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Strong Convergence of a General Iterative Algorithm for Mixed Equilibrium, Variational Inequality and Common Fixed Points Problems*

Tanakit Thianwan[1,2]
[1]Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand
[2]Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd., Bangkok, 10400, Thailand
Email: tanakit.th@up.ac.th

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ABSTRACT

The aim of this paper, is to introduce and study a general iterative algorithm concerning the new mappings which the sequences generated by our proposed scheme converge strongly to a common element of the set of solutions of a mixed equilibrium problem, the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of the variational inequality for a relaxed cocoercive mapping in a real Hilbert space. In addition, we obtain some applications by using this result. The results obtained in this paper generalize and refine some known results in the current literature.

Keywords: Nonexpansive Mapping; Mixed Equilibrium Problem; Variational Inequality; Common Fixed Points; Strong Convergence

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. A mapping $T: C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $\operatorname{Fix}(T) = \{x \in H : Tx = x\}$ the set of fixed points of T. A linear bounded operator A is strongly positive if there is a constant $\bar{\gamma} > 0$ with the property $\langle Ax, x\rangle \ge \bar{\gamma}\|x\|^2$ for all $x \in H$. A mapping $f: H \to H$ is said to be a contraction if there exists a coefficient $\alpha\,(0 < \alpha < 1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in H$. Let P_C be the nearest point projection of H onto the convex subset C (*i.e.*, for $x \in H$, P_C is the only point in C such that $\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}$. It is known that projection operator P_C is nonexpansive. It is also known that P_C satisfies $\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2$ for $x, y \in H$. The following characterizes the projection P_C Given $z \in H$ and $u \in C$. Then $u = P_C z$ if and only if there holds the relations:

$$\langle z - u, u - v\rangle \ge 0 \tag{1.1}$$

for all $v \in C$ (see [1]). Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y\rangle \ge 0$ for all $y \in C$. Let $B: C \to H$ be a nonlinear map. The classical variational inequality problem, denoted by $VI(C,B)$ is to find $u \in C$ such that

$$\langle Bu, v - u\rangle \ge 0 \tag{1.2}$$

for all $v \in C$. One can see that the variational inequality problem (1.2) is equivalent to the following fixed point problem: the element $u \in C$ is a solution of the variational inequality (1.2) if and only if $u \in C$ satisfies the relation $u = P_C(I - \lambda B)u$, where $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [2-6] and the references therein. A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in \operatorname{Fix}(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.3}$$

where A is a linear bounded operator and b is a given point in H. In [5] (see also [6]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n b + (I - \alpha_n A) S x_n,\ n \ge 0,$$

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converges strongly to the unique solution of the minimization problem (1.3) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. In 2006, Marino and Xu (see [3]) considered the following viscosity iterative method which was first introduced by Moudafi (see [7]):

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S x_n, \; n \geq 0 \quad (1.4)$$

They proved that the sequence $\{x_n\}$ generated by iterative scheme (1.4) converges strongly to the unique solution of the variational inequality $\langle (A - \gamma f) x^*, x - x^* \rangle \geq 0$, $x \in C$ which is the optimality condition for the minimization problem

$$\min_{x \in \mathrm{Fix}(S)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for α-cocoercive mapping, Takahashi and Toyoda (see [11]) introduced the following iterative process: $x_0 \in C$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n B x_n), n \geq 0, \quad (1.5)$$

where B is α-cocoercive, $\{\alpha_n\} \subset (0,1)$, and $\{\lambda_n\} \subset (0, 2\alpha)$. They showed that, if $\mathrm{Fix}(S) \cap VI(C,B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.5) converges weakly to some $z \in \mathrm{Fix}(S) \cap VI(C,B)$. In 2005, Iiduka and Takahashi (see [12]) introduced the following iterative process:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S P_C (x_n - \lambda_n B x_n), n \geq 0, \quad (1.6)$$

where $u \in C$, $\{\alpha_n\} \subset (0,1)$ and $\{x_n\} \subset (0, 2\alpha)$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (1.6) converges strongly to $z \in \mathrm{Fix}(S) \cap VI(C,B)$. In 2009, Qin, Kang and Shang, [13] introduced the following iterative algorithm given by $x_1 \in C$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) P_C S x_n, \quad n \geq 1, \quad (1.7)$$

where $C \pm C \subset C$, $T : C \to H$ a k-strict pseudo-contraction for some $0 \leq k < 1$, $S : C \to H$ defined by $Sx = kx + (1-k)Tx$, A is a strongly positive linear bounded self-adjoint operator and f is a contraction. They proved that the sequence $\{x_n\}$ generated by the iterative algorithm (1.7) converges strongly to a fixed point of T, which solves a variational inequality related to the linear operator A.

Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [14] considered the following mixed equilibrium problem: Find $x \in C$ such that

$$F(x,y) + \varphi(y) \geq \varphi(x) \quad (1.8)$$

for all $y \in C$. The set of solutions of (1.8) is denoted by $MEP(F, \varphi)$, *i.e.*,

$$MEP(F,\varphi) = \{x \in C : F(x,y) + \varphi(y) \geq \varphi(x), \forall y \in C\}.$$

It is easy to see that x is a solution of problem (1.8) implies that $x \in \mathrm{dom}\varphi = \{x \in C : \varphi(x) < +\infty\}$. Moreover, Ceng and Yao [14] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.8) and the set of common fixed points of a family of finitely nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem. If $\varphi = 0$, then the mixed equilibrium problem (1.8) becomes the following equilibrium problem:

$$F(x,y) \geq 0 \quad (1.9)$$

for all $y \in C$. The set of solutions of (1.9) is denoted by $EP(F)$, *i.e.*,

$$EP(F) = \{x \in C : F(x,y) \geq 0, \forall y \in C\}.$$

Given a mapping $T : C \to H$, let $\varphi = 0$ and $F(x,y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $z \in MEP(F, \varphi)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, *i.e.*, z is a solution of the variational inequality. Equilibrium problems have been studied extensively; see, for instance, [15,16]. The mixed equilibrium problem (1.8) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others; see for instance, [14,16-19].

Combettes and Hirstoaga (see [15]) introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. In 2007, S. Takahashi and W. Takahashi (see [20]) introduced an iterative scheme using the viscosity approximation method for finding a common element of the set of solutions of equilibrium problem (1.9) and the set of fixed points of a nonexpansive nonself-mapping in a Hilbert space. The scheme is defined as follows: $x_1 \in H$,

$$\begin{cases} F(y_n, u) + \dfrac{1}{r_n} \langle u - y_n, y_n - x_n \rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S y_n, & n \geq 1. \end{cases} \quad (1.10)$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.10) converge strongly to $z \in \mathrm{Fix}(S) \cap EP(F)$, where $z = P_{\mathrm{Fix}(S) \cap EP(F)} f(z)$. In the same year, Shang *et al.* (see [21]) introduced the following iterative scheme: $x_1 \in H$,

$$\begin{cases} F\left(y_n,u\right)+\dfrac{1}{r_n}\left\langle u-y_n,y_n-x_n\right\rangle \geq 0, & \forall u\in C, \\ x_{n+1}=\alpha_n\gamma f\left(x_n\right)+\left(1-\alpha_n A\right)Sy_n, & n\geq 1. \end{cases} \tag{1.11}$$

for finding a common element of the set of solutions of equilibrium problem (1.9) and the set of fixed points of a nonexpansive nonself-mapping in a Hilbert space. They proved that under some sufficient suitable conditions, the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.11) converge strongly to

$$q\in \operatorname{Fix}\left(S\right)\cap EP\left(F\right),$$

where

$$q=P_{\operatorname{Fix}(S)\cap EP(F)}\left(\gamma f+\left(I-A\right)\right)q,$$

which is the unique solution of the variational inequality

$$\left\langle \left(\gamma f-A\right)q,p-q\right\rangle \leq 0$$

for all $p\in \operatorname{Fix}\left(S\right)\cap EP\left(F\right)$.

