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The Zeros of a Certain Homogeneous Difference Polynomials of Meromorphic Functions*

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ABSTRACT

Let $f(z)$ be a function transcendental and meromorphic in the plane of growth order less than 1. This paper focuses on discussing and estimating the number of the zeros of a certain homogeneous difference polynomials of degree *k* in $f(z)$, and obtains that this certain homogeneous difference polynomials has infinitely many zeros.

Keywords: Meromorphic Functions; Zeros; Homogeneous Difference Polynomials

1. Introduction and the Main Result

Let $f(z)$ be a function transcendental and meromorphic in the plane. In what follow, we denote the convergence exponent of the zeros of $f(z)$ by $\lambda(f)$, the growth order of $f(z)$ by $\sigma(f)$, and the lower order of $f(z)$ by $\mu(f)$.

Following Whittaker [1], define the forward differences to be k times iteration Δ^k of the difference operator Δ, that is,

$$\Delta f(z) = f(z+1) - f(z),$$
$$\Delta^k f(z) = \Delta^{k-1} f(z+1) - \Delta^{k-1} f(z). \tag{1.1}$$

Recently, a number of papers research on complex difference equations and differences analogues of Nevanlinna's theory [2-6]. Bergweiler and Langley [7] firstly investigated the existence of zeros of $\Delta f(z)$, and obtained a result as follow.

Theorem 1.1. Let f be a function transcendental and meromorphic of lower order $\mu(f) < \mu < 1$ in the plane. Let $c \in C \setminus \{0\}$ be such that at most finitely many poles z_j, z_k of $f(z)$ satisfy $z_j - z_k = c$. Then $g(z) = f(z+c) - f(z)$ has infinitely many zeros.

In 2008, Z. X. Chen and K. H. Shon [8].

Theorem 1.2. Let $n \in N$ and f be a function transcendental and meromorphic of lower order $\mu(f) < \mu < 1$ in the plane. Let $c \in C \setminus \{0\}$ and a set $B = \{bj\}$ consist of all poles of $f(z)$, such that

$$b_j + kc \notin B(k = 1, 2, \cdots, n)$$

at most except finitely many exceptions. Then $\Delta^n f(z)$ has infinitely many zeros.

In 2009, Z. X. Chen and K. H. Shon [9] continue to investigate the existence of the zeros of the difference polynomials defined as follows

$$g(z) = f(z+c_1) + f(z+c_2) - 2f(z) \tag{1.2}$$

$$g_2(z) = f(z+c_1) \cdot f(z+c_2) - f^2(z) \tag{1.3}$$

and obtained two results.

Theorem 1.3. Let f be a function transcendental and meromorphic of growth order $\sigma(f) = \sigma < 1$, and c_1, c_2 be two complex numbers, such that $c_1, c_2 \in C \setminus \{0\}$, and $c_1 + c_2 \neq 0$. If $f(z)$ has at most finitely many poles p_j, p_s satisfying $p_j - p_s = k_1 c_1 + k_2 c_2$ $(k_d = 0, \pm 1, d = 1, 2)$, then $g(z)$ has infinitely many zeros, and $\lambda(g) = \sigma(g) = \sigma$.

In particular, if $f(z)$ has at most finitely many zeros z_j satisfying $f(z_j + c_1) + f(z_j + c_2) = 0$, then $G(z) = g(z)/f(z)$ has also infinitely many zeros, and $\lambda(G) = \sigma(G) = \sigma$.

Theorem 1.4. Let $f(z), c_1, c_2$ satisfy the conditions in Theorem 1.3, If $f(z)$ has at most finitely many poles b_j satisfying

$$f(b_j + k_1 c_1 + k_2 c_2) = 0, \infty (k_d = 0, \pm 1, d = 1, 2),$$

then $g_2(z)$ has infinitely many zeros, and $\lambda(g_2) = \sigma(g_2) = \sigma$.

