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Global Static Solutions of the Spherically Symmetric Vlasov-Einstein-Maxwell (VEM) System for Low Charge

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ABSTRACT

We consider the VEM system in the context of spherical symmetry and we try to establish a global static solution with isotropic pressure that approaches Minkowski spacetime at infinity and have a regular center. To be in accordance with numerical investigation we take here low charge particles.

Keywords: The VEM System; Isotropic Pressure; Spherical Symmetry; Particle Energy; Angular Momentum; Lebesgue's Dominated Converge Theorem

1. Introduction

In [1], the authors established static spherically symmetric solutions for the Vlasov-Einstein (VE) system by expressing the distribution function f of identical particles (stars, galaxies) on phase space as a function of the local energy and the angular momentum. This technique has already been used by J. Batt in [2] to prove existence of the static symmetric solutions of the Vlasov-Poisson (VP) system. These works concern uncharged case. Here, we couple the VE system with the Maxwell system in which the electromagnetic field reduces to its electric part that is $E(x)=\varepsilon(r)\frac{x}{r}$, once the assumption of spherical symmetry and that of regularity are considered. We have to deal with the following equations:

$$\frac{v}{\sqrt{1+v^2}}\cdot\frac{\partial f}{\partial x}-\left(\mu'\sqrt{1+v^2}-q\mathrm{e}^{2\lambda}\varepsilon\right)\frac{x}{r}\cdot\frac{\partial f}{\partial v}=0 \tag{1}$$

$$\mathrm{e}^{-2\lambda}\left(2r\lambda'-1\right)+1=8\pi r^2\rho \tag{2}$$

$$\mathrm{e}^{-2\lambda}\left(2r\mu'+1\right)-1=8\pi r^2\rho, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\mathrm{e}^{\lambda}\varepsilon\right)=qr^2\mathrm{e}^{\lambda}M, \tag{4}$$

where, $x,v\in\mathbb{R}^3, r=|x|, \lambda'=\frac{\mathrm{d}\lambda}{\mathrm{d}r}$ and

$$\rho(x)=\rho(r)=\int_{\mathbb{R}^3} f(x,v)\sqrt{1+v^2}\mathrm{d}v+\frac{1}{2}\mathrm{e}^{2\lambda(x)}\varepsilon^2(x)$$

$$p(x)=p(r)=\int_{\mathbb{R}^3}\left(\frac{x\cdot v}{r}\right)^2 f(x,v)\frac{\mathrm{d}v}{\sqrt{1+v^2}}-\frac{1}{2}\mathrm{e}^{2\lambda(x)}\varepsilon^2(x)$$

$$M(x)=M(r)=\int_{\mathbb{R}^3} f(x,v)\mathrm{d}v.$$

In the above, (1) is the Vlasov equation, (2) and (3) are a part of the Einstein equations while (4) is a part of the Maxwell system. Notice that in the Vlasov equation we have adopted the Einstein summation convention that is $v\cdot\frac{\partial f}{\partial x}=v^i\frac{\partial f}{\partial x^i}=\sum_{i=1}^{3}v^i\frac{\partial f}{\partial x^i}$, q denotes the charge of particles, λ and μ denote the metric functions. Here f is spherically symmetric if $f(Ax,Av)=f(x,v)$, for x, $v\in\mathbb{R}^3$, $A\in SO(3)$. Our spacetime we are looking for is $\mathbb{R}^4$, endowed with the metric

$$\mathrm{d}s^2=-\mathrm{e}^{2\mu}\mathrm{d}t^2+\mathrm{e}^{2\lambda}\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right),$$

in which $t\in\mathbb{R}$, $r\geq 0$, $\theta\in[0,\pi]$ and $\varphi\in[0,2\pi]$. We are also looking for the asymptotically flat solutions with a regular center that allow us to prescribe the following boundary conditions:

$$\lim_{r\to+\infty}\lambda(r)=\lim_{r\to+\infty}\mu(r)=\lambda(0)=0.$$

Again for the regularity of ε, we will need the following additional boundary conditions:

$$\lim_{r\to+\infty}\varepsilon(r)=\varepsilon(0)=0.$$

We encourage the reader to obtain more details on how to establish the above equations in [3].