Let $T_i : C\to C$, where $i=1,2,\cdots,N$, be a finite family of nonexpansive mappings. Finding an optimal point in the intersection $\cap_{i=1}^{N}\operatorname{Fix}\left(T_i\right)$ of the fixed points set of a finite family of nonexpansive mappings is a problem of interest in various branches of sciences; see [22-27] and also see [28] for solving the variational problems defined on the set of common fixed points of finitely many nonexpansive mappings. Atsushiba and Takahashi (see [29]), defined the mappings

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \lambda_{n,1}T_1U_{n,0}+\left(1-\lambda_{n,1}\right)I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1}+\left(1-\lambda_{n,2}\right)I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2}+\left(1-\lambda_{n,N-1}\right)I, \\ W_n = U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1}+\left(1-\lambda_{n,N}\right)I, \end{aligned} \tag{1.12}$$

where $\{\lambda_{n,1}\},\{\lambda_{n,2}\},\cdots,\{\lambda_{n,N}\}\subset (0,1]$. Such a mapping W_n is called the *W*-mapping generated by $T_1,T_2,\cdots,T_N$ and $\{\lambda_{n,1}\},\{\lambda_{n,2}\},\cdots,\{\lambda_{n,N}\}$. The concept of *W*-mappings was introduced in [30-33]. In 2008, Qin *et al.* (see [34]) introduced and studied the following iterative process: $x_1\in H$,

$$\begin{cases} F\left(y_n,\eta\right)+\dfrac{1}{r_n}\left\langle \eta-y_n,y_n-x_n\right\rangle \geq 0, & \forall \eta\in C, \\ x_{n+1}=\alpha_n\gamma f\left(W_nx_n\right)+\left(I-\alpha_n A\right)W_nP_C\left(I-s_nB\right)y_n, & n\geq 1, \end{cases} \tag{1.13}$$

where W_n is defined by (1.12), A is a strongly linear bounded operator and B is μ-Lipschitzian, relaxed (u,v)-cocoercive mapping of *C* into *H*. They proved that the sequences $\{x_n\}$ and $\{y_n\}$ generated by the iterative scheme (1.13) converge strongly to

$$q\in \cap_{i=1}^{N}\operatorname{Fix}\left(T_i\right)\cap EP\left(F\right)\cap VI\left(C,B\right),$$

where

$$q=P_{\cap_{i=1}^{N}\operatorname{Fix}(T_i)\cap EP(F)\cap VI(C,B)}\left(\gamma f+\left(I-A\right)\right)q,$$

which is the unique solution of the variational inequality

$$\left\langle \left(\gamma f-A\right)q,p-q\right\rangle \leq 0$$

for all

$$p\in \cap_{i=1}^{N}\operatorname{Fix}\left(T_i\right)\cap EP\left(F\right)\cap VI\left(C,B\right).$$

In the same year, Colao *et al.* (see [35]) introduced a new iterative scheme: $x_1\in H$,

$$\begin{cases} F\left(y_n,u\right)+\dfrac{1}{r_n}\left\langle u-y_n,y_n-x_n\right\rangle \geq 0, & \forall u\in H, \\ x_{n+1}=\alpha_n\gamma f\left(x_n\right)+\beta x_n+\left(\left(1-\beta\right)I-\alpha_n A\right)W_ny_n, & n\geq 1 \end{cases} \tag{1.14}$$

for approximating a common element of the set of solutions of equilibrium problem (1.9) and the set of common fixed points of a finite family of nonexpansive mappings and obtained a strong convergence theorem in a Hilbert space. In 2009, Yao *et al.* (see [36]) studied similar scheme as follows: $x_1\in H$,

$$\begin{cases} F\left(y_n,u\right)+\dfrac{1}{r_n}\left\langle u-y_n,y_n-x_n\right\rangle \geq 0, & \forall u\in H, \\ x_{n+1}=\alpha_n\gamma f\left(x_n\right)+\beta_n x_n+\left(\left(1-\beta_n\right)I-\alpha_n A\right)W_ny_n, & n\geq 1 \end{cases} \tag{1.15}$$

where $\gamma>0$, $\{\alpha_n\}$, $\{\beta_n\}\subset (0,1)$, $\{r_n\}\subset (0,\infty)$ and W_n is the *W*-mapping defined by (1.12). They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ and $\{\lambda_{n,i}\}$ $\left(\forall i=1,2,\cdots,N\right)$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.15) converge strongly to

$$x^* \in \cap_{i=1}^N \operatorname{Fix}(T_i) \cap EP(F),$$

where

$$x^* = P_{\cap_{i=1}^N \operatorname{Fix}(T_i) \cap EP(F)} \left(\gamma f + (I - A)\right) x^*,$$

which is the unique solution of the variational inequality $\left\langle (\gamma f - A) x^*, x - x^* \right\rangle \leq 0$ for all $x \in \cap_{i=1}^N \operatorname{Fix}(T_i) \cap EP(F)$.