In particular, if $f(z)$ has at most finitely many zeros z_j, z_s such that $z_j - z_s = c_1, c_2$, then $G_2(z) = g_2(z)/f^2(z)$ has also infinitely many zeros, and

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$$\lambda(G_2)=\sigma(G_2)=\sigma .$$

It is not difficult to understand that $g(z)$ defined by (1.2) is more general difference polynomials than $\Delta f(z)$ or $\Delta^2 f(z)$ and Theorem 1.3 extends Theorem 1.1. Therefore, we pose naturally one question whether more general difference polynomials than $g_2(z)$ defined by (1.3) has also infinitely many zeros. In this paper, we focus on research a certain homogeneous difference polynomials and affirm to answer this problem.

Theorem 1.5. Suppose that k is a positive integer, $k\geq 1$. Let $f(z)$ be a function transcendental and meromorphic of growth order $\sigma(f)=\sigma<1$, and there exists k complex numbers $c_j\in C\setminus\{0\}$ $j=1,2,\cdots,k$ such that $\sum_{j=1}^{k}c_j\neq 0$. If $f(z)$ has at most finitely many poles b_j satisfying

$$f\left(b_j+l_1c_1+l_2c_2+\cdots+l_kc_k\right)=0,\infty,$$
$$l_d=0,\pm 1, d=1,2,\cdots,k.$$

Then $H_k(f)=\prod_{i=1}^{k}f(z+c_i)-f^k(z)$ has infinitely many zeros, and $\lambda(H_k)=\sigma(H_k)=\sigma$.

In particular, if $f(z)$ has at most finitely many zeros z_j, z_s satisfying $z_j-z_s=c_1,c_2,\cdots,c_k$, then $\psi_k(f)=H_k(f)/f^k(z)$ has also infinitely many zeros, and $\lambda(\psi_k)=\sigma(\psi_k)=\sigma$.

2. Lemmas

Lemma 2.1. (see [7]) Let f be a function transcendental and meromorphic in the plane of growth order less than 1, and $h>0$. Then there exists an $\varepsilon-$set E such that

$$f(z+c)-f(z)=cf'(z)(1+o(1)), \tag{2.1}$$

as $z\to\infty, z\in C\setminus E$, uniformly in c for $|c|\leq h$.

Lemma 2.2. (see [7]) Let $f(z)$ be a function transcendental and meromorphic in the plane of lower order $\mu(f)=\mu<1$. Then there exists arbitrarily large R with the following properties. First,

$$T(32R, f')<R^{\mu}. \tag{2.2}$$

Second, there exists a set $J_R\subseteq[R/2,R]$ of linear measure $m(J_R)=\int_{J_R}\frac{\mathrm{d}r}{R-r}=[1-o(1)]R/2$, such that for $r\in J_R$,

$$f(z+1)-f(z)\sim f'(z) \tag{2.3}$$

on $|z|=r$.

Lemma 2.3. Let $f(z)$ be a function transcendental and meromorphic in the plane with growth order $\sigma(f)=\sigma<1$. Supposed that $\sum_{j=1}^{k}c_j\neq 0$. If the homogeneous difference polynomials

$$H_k(f)=\prod_{j=1}^{k}f(z+c_j)-f^k(z),$$

or quotient of difference polynomials

$$\psi_k(f)=H_k(f)/f^k(z)$$

is rational functions, then $f(z)$ has at most finite many poles.

Proof. Without loss of generality, we assume that c_1 = 1. Because that the homogeneous difference polynomials $H_k(f)$ is rational, there exists a rational functions $R(z)$ such that

$$H_k(f)=\prod_{j=1}^{k}f(z+c_j)-f^k(z)=R(z). \tag{2.4}$$

Set

$$B=\left\{bj=xj+iyj \mid R(bj)=\infty,\ j=1,2,\cdots,s\right\},$$

and

$$M=\max\left\{|xj|+|yj|+1+\sum_{j=1}^{k}|c_j|:1\leq j\leq s\right\}.$$

So there exists no poles of $R(z)$ in the domain

$$D_1=\{z:\operatorname{Re}z>M\},$$
$$D_2=\{z:\operatorname{Re}z<-M\},$$
$$D_3=\{z:\operatorname{Im}z>M\}$$

and

$$D_4=\{z:\operatorname{Im}z<-M\}.$$

Now we complete the proof of the conclusion that $f(z)$ has at most finite.