Next, in the related literature, the initial value problem for the corresponding time dependent is investigating in [3]. Again the Newtonian limit of the spherically symmetric VEM is discussed in [4] and this work extends the work that is done in [5]. Moreover, in [6] the authors prove the existence of a globally defined smooth static solution for the Einstein-Yang-Mills equations with $SU(2)$ gauge group. Also, global static solutions are established in [7] for the VP system. We also notice that stationary axially symmetric solutions have been found by G. Rein in [8] for the VP System. A construction by numerical means has been made by H. Andréasson in [9] for the spherically symmetric VEM system. In this paper numerical solutions are obtained only for the low charge particles and we try in what follows to obtain the same result by means of analytical arguments.

Now, why our problem is interesting? In the uncharged case, the authors reduce the EV system in a single non-linear integrodifferential equation in μ and with the monotonicity of sources terms ρ and p of the field equations, they extend the local solution to the global one. But with the contribution of the electric field, things seem to be more complicated, since none of these properties hold. So, we try to deduce the global solution for the local one using the same techniques that were developed in [10] when constructing solutions that satisfying the constraints for the spherically symmetric EVM system. We recall that this method is based on the ODE techniques, since the charge q of particle is taking as a parameter.

The present work proceeds as follows: in Section 2, considering f as function of two news variables E and L we write down the corresponding sources terms of the fields equations and then we obtain the reduced system. In Section 3, we try to prove the existence of solutions and we summarize this work in Section 4.

2. Conserved Quantities and Reduction of the Problem

We aim to express the full system as a nonlinear integro-differential system for λ, μ and ε. Now, the characteristic system that corresponds to the Vlasov equation reads

$$\dot{x}=\frac{v}{1+v^2},\ \dot{v}=-\left(\mu'\sqrt{1+v^2}-q\mathrm{e}^{2\lambda}\varepsilon\right)\frac{x}{r}\cdot$$

Next, the straightforward calculation shows that the following quantities

$$E:=\mathrm{e}^{\mu(r)}\sqrt{1+v^2}-q\int_0^r \mathrm{e}^{\mu+2\lambda}\mathrm{d}s;\ L:=r^2v^2-(x.v)^2;$$

are conserved along the characteristics. We recall that E is the particle energy [7] and $\sqrt{L}$ is the angular momentum. We now set $f(x,v)=\Phi(E,L)$, for a fixed function Φ. Then, f satisfies the Vlasov equation and we can write using the polar coordinates:

$$\rho(r)=\frac{2\pi}{r^2}\int_1^{+\infty}\int_0^{r^2(\tau^2-1)}\frac{\tau^2}{\sqrt{\tau^2-1-L/r^2}}\Phi(E,L)\mathrm{d}L\mathrm{d}\tau+\frac{1}{2}\mathrm{e}^{2\lambda(r)}\varepsilon^2(r)$$

$$p(r)=\frac{2\pi}{r^2}\int_1^{+\infty}\int_0^{r^2(\tau^2-1)}\Phi(E,L)\sqrt{\tau^2-1-L/r^2}\,\mathrm{d}L\mathrm{d}\tau-\frac{1}{2}\mathrm{e}^{2\lambda(r)}\varepsilon^2(r)$$

$$M(r)=\frac{2\pi}{r^2}=\frac{2\pi}{r^2}\int_1^{+\infty}\int_0^{r^2(\tau^2-1)}\frac{\tau}{\sqrt{\tau^2-1-L/r^2}}\Phi(E,L)\mathrm{d}L\mathrm{d}\tau$$

We are looking for solutions with an isotropic pressure, this means that pressure does not depend on the direction. So we take f in the form $f(x,v)=\Phi(E)$. Once again, f defines a solution of the Vlasov equation and we obtain:

$$\rho(r)=g_\Phi(\mu(r),\lambda(r),\varepsilon(r));$$

$$p(r)=h_\Phi(\mu(r),\lambda(r),\varepsilon(r));$$

$$M(r)=l_\Phi(\mu(r),\lambda(r),\varepsilon(r)),$$

where

$$g_\Phi(u,v,w):=4\pi\int_1^{+\infty}\tau^2\Phi(E)\sqrt{\tau^2-1}\mathrm{d}\tau+\frac{1}{2}\mathrm{e}^{2v}w^2 \quad (5)$$

$$h_\Phi(u,v,w):=\frac{4\pi}{3}\int_1^{+\infty}\Phi(E)\left(\tau^2-1\right)^{3/2}\mathrm{d}\tau-\frac{1}{2}\mathrm{e}^{2v}w^2 \quad (6)$$

$$l_\Phi(u,v,w):=4\pi\int_1^{+\infty}\tau\Phi(E)\sqrt{\tau^2-1}\mathrm{d}\tau. \quad (7)$$