If $\beta_n \equiv \beta$ for some $\beta \in (0,1)$, then (1.15) reduces to the iterative scheme (1.14). Very recently, Kangtunyakarn and Suantai (see [37]) defined the new mappings

$$\begin{aligned} &U_{n,0} = I, \\ &U_{n,1} = \lambda_{n,1} T_1 U_{n,0} + (1 - \lambda_{n,1}) I, \\ &U_{n,2} = \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) U_{n,1}, \\ &\vdots \\ &U_{n,N-1} = \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) U_{n,N-2}, \\ &K_n = U_{n,N} = \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) U_{n,N-1}, \end{aligned} \tag{1.16}$$

where $\{\lambda_{n,i}\}_i^N \subset (0,1]$. Such a mapping K_n is called the K-mapping generated by $T_1, T_2, \cdots, T_N$ and $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \cdots, \{\lambda_{n,N}\}$. Nonexpansivity of each T_i ensures the nonexpansivity of K_n Also following they defined the new mappings

$$\begin{aligned} &U_0 = I, \\ &U_1 = \lambda_1 T_1 U_0 + (1 - \lambda_1) I, \\ &U_2 = \lambda_2 T_2 U_1 + (1 - \lambda_2) U_1, \\ &\vdots \\ &U_{N-1} = \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) U_{N-2}, \\ &K = U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N) U_{N-1}, \end{aligned} \tag{1.17}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_N \in (0,1]$ such that $0 < \lambda_i < 1$ for all $i = 1, 2, \cdots, N-1$ and $0 < \lambda_N \leq 1$. Such a mapping K is called the K-mapping generated by $T_1, T_2, \cdots, T_N$ and $\lambda_1, \lambda_2, \cdots, \lambda_N$. In [37], Lemma 2.9 and Lemma 2.10, its shown that

$$\operatorname{Fix}(K) = \cap_{i=1}^N \operatorname{Fix}(T_i)$$

and $\lim_{n\to\infty} \|K_n x - Kx\| = 0$ for all $x \in C$, where K_n and K are the K-mappings defined by (1.16) and (1.17), respectively. Its important tool for the proof of the main results in this paper. Moreover, Kangtunyakarn and Suantai (see [37]) introduced a new iterative scheme: $x_1 \in H$ and $n \geq 1$,

$$\begin{cases} F(y_n, \eta) + \dfrac{1}{r_n} \langle \eta - y_n, y_n - x_n \rangle \geq 0, \ \forall \eta \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + \left((1 - \beta_n) I - \alpha_n A\right) K_n y_n, \end{cases} \tag{1.18}$$

where $\gamma > 0$, $\beta \in (0,1)$, $\{\alpha_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$ and K_n is the K-mapping defined by (1.16). They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{r_n\}$ and $\{\lambda_{n,i}\}$ $(\forall i = 1, 2, \cdots, N)$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.18) converge strongly to

$$x^* \in \cap_{i=1}^N \operatorname{Fix}(T_i) \cap EP(F),$$

where

$$x^* = P_{\cap_{i=1}^N Fix(T_i) \cap EP(F)} \left(\gamma f + (I - A)\right) x^*.$$

Motivated by the recent works, we introduce a more general iterative algorithm for finding a common element of the set of common fixed points of a finite family of nonexpansive mappings, the set of solutions of a mixed equilibrium problem, and the set of solutions of the variational inequality problem for a relaxed cocoercive mapping in a real Hilbert space. The scheme is defined as follows: $x_1 \in H$ and $\forall n \geq 1$,

$$\begin{cases} F(y_n, \eta) + \varphi(\eta) - \varphi(y_n) + \dfrac{1}{r_n} \langle \eta - y_n, y_n - x_n \rangle \geq 0, \\ \qquad \forall \eta \in C, \\ x_{n+1} = \alpha_n \gamma f(K_n x_n) + \beta_n x_n \\ \qquad + \left((1 - \beta_n) I - \alpha_n A\right) K_n P_C (I - s_n B) y_n, \end{cases} \tag{1.19}$$