Now we complete the proof of the conclusion that $f(z)$ has at most finite many poles. Suppose not, there exists one domain D_j, for example D_1, in which $f(z)$ has infinitely many poles. We assume that the set $A=\{z_j\}$ consists of all poles of $f(z)$ in D_1 and $M<|z_1|\leq|z_2|\leq\cdots$ and divide it into two cases:

Case 1. There exists $z_d\in A$, such that for an arbitrary $b_j\in B$, there does not exist $m_1,m_2,\cdots,m_k\in N$ such that $b_j=z_d+m_1+\sum_{r=2}^{k}m_rc_r$, that is, for an arbitrary $m_1,m_2,\cdots,m_k\in N$, we have $b_j\neq z_d+m_1+\sum_{r=2}^{k}m_rc_r$. In fact, since $\operatorname{Re}b_j<M$ and $\operatorname{Re}z_d>M$, this case appears whenever $\operatorname{Re}c_j>0$ for every $i=2,3,\cdots,k$. Therefore, we know $R\left(z_d+m_1+\sum_{r=2}^{k}m_rc_r\right)\neq\infty$ and that there exists a unbounded subsequence set $A_1=\{z_d+m_1+m_2c_2+\cdots+m_kc_k\}$, in which every

$z_d + m_1 + \sum_{r=2}^{k} m_r c_r$ is the poles of $f(z)$. Hence we know that there are at least one in these signs $m_1, m_2, \cdots, m_k$, which takes every positive integer, for instance, m_1 takes every positive integer.

Thus, $\lambda(f) = 1$, which contradicts the hypothesis of Lemma 2.3.

Case 2. There exists $b_0 \in B$, such that for every $z_j \in A$, there exists $m_{j1}, m_{j2}, \cdots, m_{jk} \in N$, such that $b_0 = z_j + m_{j1} + m_{j2}c_2 + \cdots + m_{jk}c_k$. From $\operatorname{Re}(z_j) > M$ and $\operatorname{Re}(b_0) < M$, we have that $\sum_{r=2}^{k} m_{jr} \operatorname{Re} c_r < 0$. As the set A is infinite and B has only a finite elementary, there exists $b_0 \in B$, satisfying

$$b_0 = z_1 + m_{11} + \sum_{r=2}^{k} m_{1r}c_r = z_2 + m_{21} + \sum_{r=2}^{k} m_{2r}c_r = \cdots \quad (2.5)$$

By putting $\left\{ z_j + m_{j1} + \sum_{r=2}^{k} m_{jr}c_r \right\}$ in order again, we have the following express

$$m_{jl} \le m_{j+1,l}, l = 1,2,\cdots,k;\ j = 1,2,\cdots,$$

and

$$z_{j+1} = z_j + \left(m_{j1} - m_{j+1,1}\right) + \sum_{r=2}^{k}\left(m_{jr} - m_{j+1,r}\right)c_r,$$
$$j = 1,2,\cdots,$$

where

$$0 \ge m_{1r} - m_{jr} \ge m_{1r} - m_{j+1,r}, r = 1,2,\cdots,k;\ j = 1,2,\cdots,.$$

Now set

$$z_{3ij\cdots s} = z_1 + \left(m_{11} - m_{31} + i\right) + \left(m_{12} - m_{32} + j\right)c_2 + \cdots + \left(m_{1k} - m_{3k} + s\right)c_k,$$

where

$$i = 0,1,2,\cdots,\left(m_{11} - m_{21}\right) - \left(m_{11} - m_{31}\right)$$
$$j = 0,1,2,\cdots,\left(m_{12} - m_{22}\right) - \left(m_{12} - m_{32}\right)$$
$$\vdots$$
$$s = 0,1,2,\cdots,\left(m_{1k} - m_{2k}\right) - \left(m_{1k} - m_{3k}\right).$$