Before continuing our investigation, we give details on how to establish for instance the expression given by (5). Once this is done the reader could applied the same method to establish (6) and (7). We will focus on the first term on the right hand side of $\rho(r)$, that is denoted by A. So in this expression we take $\Phi(E,L)=\Phi(E)$ and we can write:

$$A=\frac{2\pi}{r^2}\int_1^{+\infty}\tau^2\Phi(E)\mathrm{d}\tau\int_0^{r^2(\tau^2-1)}\frac{\mathrm{d}L}{\sqrt{\tau^2-1-L/r^2}}$$
$$=4\pi\int_1^{+\infty}\tau^2\Phi(E)\sqrt{\tau^2-1}\mathrm{d}\tau$$

where we have made the change of the variable: $L'=\tau^2-1-L/r^2$, and (5) is deduced. We also set $E=\tau\mathrm{e}^{\mu(r)}+j(\mu(r),\lambda(r),\varepsilon(r))$ with

$$j\left(\mu(r),\lambda(r),\varepsilon(r)\right):=-q\int_0^r \mathrm{e}^{\mu+2\lambda}\varepsilon \mathrm{d}s.$$

So, the VEM system reduces to the following equations:

$$\mathrm{e}^{-2\lambda}\left(2r\lambda'-1\right)+1=8\pi r^2 g_\Phi\left(\mu,\lambda,\varepsilon\right) \tag{8}$$

$$\mathrm{e}^{-2\lambda}\left(2r\mu'+1\right)-1=8\pi r^2 h_\Phi\left(\mu,\lambda,\varepsilon\right) \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\mathrm{e}^{\lambda}\varepsilon\right)=qr^2\mathrm{e}^{\lambda}l_\Phi\left(\mu,\lambda,\varepsilon\right). \tag{10}$$

The integration of (8) on $[0,r]$ with $\lambda(0)=0$, yields:

$$\mathrm{e}^{-2\lambda}=1-\frac{8\pi}{r}\int_0^r s^2 g_\Phi\left(\mu(s),\lambda(s),\varepsilon(s)\right)\mathrm{d}s, \tag{11}$$

and inserting this in (9), one has:

$$\mu'(r)=\frac{4\pi r h_\Phi\left(\mu(r),\lambda(r),\varepsilon(r)\right)}{1-\frac{8\pi}{r}\int_0^r s^2 g_\Phi\left(\mu(s),\lambda(s),\varepsilon(s)\right)\mathrm{d}s}+\frac{4\pi\int_0^r s^2 g_\Phi\left(\mu(s),\lambda(s),\varepsilon(s)\right)\mathrm{d}s}{r^2\left(1-\frac{8\pi}{r}\int_0^r s^2 g_\Phi\left(\mu(s),\lambda(s),\varepsilon(s)\right)\mathrm{d}s\right)}. \tag{12}$$

Next (10) yields by the integration on $[0,r]$:

$$\varepsilon(r)=\frac{q}{r^2}\mathrm{e}^{-\lambda}\int_0^r s^2 l_\Phi\left(\mu(s),\lambda(s),\varepsilon(s)\right)\mathrm{d}s. \tag{13}$$

In the sequel, we try to solve the reduced system (11)-(13) globally on $[0,+\infty[$.

3. Existence of Solutions

First of all we show that for a large class of Φ, the functions g_Φ, h_Φ and l_Φ are C^1. This will allow us to conclude that a solutions of our reduced system will be a regular one.

Lemma 3.1. *Let* $\Phi:]0,+\infty[\to]0,+\infty[$ *be a mesurable function with*

$$\Phi(E)\le C(1+E)^{-\alpha}, E>0,$$

for some constant $C>0$, *and* $\alpha>4$. *Then the sources given by system* (5)-(7) *belong to* $C^1\left(\mathbb{R}^3\right)$.

