Gp\right)=d\left(Ip,Jp\right)$.

We now show that $Ip=Jp$. For instance, suppose $Ip\ne Jp$ then from $\left(d\right)$, we have

$$\psi\left(d\left(Ip,Jp\right)\right)=\psi\left(d\left(Fp,Gp\right)\right)\le\alpha\psi\left(\max\left\{d\left(Ip,Jp\right),D\left(Ip,Fp\right),D\left(Jp,Gp\right),\frac{D\left(Ip,Gp\right)+D\left(Ip,Fp\right)}{2}\right\}\right)$$
$$=\alpha\psi\left(\max\left\{d\left(Ip,Jp\right),0,0,\frac{d\left(Ip,Jp\right)+0}{2}\right\}\right)$$

Implies as above $d\left(Ip,Jp\right)\to 0$ as $n\to\infty$. Hence $Ip=Jp$ and therefore $Fp=Gp=\left\{Ip\right\}=\left\{Jp\right\}.$

We now show $Fp=\left\{p\right\}$. For, suppose $Fp\ne\left\{p\right\}$. For this let $Fp\ne Gv$ in $\left(d\right)$, we have

$$\psi\left(d\left(Fp,Gv\right)\right)\le\alpha\psi\left(\max\left\{d\left(Ip,Jv\right),D\left(Ip,Fp\right),D\left(Jv,Gv\right)\frac{D\left(Ip,Gv\right)+D\left(Iv,Fp\right)}{2}\right\}\right)$$
$$=\alpha\psi\left(\max\left\{d\left(Fp,Gv\right),D\left(Fp,Fp\right),D\left(Gv,Gv\right),\frac{d\left(Fp,Gv\right)+d\left(Fv,Fp\right)}{2}\right\}\right),$$

or

$$\psi\left(d\left(Fp,Gv\right)\right)\le\alpha\psi d\left(Fp,Gv\right)\le\alpha\psi d\left(Fp,Gv\right),$$

which is a contradiction. Consequently $d\left(Fp,Gv\right)\to 0$ as $n\to\infty.$ Therefore $Fp=Gv=\left\{p\right\}$ and hence

$$Fp=Gp=\left\{Ip\right\}=\left\{Jp\right\}=\left\{p\right\}.$$

Let $q\in X$ be any point satisfying

$$Fq=Gq=\left\{Iq\right\}=\left\{Jq\right\}=\left\{q\right\}.$$

Suppose $q\ne p$ then from $\left(d\right)$, we have

$$\psi\left(d\left(p,q\right)\right)=\psi\left(d\left(Fp,Gq\right)\right)\le\alpha\psi\left(\max\left\{d\left(Ip,Jq\right),D\left(Ip,Fp\right),D\left(Jq,Gq\right),\frac{D\left(Ip,Gq\right)+D\left(Iq,Fp\right)}{2}\right\}\right),$$

in view of ψ, $d\left(p,q\right)\to 0$ as $n\to\infty.$ Hence $q=p$.

Corollary 2.1. Let I be a self mapping of a metric space $\left(X,d\right)$ and $f:X\to B\left(X\right)$ a set valued mapping satisfying

1)' $f\left(X\right)\subset\left(X\right)$,

2)' $\left\{f,I\right\}$ are weakly compatible,

3)' $\psi\left(d\left(fx,fy\right)\right)\le\alpha\psi\left(\max\left\{d\left(Ix,Iy\right),D\left(Ix,fx\right),D\left(Iy,fy\right),\frac{D\left(Ix,fy\right)+D\left(Iy,fx\right)}{2}\right\}\right)$,

for all comparable $x,y\in X$, where $0<\alpha<1$ and ψ is an altering distance function. If $I\left(X\right)$ is complete subspace of X, there exists a unique point $p\in X$ such that $fp=\left\{Ip\right\}=\left\{p\right\}$.

Proof: Taking $I = J$ and $F = G = f$ in Theorem 2.2.

Taking I = identity mapping in Corollary 2.1, we get the new corollary as follows:

Corollary 2.2. Let (X,d) be a complete metric space and $f: X \to B(X)$ a set valued mapping satisfying

$$\psi\left(d\left(fx, fy\right)\right) \le \alpha\psi\left(\max\left\{d\left(x,y\right), D\left(x,fx\right), D\left(y,fy\right), \frac{D\left(x,fy\right)+D\left(y,fx\right)}{2}\right\}\right)$$

Then f has a unique fixed point in X.

Proof. Obvious.