Since $\operatorname{Re}\left(z_{3ij\cdots s}\right)$ are between $\operatorname{Re}(z_2)$ and $\operatorname{Re}(z_3)$, $\operatorname{Im}\left(z_{3ij\cdots s}\right)$ are between $\operatorname{Im}(z_2)$ and $\operatorname{Im}(z_3)$, we know that $z_{3ij\cdots s} \in D_1$, that is, $R\left(z_{3ij\cdots s}\right) \ne \infty$. From $f(z_3) = \infty$, $R\left(z_{3ij\cdots s}\right) \ne \infty$, and (2.4), we know that one of $z_{310\cdots0}, z_{301\cdots0}, \cdots$, and $z_{310\cdots0}$ is the pole of $f(z)$. If $z_{310\cdots0}$ is the pole of $f(z)$, then from the some argument above we have one of $z_{320\cdots0}, z_{311\cdots0}, \cdots$, and $z_{310\cdots1}$ is also the pole of $f(z)$. If $z_{301\cdots0}$ is the pole of $f(z)$, then one of $z_{311\cdots0}, z_{302\cdots0}, \cdots$, and $z_{301\cdots1}$ is also the pole of $f(z)$. On the analogy of this, it is not difficult to find there exists at least one of $i, j, \cdots, s$, for instance, we assume that is *j*, such that *j* takes all value of $0,1,2,\cdots,\left(m_{12} - m_{22}\right) - \left(m_{12} - m_{32}\right)$. From z_4 to z_3, z_5 to z_4, and z_n to z_{n-1}, repeating above proceeding, we have

$$z_{nij\cdots s} = z_{n-1} + \left(m_{n-1,1} - m_{n1} + i\right) + \left(m_{n-1,r} - m_{nr}\right)c_r + \cdots + \left(m_{n-1k} - m_{nk}\right)c_k, n = 2,3,\cdots,$$

where

$$i = 0,1,2,\cdots,m_{n-1,1} - m_{n1}$$
$$j = 0,1,2,\cdots,m_{n-1,2} - m_{n2}$$
$$\vdots$$
$$s = 0,1,2,\cdots,m_{n-1,k} - m_{nk}.$$

Therefore, we can see that there exist infinite many poles of $f(z)$ whose expressions are as follows

$$z_{nij\cdots s} = z_1 + \left(m_{11} - m_{n1} + i\right) + \left(m_{1r} - m_{nr} + j\right)c_r + \cdots + \left(m_{1k} - m_{nk} + s\right)c_k, n = 2,3,\cdots,$$

where

$$i = 0,1,2,\cdots,m_{n,1} - m_{11}$$
$$j = 0,1,2,\cdots,m_{n,2} - m_{12}$$
$$\vdots$$
$$s = 0,1,2,\cdots,m_{n,k} - m_{1k}.$$

in which we can find that one of $i, j, \cdots, s$ takes every positive integer. Thus, $\lambda(f)$, which still contradict the hypothesis on the growth order of $f(z)$ in Lemma 2.3.

By the similar method to above, it is easy to prove that $f(z)$ has at most finite many poles whenever quotient of difference polynomials

$$\psi_k(f) = H_k(f) / f^k(z)$$

is rational functions. □

Lemma 2.4. Let $f(z)$ be a function transcendental and meromorphic in the plane with growth order $\sigma(f) = \sigma < 1$. Supposed that $\sum_{j=1}^{k} c_j \ne 0$, then the homogeneous difference polynomials

$$H_k(f) = \prod_{j=1}^{k} f\left(z + c_j\right) - f^k(z)$$

and

$$H_k(f) / f^k(z)$$

also are transcendental.