Proof: *It will be enough if we prove that* g_Φ, h_Φ and l_Φ *are* C^1. *Next using the decay property of* Φ, *these functions are well defined. Besides, with the help of the change of variables, one obtains*:

$$g_\Phi(u,v,w)=4\pi\mathrm{e}^{-4u}\tilde{g}_u\left(\mathrm{e}^u+j(u,v,w)\right)+\frac{1}{2}\mathrm{e}^{2v}w$$

$$h_\Phi(u,v,w)=\frac{4\pi}{3}\mathrm{e}^{-4u}\tilde{h}_u\left(\mathrm{e}^u+j(u,v,w)\right)-\frac{1}{2}\mathrm{e}^{2v}w^2$$

$$l_\Phi(u,v,w)=4\pi\mathrm{e}^{-3u}\tilde{l}_u\left(\mathrm{e}^u+j(u,v,w)\right)$$

where

$$\tilde{g}_u(t)=\int_t^{+\infty}\phi(E)\left(E+\mathrm{e}^u-t\right)^2\sqrt{\left(E+\mathrm{e}^u-t\right)^2-\mathrm{e}^{2u}}\,\mathrm{d}E$$

$$\tilde{h}_u(t)=\int_t^{+\infty}\Phi(E)\left[\left(E+\mathrm{e}^u-t\right)^2-\mathrm{e}^{2u}\right]^{3/2}\mathrm{d}E$$

$$\tilde{l}_u(t)=\int_t^{+\infty}\Phi(E)\left(E+\mathrm{e}^u-t\right)\sqrt{\left(E+\mathrm{e}^u-t\right)^2-\mathrm{e}^{2u}}\,\mathrm{d}E,$$

with $t>0$ *and* $j\in C^1\left(\mathbb{R}^3\right)$ *is deduced from the definition of j, replacing* μ, λ *and* ε *by* u, v *and* w *respectively*. *We now prove that the function* $\tilde{g}_u,\tilde{h}_u$ *and* $\tilde{l}_u$ *are* C^1 *on* $\mathbb{R}$ *and with this we can conclude that the same property holds for* g_Φ,h_Φ *and* l_Φ *on* $\mathbb{R}^3$. *Next for* $t>0$, $\Delta t>0$ *such that* $t-\Delta t>0$, *one has, for*

$$A:=\frac{1}{\Delta t}\left(\tilde{h}_u(t-\Delta t)-\tilde{h}_u(t)\right),$$

$$\begin{aligned}A=&\frac{1}{\Delta t}\int_{t-\Delta t}^{t}\Phi(E)\left[\left(E+\mathrm{e}^u-t+\Delta t\right)^2-\mathrm{e}^{2u}\right]^{3/2}\mathrm{d}E\\&+\int_t^{+\infty}\frac{\Phi(E)}{\Delta t}\left\{\left[\left(E+\mathrm{e}^u-t+\Delta t\right)^2-\mathrm{e}^{2u}\right]^{3/2}\right.\\&\left.-\left[\left(E+\mathrm{e}^u-t\right)^2-\mathrm{e}^{2u}\right]^{3/2}\right\}\mathrm{d}E:=I_1+I_2.\end{aligned}$$

Using the decay property of Φ *and the mean value theorem, one observes that* $\lim_{\Delta t\to 0^-}I_1=0$. *On the one hand, using Lebesgue's dominated convergence theorem, one concludes that* $\lim_{\Delta t\to 0^-}I_2$ *exists and the left derivative function of* $\tilde{h}_u$ *is in the form*:

$$\tilde{h}_u'(t)=3\int_t^{+\infty}\Phi(E)\left(E+\mathrm{e}^u-t\right)\left[\left(E+e^u-t\right)^2-\mathrm{e}^{2u}\right]^{1/2}\mathrm{d}E.$$

On the second hand, the same argument is valid for $\Delta t\to 0^+$ *and the corresponding right derivative function exists with its expression being the same as the one above. Thus* $\tilde{h}_u$ *is differentiable on* $\mathbb{R}$ *and using once again the Lebesgue dominated convergence theorem, its derivative is continuous. So this function is* C^1 *and one can proceed as above to obtain the same result for both functions* $\tilde{g}_u$ *and* $\tilde{l}_u$. *Next we state and prove the local existence of* λ, μ *and* ε:

Theorem 3.1. (*Local existence*) *Let* $\Phi:]0,+\infty[\to[0,+\infty[$ *be a* C^∞ *function with*

$$\Phi(E)\le C(1+E)^{-\alpha}, E>0,$$

for some constant $C>0$, $\alpha>4$ *and let* g_Φ, h_Φ *and* l_Φ *be defined by* (5)-(7). *Then for every* $\mu_0\in\mathbb{R}$, *there exists a number* $T>0$ *and a unique solution*,

$\lambda,\mu\in C^2([0,T])$, $\varepsilon\in C^1([0,T])$ *of system* (8)-(10) *with* $\lambda(0)=0$, $\mu(0)=0$ *and* $\varepsilon(0)=0$. *Moreover, the above solution depends regularly on the parameter q*.

Proof: *Let T be a function that is defined on some set* $\mathcal{C}$ *by*

$$T(\mu,\tilde\lambda,\varepsilon):=\left(T_1(\mu,\tilde\lambda,\varepsilon),T_2(\mu,\tilde\lambda,\varepsilon),T_3(\mu,\tilde\lambda,\varepsilon)\right)$$

where for $r\in[0,\delta]$,

$$T_1(\mu,\tilde\lambda,\varepsilon)(r)=\mu_0+\int_0^r\frac{4\pi s h_\Phi(\mu(s),\tilde\lambda(s),\varepsilon(s))}{1-\frac{8\pi}{s}\int_0^s\sigma^2 g_\Phi(\mu(\sigma),\tilde\lambda(\sigma),\varepsilon(\sigma))\mathrm{d}\sigma}\mathrm{d}s$$
$$+\int_0^r\frac{4\pi}{s^2\left(1-\frac{8\pi}{s}\int_0^s\sigma^2 g_\Phi(\mu(\sigma),\tilde\lambda(\sigma),\varepsilon(\sigma))\mathrm{d}\sigma\right)}\int_0^s\sigma^2 g_\Phi(\mu(\sigma),\tilde\lambda(\sigma),\varepsilon(\sigma))\mathrm{d}\sigma\mathrm{d}s$$

$$T_2(\mu,\tilde\lambda,\varepsilon)(r)=1-\frac{8\pi}{r}\int_0^r s^2 g_\Phi(\mu(s),\tilde\lambda(s),\varepsilon(s))\mathrm{d}s$$

$$T_3(\mu,\tilde\lambda,\varepsilon)(r)=\frac{q}{r^2}\left(\tilde\lambda(r)\right)^{1/2}\int_0^r s^2\left(\tilde\lambda(s)\right)^{-1/2}l_\Phi(\mu(s),\tilde\lambda(s),\varepsilon(s))\mathrm{d}s$$

with the closed set $\mathcal{C}$ *of the Banach space* $(L^\infty)^3$, *described by the set of functions* μ, $\tilde\lambda$ *and* ε *such that* $\mu(0)=\mu_0$, $|\mu(r)|\le|\mu_0|+1$, $\tilde\lambda(0)=1$, $\varepsilon(0)=0$, $|\varepsilon(r)|\le 1$, $\frac12\le\tilde\lambda(r)\le 2$, *with*

$$\frac{8\pi}{r}\int_0^r s^2 g_\Phi(\mu(s),\tilde\lambda(s),\varepsilon(s))\mathrm{d}s\le\frac12\qquad r\in[0,\delta].$$

In the above we have set $\tilde\lambda:=\mathrm{e}^{-2\lambda}$. *On* $(L^\infty)^3$, *we consider the norm* $\|(u,v,w)\|=\|u\|+\|v\|+\|w\|$. *Next, we deduce from the following inequalities*

$$\frac12\le T_2(\mu,\tilde\lambda,\varepsilon)(r)\le 1+8\pi C\delta^2,$$
$$\left|T_1(\mu,\tilde\lambda,\varepsilon)(r)\right|\le|\mu_0|+8\pi C\delta^2,$$
$$\left|T_3(\mu,\tilde\lambda,\varepsilon)(r)\right|\le C\delta,$$