Corollary 2.3. Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $I,J: X \to X$ be single valued and $F,G: X \to CB(X)$ be multivalued mappings such that the following conditions are satisfied:

1)" $\bigcup F(X) \prec_1 J(X)$ and $\bigcup G(X) \prec_1 I(X)$,

2)" $\{F,I\}$ and $\{G,J\}$ are weakly compatible,

3)" if $x_n \to x$ is a strictly decreasing sequence in X, then $x_n \preccurlyeq x$, for all n,

4)" $$\psi\left(d\left(Fx,Gy\right)\right) \le \alpha\psi\left(\max\left\{d\left(Ix,Jy\right), D\left(Ix,Fx\right), D\left(Jy,Gy\right), \frac{D\left(Ix,Gy\right)+D\left(Ix,Fy\right)}{2}\right\}\right),$$

for all comparable $x,y \in X$, $\text{diam}Fx = \text{diam}Gy$, where $0 < \alpha < 1$ and ψ is an Altering distance function and suppose that one of $I(X)$ or $J(X)$ is complete. Then there exists a unique point $p \in X$ such that

$$Fp = Gp = \{Ip\} = \{Jp\} = \{p\}$$

Example 2.1. Let $X = \left\{(0,0), \left(0,-\frac{1}{3}\right), \left(-\frac{1}{6},0\right)\right\}$ be a sub set of R^2 with the order $\preccurlyeq$ defined as for

$$(x_1,y_1),(x_2,y_2) \in X, (x_1,y_1) \preccurlyeq (x_2,y_2)$$

if and only if $x_1 \le y_1, x_2 \le y_2$. Let $d: X \times X \to R$ be given as

$$d(x,y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

for $x = (x_1,y_1)$, $y = (x_2,y_2) \in X$.

The (X,d) is a complete metric space with the required properties of Theorem 2.2.

Let $I,J: X \to X$ and $F,G: X \to CBX$, be defined as follows:

$Ix = x$, $Jx = x/2$,

$$Fx = Gx = \begin{cases} \{(0,0)\}, & \text{if } x = (0,0), \\ \left\{(0,0), \left(-\frac{1}{6},0\right)\right\}, & \text{if } x = \left(0,-\frac{1}{3}\right), \\ \{(0,0)\}, & \text{if } x = \left(-\frac{1}{6},0\right), \end{cases}$$

Let $\psi: [0,\infty) \to [0,\infty)$ defined as $\psi(t) = t^2$, and $\alpha = 1/2$. Then all the conditions in the Theorem 2.2 satisfied. Without loss of generality, we assume that $x \preccurlyeq y$, we discuss the following cases.

1) If $x = \left(0,-\frac{1}{3}\right)$, $y = (0,0)$, then $\delta(Fx,Gy) = -\frac{1}{6}$ and

$$\max\left\{d\left(Ix,Jy\right), D\left(Ix,Fx\right), D\left(Jy,Gy\right), \frac{D\left(Ix,Gy\right)+D\left(Iy,Fx\right)}{2}\right\} = \frac{1}{3}$$

2) If $x = \left(-\frac{1}{6},0\right), y = (0,0)$, then $\delta(Fx,Gy) = 0$, and

$$\max\left\{d\left(Ix,Jy\right), D\left(Ix,Fx\right), D\left(Jy,Gy\right), \frac{D\left(Ix,Gy\right)+D\left(Iy,Fx\right)}{2}\right\} = \frac{1}{6}$$

3) If $x = (0,0), y = (0,0)$, then $\delta(Fx,Gy) = 0$, and

$$\max\left\{d\left(Ix,Jy\right), D\left(Ix,Fx\right), D\left(Jy,Gy\right), \frac{D\left(Ix,Gy\right)+D\left(Iy,Fx\right)}{2}\right\} = 0$$

4) If $x = \left(-\frac{1}{6},0\right), y = \left(-\frac{1}{6},0\right)$, then $\delta(Fx,Gy) = 0$, and

$$\max\left\{d\left(Ix,Jy\right),D\left(Ix,Fx\right),D\left(Jy,Gy\right),\frac{D\left(Ix,Gy\right)+D\left(Iy,Fx\right)}{2}\right\}=\frac{1}{6}$$

5) If $x=\left(0,-\frac{1}{3}\right),y=\left(0,-\frac{1}{3}\right)$, then $\delta\left(Fx,Gy\right)=-\frac{1}{6}$ and

$$\max\left\{d\left(Ix,Jy\right),D\left(Ix,Fx\right),D\left(Jy,Gy\right),\frac{D\left(Ix,Gy\right)+D\left(Iy,Fx\right)}{2}\right\}=\frac{1}{3}$$

In all above cases, it is clearly shown that

$$\psi\left(d\left(Fx,Gy\right)\right)\le\alpha\psi\left(\max\left\{d\left(Ix,Jy\right),D\left(Ix,Fx\right),D\left(Jy,Gy\right),\frac{D\left(Ix,Gy\right)+D\left(Iy,Fx\right)}{2}\right\}\right),$$

Hence the conditions of Theorem 2.2 are satisfied and shown that $\{(0,0)\}$ is a fixed point of *I*, *J*, *F*, and *G*.

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