Proof. Suppose first that there exists a rational function $R(z)$, such that

$$H_k(f) = \prod_{j=1}^{k} f\left(z + c_j\right) - f^k(z) = R(z) \quad (2.6)$$

By Lemma 2.3, $f(z)$ has at most finite many poles.

Again from Lemma 2.1, there exists $\varepsilon-\text{set}E$ such that as $z\to\infty\left(z\in C\setminus E\right)$, we have

$$f\left(z+c_j\right)-f\left(z\right)=c_j f'(z)\left(1+o(1)\right),\ j=1,2,\cdots,k \quad (2.7)$$

It follows that from (2.6) and (2.7)

$$\begin{aligned} f'(z)\Big\{ & c_1c_2\cdots c_k\left(f'(z)\right)^{k-1}\left(1+o(1)\right) \\ +\cdots+ & \sum_{1\le j_1\le\cdots\le j_m\le k} c_{j_1}c_{j_2}\cdots c_{j_m}\left(f'(z)\right)^{m-1}\left(f(z)\right)^{k-m}\left(1+o(1)\right) \\ +\cdots+ & \sum_{1\le j\le k} c_j\left(f(z)\right)^{k-1}\left(1+o(1)\right)\Big\}=R(z) \end{aligned} \quad (2.8)$$

We write $d(z)$ for a polynomial formed by the pole of $f(z)$, and $f_0(z)=f(z)d(z)$. So $f_0(z)$ is an entire function, and $\sigma(f_0)=\sigma(f)=\sigma<1$. With the standard result in the Wiman-Valiron Theory, we know that there exists a subset $F\subset(1,+\infty)$ with finite logarithmic measure $\int_F\frac{\mathrm{d}r}{r}<+\infty$, in which for an sufficiently large $r\notin F, \left|f_0(z)\right|=M(r,f_0), |z|=r,$ the following equality holds

$$\frac{f_0'(z)}{f_0(z)}=\frac{\upsilon(r)}{z}\left(1+o(1)\right).$$

Thus,

$$\frac{f'(z)}{f(z)}=\frac{f_0'(z)}{f_0(z)}-\frac{d'(z)}{d(z)}=\frac{\upsilon(r)}{z}\left(1+o(1)\right) \quad (2.9)$$

where $\upsilon(r)/z\to\infty$, and $\upsilon(r)\to\infty$, as $z\to\infty$. Set $F_1=\left\{|z|:z\in E\right\}$. Since E is $\varepsilon-\text{set}$, we have that F_1 also is of finite logarithmic measure. Therefore, for all z, $|z|\notin[0,1]\cup F\cup F_1$, and

$$\left|f_0(z)\right|=M(r,f_0),$$

we immediately deduce that from (2.8) and (2.9)

$$\begin{aligned} &\sum_{m=1}^{k}\sum_{1\le j_1\le\cdots\le j_m\le k}\left(\prod_{t=1}^{m}c_{j_t}\right)\left(\frac{\upsilon(r)}{z}\right)^{m-1}\left(1+o(1)\right) \\ &=\frac{R(z)d^k z}{M^k(r,f_0)}\cdot\frac{1}{\upsilon(r)}\left(1+o(1)\right) \end{aligned} \quad (2.10)$$

Since $\sigma(f_0)=\sigma<1$ and f_0 is transcendental, there exists a sequence $\left|r\notin[0,1]\cup F\cup F_1 r_n\to\infty\right.$, such that for arbitrary $\varepsilon>0$, we have that

$$\exp\left(kr_n^{\sigma-\varepsilon}\right)<M\left(r_n,f_0\right)^k<\exp\left(kr_n^{\sigma+\varepsilon}\right) \quad (2.11)$$

$$\upsilon(r)/z\to0, \upsilon(r)\to\infty\left(z\to\infty\right). \quad (2.12)$$

Then, we induce that from (2.4) and (2.11)

$$\frac{r_n^{\tau}\left(1+o(1)\right)}{\exp\left(kr_n^{\sigma+\varepsilon}\right)}<\frac{\left|R(z)d(z)^k z\right|}{M\left(r_n,f_0\right)^k}<\frac{r_n^{\tau}\left(1+o(1)\right)}{\exp\left(kr_n^{\sigma-\varepsilon}\right)}. \quad (2.13)$$