with a constant $C=C(\|g_\Phi\|,\|h_\Phi\|,\|l_\Phi\|)$, *that one can choose* δ *small enough such a way that T is a function of* $\mathcal{C}$ *into itself. We now prove that T is a contraction mapping. To achieve this goal, we fix two elements* $(\mu,\tilde\lambda,\varepsilon)$ *and* $(\mu_1,\tilde\lambda_1,\varepsilon_1)$ *of* $\mathcal{C}$ *and we write*:

$$A_1(r):=T_1(\mu(r),\tilde\lambda(r),\varepsilon(r))-T_1(\mu_1(r),\tilde\lambda_1(r),\varepsilon_1(r))$$
$$=4\pi\int_0^r\left[\frac{1}{1-\frac{8\pi}{s}\int_0^s\sigma^2 g_\Phi(\mu,\tilde\lambda,\varepsilon)\mathrm{d}\sigma}-\frac{1}{1-\frac{8\pi}{s}\int_0^s\sigma^2 g_\Phi(\mu_1,\tilde\lambda_1,\varepsilon_1)\mathrm{d}\sigma}\right]\left(s h_\Phi(\mu,\tilde\lambda,\varepsilon)+\frac{1}{s^2}\int_0^s\sigma^2 g_\Phi(\mu,\tilde\lambda,\varepsilon)\mathrm{d}\sigma\right)\mathrm{d}s$$
$$+4\pi\int_0^r\frac{s\left(h_\Phi(\mu,\tilde\lambda,\varepsilon)-h_\Phi(\mu_1,\tilde\lambda_1,\varepsilon_1)\right)}{1-\frac{8\pi}{s}\int_0^s\sigma^2 g_\Phi(\mu,\tilde\lambda,\varepsilon)\mathrm{d}\sigma}\mathrm{d}s+4\pi\int_0^r\frac{1}{1-\frac{8\pi}{s}\int_0^s\sigma^2 g_\Phi(\mu,\tilde\lambda,\varepsilon)\mathrm{d}\sigma}\frac{1}{s^2}\int_0^s\sigma^2\left(g_\Phi(\mu,\tilde\lambda,\varepsilon)-g_\Phi(\mu_1,\tilde\lambda_1,\varepsilon_1)\right)\mathrm{d}\sigma\mathrm{d}s,$$

$$A_2(r)=T_2(\mu(r),\tilde\lambda(r),\varepsilon(r))-T_2(\mu_1(r),\tilde\lambda_1(r),\varepsilon_1(r))=\frac{-8\pi}{r}\int_0^r s^2\left(g_\Phi(\mu(s),\tilde\lambda(s),\varepsilon(s))-g_\Phi(\mu_1(s),\tilde\lambda_1(s),\varepsilon_1(s))\right)\mathrm{d}s$$

$$A_3(r):=T_3(\mu(r),\tilde\lambda(r),\varepsilon(r))-T_3(\mu_1(r),\tilde\lambda_1(r),\varepsilon_1(r))=\frac{q}{r^2}\left[\left(\tilde\lambda(r)\right)^{1/2}-\left(\tilde\lambda_1(r)\right)^{1/2}\right]\int_0^r s^2\left(\tilde\lambda(s)\right)^{-1/2}l_\Phi(\mu(s),\tilde\lambda(s),\varepsilon(s))\mathrm{d}s$$
$$+\frac{q}{r^2}\left(\tilde\lambda_1(r)\right)^{1/2}\int_0^r s^2\left[\left(\tilde\lambda(s)\right)^{-1/2}-\left(\tilde\lambda_1(s)\right)^{-1/2}\right]l_\Phi(\mu(s),\tilde\lambda(s),\varepsilon(s))\mathrm{d}s$$
$$+\frac{q}{r^2}\left(\tilde\lambda_1(r)\right)^{1/2}\int_0^r s^2\left(\tilde\lambda_1(r)\right)^{-1/2}\left(l_\Phi(\mu(s),\tilde\lambda(s),\varepsilon(s))-l_\Phi(\mu_1(s),\tilde\lambda_1(s),\varepsilon_1(s))\right)\mathrm{d}s,$$

and using the mean value theorem, one has the following estimates:

$$\left|\left(\tilde{\lambda}(r)\right)^{1/2}-\left(\tilde{\lambda}_1(r)\right)^{1/2}\right|\le\frac{1}{\sqrt{2}}\left|\tilde{\lambda}(r)-\tilde{\lambda}_1(r)\right|\le\frac{1}{\sqrt{2}}\left\|\tilde{\lambda}-\tilde{\lambda}_1\right\|_{L^\infty}$$

$$\left|\left(\tilde{\lambda}(r)\right)^{-1/2}-\left(\tilde{\lambda}_1(r)\right)^{-1/2}\right|\le\sqrt{2}\left\|\tilde{\lambda}-\tilde{\lambda}_1\right\|_{L^\infty},$$

$$\left|T_1\left(\mu(r),\tilde{\lambda}(r),\varepsilon(r)\right)-T_1\left(\mu_1(r),\tilde{\lambda}_1(r),\varepsilon_1(r)\right)\right|$$
$$\le C\delta^3\left\|\left(\mu-\mu_1,\tilde{\lambda}-\tilde{\lambda}_1,\varepsilon-\varepsilon_1\right)\right\|_{\left(L^\infty\right)^3},$$

$$\left|T_2\left(\mu(r),\tilde{\lambda}(r),\varepsilon(r)\right)-T_2\left(\mu_1(r),\tilde{\lambda}_1(r),\varepsilon_1(r)\right)\right|$$
$$\le C\delta^2\left\|\left(\mu-\mu_1,\tilde{\lambda}-\tilde{\lambda}_1,\varepsilon-\varepsilon_1\right)\right\|_{\left(L^\infty\right)^3},$$

$$\left|T_3\left(\mu(r),\tilde{\lambda}(r),\varepsilon(r)\right)-T_3\left(\mu_1(r),\tilde{\lambda}_1(r),\varepsilon_1(r)\right)\right|$$
$$\le C\delta\left\|\left(\mu-\mu_1,\tilde{\lambda}-\tilde{\lambda}_1,\varepsilon-\varepsilon_1\right)\right\|_{\left(L^\infty\right)^3},$$

So, using the above inequalities, one obtains:

$$\left|T\left(\mu,\tilde{\lambda},\varepsilon\right)-T\left(\mu_1,\tilde{\lambda}_1,\varepsilon_1\right)\right|_{\left(L^\infty\right)^3}$$
$$\le C\delta^3\left\|\left(\mu-\mu_1,\tilde{\lambda}-\tilde{\lambda}_1,\varepsilon-\varepsilon_1\right)\right\|_{\left(L^\infty\right)^3},$$

and thus, δ *is chosen small enough to force T to be a contraction mapping. Hence, we obtain a local solution* $\mu,\lambda,\varepsilon\in C^1\left(\left[0,\delta\right]\right)$ *of the system* (11)-(13) *that can be extended on the right maximal interval* $\left[0,R\right]$, *on which this solution is unique, since we are away from the center* $r=0$, *in which a singularity may occur. We also notice that the regularity of* μ, λ *and* ε *is deduced from that of* g_Φ, h_Φ *and* l_Φ. *So*, λ, $\mu\in C^2\left(\left]0,R\right[\right)$, $\varepsilon\in C^1\left(\left]0,R\right[\right)$, *with* $\mu'(0)=\lambda(0)=0$ *and* $\varepsilon'(0)$ *that exists. Next, to prove that our solution depends regularly on q, one can write* (8) *and* (10) *in the form*

$$r\frac{\mathrm{d}\Psi}{\mathrm{d}r}+2\Psi=rG\left(r,\Psi(r)\right) \tag{14}$$

where $\Psi=\begin{pmatrix}L\\ \varepsilon\end{pmatrix},\lambda(r)=rL(r)$, *and applying Theorem* 3.2 *of* [10] *to* (14) *to obtain the needed result, and the proof is complete*.

Remark 3.1. *If a solution* $\left(\lambda,\varepsilon\right)$ *of the system* (8)-(10) *is given, then* μ *will be determined via Equation* (9).