where $\gamma > 0$, $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$, $\{s_n\} \subset [0,\infty)$, $B: C \to H$ is a μ-Lipschitzian, relaxed (u,v)-cocoercive mapping, f is a contraction of H into itself with a coefficient $\alpha (0 < \alpha < 1)$, P_C is a projection of H onto C, A is a strongly positive linear bounded operator on H, F is a mixed equilibrium bifunction, $\varphi: C \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous and convex function and K_n is the K-mapping generated by $T_1, T_2, \cdots, T_N$ and $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \cdots, \{\lambda_{n,N}\}$. We prove that the sequences $\{x_n\}$ and $\{y_n\}$ generated by the iterative scheme (1.19) converge strongly to

$$q \in \cap_{i=1}^N \operatorname{Fix}(T_i) \cap MEP(F, \varphi) \cap VI(C, B),$$

where

$$q = P_{\cap_{i=1}^N \operatorname{Fix}(T_i) \cap MEP(F,\varphi) \cap VI(C,B)} \left(\gamma f + (I - A)\right) q,$$

which is the unique solution of the variational inequality for all $\left\langle (\gamma f - A) q, p - q \right\rangle \leq 0$

$$p \in \cap_{i=1}^N \operatorname{Fix}(T_i) \cap MEP(F, \varphi) \cap VI(C, B),$$

and is also the optimality condition for the minimization problem

$$\min_{x \in \cap_{i=1}^N \operatorname{Fix}(T_i) \cap MEP(F,\varphi) \cap VI(C,B)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

2. Preliminaries and Lemmas

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

A mapping B is called v-strongly monotone, if each $x, y \in C$, we have

$$\langle Bx - By, x - y \rangle \geq v \|x - y\|^2,$$

for a constant $v > 0$, which implies that $\|Bx - By\| \geq v\|x - y\|$, so that B is v-expansive and when $v = 1$, it is expansive. B is said to be v-cocoercive (see [8] and [9]), if for each $x, y \in C$, we have

$$\langle Bx - By, x - y \rangle \geq v \|Bx - By\|^2,$$

for a constant $v > 0$. Clearly, every v-cocoercive mapping B is $\frac{1}{v}$-Lipschitz continuous. B is called relaxed u-co-coercive, if there exists a constant $u > 0$ such that

$$\langle Bx - By, x - y \rangle \geq (-\mu) \|Bx - By\|^2,$$

for all $x, y \in C$. B is said to be relaxed (u, v)-cocoercive, if there exist two constants u, $v > 0$ such that

$$\langle Bx - By, x - y \rangle \geq (-u) \|Bx - By\|^2 + v\|x - y\|^2$$

for all $x, y \in C$, for $u = 0$, B is v-strongly monotone.

It is worth mentioning that the class of mappings which are relaxed (u, v)-cocoercive more general than the class of strongly monotone mappings. It is easy to see that if B is a v-strongly monotone mapping, then it is a relaxed (u, v)-cocoercive mapping (see [10]).

It is well known that for all $x, y \in H$ and $\lambda \in [0,1]$ there holds

$$\begin{aligned}&\|\lambda x + (1-\lambda) y\|^2 \\ &= \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2.\end{aligned}$$

Recall that a space X is said to satisfy Opial's condition (see [38]) if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ for all $y \in X$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all x, $y \in H$, $u \in Tx$ and $v \in Ty$ imply $\langle x - y, u - v \rangle \geq 0$.

A monotone mapping $T : H \to 2^H$ is maximal if graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$ for every $(y, v) \in G(T)$ implies $u \in Tx$. Let B be a monotone mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, *i.e.*,

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \varnothing, & v \notin C \end{cases}$$

Then T is a maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [39].

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.1. (see [4,5]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n,$$

where γ_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. (see [3]). Assume A is a strong positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 2.3. (see [40]). Let $\{x_n\}$ and $\{y_n\}$, be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$.