Therefore, from (2.12) and (2.13) we have

$$\frac{R(z)d^k z}{M^k(r,f_0)}\to0\left(r_n\to\infty\right) \quad (2.14)$$

By (2.10), (2.12), and (2.14), we deduce easily that $c_1+c_2+\cdots+c_k=0$, which contradicts the assumption on $c_1+c_2+\cdots+c_k\ne0$, that is, $H_k\left(f(z)\right)$ transcendental.

Lemma 2.5. Let $f(z)$ be a function transcendental and meromorphic in the plane, whose growth orde $\sigma(f)=\sigma<1$. Supposed that $a_1,a_2,\cdots,a_k\in C\setminus\{0\}$, and $\overline{\lambda}(1/f)=\lambda(1/f)$. Then

$$\max\left\{\lambda(f'),\lambda(a_1\left(f'\right)^{k-1}+a_2\left(f'\right)^{k-2}f+\cdots+a_k f^{k-1}\right\}=\sigma$$

Proof. For $f(z)$ of growth order $\sigma(f)=\sigma<1$, from Hadamard's factorization theorem we have

$$\begin{aligned} f(z)&=p(z)q(z), \\ f'(z)&=p_1(z)q_1(z) \end{aligned}, \quad (2.15)$$

where $p(z)\left(p_1(z)\right)$ and $q(z)\left(q_1(z)\right)$ are respectively the canonical product of zeros and poles of $f(z)\left(f'(z)\right)$, satisfying

$$\left(p(z),q(z)\right)=1\left(\left(p_1(z),q_1(z)\right)=1\right).$$

From (2.15), we have

$$\begin{aligned} \sigma(f')&=\max\left(\sigma\left(p_1(z)\right),\sigma\left(q_1(z)\right)\right) \\ &=\max\left(\lambda\left(f'(z)\right),\lambda\left(1/f'(z)\right)\right)=\sigma. \end{aligned}$$

Therefore, if $\lambda(f')<\sigma$, we deduce that $\lambda(1/f')=\sigma$. For $\overline{\lambda}(1/f)=\lambda(1/f)$, the following equations hold

$$\lambda(1/f')=\lambda(1/f)=\overline{\lambda}(1/f)=\lambda(f)=\sigma \quad (2.16)$$

We have that from $\lambda(f')<\sigma$ and (2.16)

$$\begin{aligned} \sigma(p)&=\sigma(p_1)=\sigma(f), \\ \sigma(q)&=\sigma(q_1)<\sigma(f) \end{aligned}. \quad (2.17)$$

If z_0 is a poles of $f(z)$ with multiplicity m, then z_0 must be a poles of $f'(z)$ with multiplicity $m+1$, so that we denote $q_1(z)$ by $q(z)d(z)$, that is,

$$q_1(z)=q(z)d(z), \quad (2.18)$$

where $d(z)$ is a canonical product of distinct poles of $d(z)$. By (2.16), we obtain that

$$\sigma(d)=\lambda(d)=\overline{\lambda}(f)=\sigma(f)=\sigma. \quad (2.19)$$

From (2.15) and (2.18), we deduce that

$$a_1(f')^{k-1}+a_2(f')^{k-2}f^1+\cdots+a_kf^{k-1}=\frac{a_1(p_1)^{k-1}+a_2(p_1)^{k-2}(pd)^1+\cdots+a_k(pd)^{k-1}}{q_1^{k-1}}. \quad (2.20)$$