We now state the global existence theorem for our system:

Theorem 3.2. (*Global existence*) *Let* $\Phi:\left]0,+\infty\right[\to\left[0,+\infty\right[$ *be a* C^∞ *function that is compactly supported, with*

$$\Phi(E)\le C\left(1+E\right)^{-\alpha},E>0,$$

for a constant $C>0$ *and* $\alpha>4$. *Let* g_Φ *and* l_Φ *be defined by* (5) *and* (7) *respectively. Then, for q small, the system given by* (8) *and* (10) *has a unique global and regular solution* $\left(\lambda,\varepsilon\right)$ *defined on* $\left[0,+\infty\right[$ *that satisfies* $\lambda(0)=\varepsilon(0)=0$.

Proof: *We will follow the proof of Theorem* 3.3 *in* [10]. *Let* Φ, g_Φ and l_Φ *be as stated in Theorem* 3.2. *Using Theorem* 3.1, *the Equations* (8) *and* (10) *have a unique local regular solution in some interval* $\left[0,R\right]$, $R>0$. *Again, the O.D.E techniques allow us to ensure existence of a number* $\alpha>0$ *such that for* $q\in\left[-\alpha,\alpha\right]$, *R can be chosen uniformly and the solution on* $\left[0,R\right]$ *depends continuously on the parameter q. Now, for fixed* Φ *and q, the solution has a right maximal interval of existence*, $\left[0,R_*\right[$, $R_*=R_*\left(\Phi,q\right)$. *We have to prove that* $R_*=+\infty$. *First of all, one observes that for* $q=0$, *the second term on the right hand side of* (5) *vanishes as one can see by integrating* (10) *over*, $\left[0,r\right]$. *Thus, for* $q=0$, (8) *and* (10) *have a global solution that corresponds to the one of the VE system. Then, by the stability theorem for O.D.E, for every* $R>0$, *there exists* $\alpha>0$, *such that, for* $q\in\left[-\alpha,\alpha\right]$, *the system* (14) *has a solution* Ψ_α *that exists on* $\left[0,R\right]$. *Thus*, $R>R_*$. *Now, we choose R large so that* $R>E$, *with* $supp\Phi\subset\left[0,E\right]$. *So, if* E_0 *is the radius of the support of* Φ, *then R may be chosen to be bigger than* $\tilde{M}=m\left(R_0\right)+Q^2/\left(8\pi R_0\right)$, *with* $R_0>E_0$, *for all q in the interval* $\left[-\alpha,\alpha\right]$, *with*

$$Q:=4\pi q\int_0^{+\infty}s^2\mathrm{e}^{\lambda(s)}\mathrm{d}s\int_{\mathbb{R}^3}f\left(s,v\right)\mathrm{d}v$$
$$=16\pi q\int_0^{+\infty}s^2\mathrm{e}^{\lambda(s)}\mathrm{d}s\int_1^{+\infty}\tau\Phi\left(E\right)\sqrt{\tau^2-1}\mathrm{d}\tau$$

$$m(r)=4\pi\int_0^r s^2\rho(s)\mathrm{d}s,$$

where Q and m are respectively the total charge of the system and the mass function whose limit as $r\to+\infty$ *is M the ADM mass of the system. We deduce, as it is the case in the proof of Theorem* 3.3 *in* [10], *that the exterior region* $R>2\tilde{M}$ *can be filled by the Reisner Nordström solution that extends our solution to the globale one. Thus the proof of Theorem* 3.2 *is complete*.

Remark 3.2. *In the isotropic case* (*i.e.* $f\left(x,v\right)=\Phi\left(E\right)$), *the regularity of f depends on that of* Φ. *So, for instance if* Φ *is a* C^1 *function, then f will be a* C^1 *one too. Thus*, $\left(\Phi\left(E\right),\lambda,\mu,\varepsilon\right)$ *is a regular solution of the full EVM system.*

4. Conclusion

Our goal in this work was to look for a global static solutions for the spherically symmetric EVM system. To achieve this, a first step has consisted of establishing in Theorem 3.1 a local existence of solutions, using the contraction mapping theorem on a complete metric space. We have also prove in Theorem 3.2 that these local solutions can be extended to the global ones, if the assumption of compactness is added to the decay property of the distribution function Φ. We obtain as it is the case for the uncharged particles that our spacetime is asymptotically flat, since the exterior region is filled by the Reisner Norsdtröm solution. One can also prove that this spacetime is geodesically complete.

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