Thus, if z_0 is the pole of $f'(z)$ (that is, $q_1(z_0)=0$), then $d(z_0)=0$, $p(z_0)\neq 0,\infty$, but $p_1(z_0)\neq 0,\infty$. Hence, we have that z_0 is not the zero of

$$a_1(f')^{k-1}+a_2(f')^{k-2}f+\cdots+a_kf^{k-1}.$$

So that

$$\left(a_1(p_1)^{k-1}+a_2(p_1)^{k-2}(pd)^1+\cdots+a_k(pd)^{k-1},q_1^{k-1}\right)=1,$$

and

$$\begin{aligned}&\lambda\left(a_1(f')^{k-1}+a_2(f')^{k-2}f+\cdots+a_kf^{k-1}\right)\\&=\lambda\left(a_1(p_1)^{k-1}+a_2(p_1)^{k-2}(pd)^1+\cdots+a_k(pd)^{k-1}\right)\\&=\sigma\left(a_1(p_1)^{k-1}+a_2(p_1)^{k-2}(pd)^1+\cdots+a_k(pd)^{k-1}\right)\\&\geq\sigma(d)=\sigma(f).\end{aligned}$$

This completes the proof of Lemma 2.5. □

3. Proofs of Theorem 1.5

From Lemma 2.2 we see that there exists a sufficiently large R, a positive number $\sigma_1(\sigma<\sigma_1<1)$ such that

$$T(32R,f')<R^{\sigma_1} \quad (2.21)$$

and there exists a set $J_R\subseteq[R/2,R]$ with linear measure $(1-o(1))R/2$, such that for any $r\in J_R,|z|=r$, we have the following equation

$$H_k(f)=\prod_{i=1}^{k}f(z+c_i)-f^k(z)=F(z)\left(1+o(1)\right), \quad (2.22)$$

where $F(z)$ satisfies the following express,

$$F(z)=f'(z)\phi(z), \quad (2.23)$$

here $\phi(z)=\sum_{m=1}^{k}\sum_{1\leq j_1\leq j_2\leq\cdots\leq k}c_{j_1}c_{j_2}\cdots c_{c_{j_m}}(f')^{m-1}f^{k-m}$.

On the other hand, under the condition of Theorem 1.5 and from Lemma 2.4 we know $H_k(f)$ transcendental.

Suppose that $\varepsilon-$set E concludes all of zeros and poles of $H_k(f),f(z),f(z+c_1),\cdots,f(z+c_k)$, and $f'(z)$. Setting

$$E_R=\{r:r\in E,|z|=r<R\},$$
$$E_\infty=\{r:r\in E,|z|=r<\infty\}.$$

Since the property of $\varepsilon-$set and $\sigma_1<1$, we have that E_∞ is with finite logarithmic measure, and E_R has linear measure $o(1)R/2$ for sufficiently large R.

We assume that F_R is a set, such that

$$F_R=\left\{r:r\in\left[\frac{R}{2},R\right],n(r,f)=n\left(r-\sum_{j=1}^{k}|c_j|,f\right)\right\}. \quad (2.24)$$

Noting that there exists $o(R)$ many points $q_l\in\left[\frac{R}{3},R\right]$ at most from (2.22), at which $n(t,f)$ is not continuous, and also for any $r\in[R/2,R]$, $r\in\left[q_l,q_l+\sum_{j=1}^{k}|c_j|\right]$ holds for some l whenever $n(r,f)>n\left(r-\sum_{j=1}^{k}|c_j|,f\right)$. Therefore, F_R has linear measure

$$m(F_R)\geq(1-o(1)R/2), \quad (2.25)$$

From (2.23)-(2.25), we know that there exists $r\in F_R\cap J_R\setminus E_R$ such that $H_k(f)$, $f(z)$, $f(z+c_j)$, $j=1,2,\cdots,k$, and $f'(z)$ have no zeros and poles on the circle $|z|=r$. Therefore,

$$|H_k(z)-F(z)|<|o(1)F(z)|<|F(z)|. \quad (2.26)$$

Applying Rouché's Theorem to $H_k(f)$ and $F(z)$, we obtain the following equation

$$n\left(r,\frac{1}{H_k(f)}\right)=n\left(r,\frac{1}{F}\right)-n(r,F)+n(r,H_k(f)). \quad (2.27)$$

Without loss of generality, we may assume that

$$f\left(z_0+\sum_{j=1}^{k}l_jc_j\right)\neq 0,\infty\left(l_j=0,\pm1,j=1,2,\cdots,k\right)$$

for all poles z_0 of $f(z)$. From the assumption in Theorem 1.3, we know that there exists positive number $r_0>0$, which does not depend on R and r, such that if z_0 is a pole of $f(z)$ with multiplicity n,

$$r_0<|z_0|<r-\sum_{j=1}^{k}|c_j|,$$

then by the expression of $H_k(f)$ and $H_k\left(f(z-c_j)\right)$,

$$H_k\left(f(z-c_j)\right)=\sum_{t=1}^{k}f(z+c_t-c_j)-f^k(z-c_j)$$

we see that z_0, $z_0-c_j(j=1,2,\cdots,k)$ are respectively the pole of $H_k(f)$ with multiplicity kn,n. Therefore, we deduce that

$$n(r,H_k(f))\geq 2kn(r,f)+O(1). \quad (2.28)$$

Since the pole z_0 of $F(z)$ has multiplicity $k(n+1)$, we have the following equality

$$n(r,F)=k\left(n(r,f)+\bar{n}(r,f)\right). \quad (2.29)$$

And obviously,

$$n\left(r,\frac{1}{F}\right)=n\left(r,\frac{1}{f}\right)+n\left(r,\frac{1}{\phi(f)}\right). \quad (2.30)$$

Substituting (2.28), (2.30) into (2.27), we obtain

$$n\left(r,\frac{1}{H_k(f)}\right)\geq n\left(r,\frac{1}{f'}\right)+n\left(r,\frac{1}{\phi(f)}\right)+k\left[n(r,f)-\overline{n}(r,f)\right]+O(1), \quad (2.31)$$

If $\overline{\lambda}\left(\frac{1}{f}\right)<\lambda\left(\frac{1}{f}\right)$, then $n(r,f)=o(n(r,f))$. Thus, we have that by (2.31)

$$n\left(r,\frac{1}{H_k(f)}\right)\geq n\left(r,\frac{1}{f'}\right)+n(r,f)+O(1), \quad (2.32)$$

then $\lambda(H_k(f))=\sigma(H_k(f))=\sigma(f)$.

If $\overline{\lambda}\left(\frac{1}{f}\right)=\lambda\left(\frac{1}{f}\right)$, we have that from (2.31)

$$n\left(r,\frac{1}{H_k(f)}\right)\geq n\left(r,\frac{1}{f'}\right)+n\left(r,\frac{1}{\phi(f)}\right)+O(1). \quad (2.33)$$

By Lemma 2.5 and (2.33), we deduce that

$$\lambda(H_k(f))=\sigma(H_k(f))=\sigma(f).$$

In particular, if z_0 is the zero of

$$\psi_k(f)=H_k(f)/f^k(z),$$

then z_0 is, also the zero of $H_k(f)$. On the other hand, if z_1 is the zero of $H_k(f)$, but not the zero of $\psi_k(f)$, then z_1 must be the zero of $f(z)$, that is, $f(z_1+c_j)=0$ for some j. From the assumption in Theorem 1.5 that $f(z)$ has at most finitely many zeros z_j, z_s satisfying $z_j-z_s=c_1,c_2,\cdots,c_k$, we have

$$n\left(r,\frac{1}{\psi(f)}\right)=n\left(r,\frac{1}{H_k(f)}\right)+O(1).$$

Therefore, $\lambda(\psi(f))=\sigma(\psi(f))=\sigma(f)$.

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