Then

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.4. (see [37]). Let C be a nonempty closed convex set of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N \mathrm{Fix}(T_i) \neq \varnothing$. and let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \cdots, N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, \cdots, T_N$ and $\lambda_1, \lambda_2, \cdots, \lambda_N$. Then $\mathrm{Fix}(K) = \cap_{i=1}^N \mathrm{Fix}(T_i)$.

Lemma 2.5. (see [37]). Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ be sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$ $(i = 1, \cdots, N)$. Moreover for every $n \in \mathbb{N}$, let K and K_n be the K-mappings generated by $T_1, T_2, \cdots, T_N$ and $\lambda_1, \lambda_2, \cdots, \lambda_N$ and $T_1, T_2, \cdots, T_N$ and $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \cdots, \{\lambda_{n,N}\}$, respectively. Then for every $x \in C$, it follows that

$$\lim_{n\to\infty} \|K_n x - Kx\| = 0.$$

For solving the mixed equilibrium problem, let us give the following assumptions for a bifunction F, φ and the set C:

(A1) $F(x,x)=0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y)+F(y,x) \le 0$ for all $x, y \in C$;

(A3) For each $x, y, z \in C$,

$$\lim_{t \to 0} F(tz+(1-t)x, y) \le F(x,y);$$

(A4) For each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous;

(B1) For each $x \in H$ and $r>0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in CD_x$,

$$F(z,y)+\varphi(y_x)+\frac{1}{r}\langle y_x - z, z-x\rangle < \varphi(z);$$

(B2) C is a bounded set.

By a similar argument as in the proof of Lemma 2.3 in [18], we have the following result.

Lemma 2.6. Let C be a nonempty closed convex subset of a Hilbert space H and let F be a mixed equilibrium bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)-(A4) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r>0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \Big\{ z \in C : F(z,y)+\varphi(y) + \frac{1}{r}\langle y-z, z-x\rangle \ge \varphi(z), \forall y \in C \Big\}$$

for all $x \in H$. Then T_r is well defined and the following hold:

1) T_r is single-valued;
2) T_r is firmly nonexpansive, *i.e.*, for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x-y\rangle;$$

3) $\mathrm{Fix}(T_r) = MEP(F,\varphi)$;
4) $MEP(F,\varphi)$ is closed and convex.

Remark 2.7. We remark that Lemma 1.6 is not a consequence of Lemma 3.1 in [14], because the condition of the sequential continuity from the weak topology to the strong topology for the derivative K' of the function $K : C \to \mathbb{R}$ does not cover the case

$$K(x) = \frac{\|x\|^2}{2}.$$

The following lemma is well known.

Lemma 2.8. In a real Hilbert space H, there holds the following inequality

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$$

for all $x, y \in H$.

3. Main Results

Theorem 3.1. Let H be a real Hilbert space, C a nonempty closed convex subset of H, B a μ-Lipschitzian, relaxed (u,v)-cocoercive mapping of C into H, F a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4), $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function and $T_1, T_2, \cdots, T_N$ a finite family of nonexpansive mappings of C into H such that the common fixed points set

$$\cap_{i=1}^N \mathrm{Fix}(T_i) \cap MEP(F,\varphi) \cap VI(C,B) \ne \varnothing.$$

Let f be a contraction of H into itself with a coefficient $\alpha\,(0<\alpha<1)$ and A a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}>0$ such that $\|A\| \le 1$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and either (B1) or (B2) holds. Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be real numbers such that $0<\lambda_i<1$ for every $i=1,2,\cdots,N-1$ and $0<\lambda_N \le 1$, $\lambda_{n,i} \to \lambda_i\,(i=1,\cdots,N)$, $\{r_n\} \subset (0,\infty)$, $\{s_n\} \subset [0,\infty)$ and $\{\alpha_n\}$, $\{\beta_n\}$ two real sequences in (0, 1) satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(C3) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$ (this is weaker than the condition); $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(C4) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(C5) $\{s_n\} \subset [a,b]$ for some a, b with $0 \le a \le b \le \frac{2(v-u\mu^2)}{\mu^2}, v \ge u\mu^2$;

(C6) $\sum_{n=0}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$ for all $i = 1,2,\cdots,N$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ generated iteratively by (1.19) converge strongly to

$$q \in \cap_{i=1}^N \mathrm{Fix}(T_i) \cap MEP(F,\varphi) \cap VI(C,B),$$

where

$$q = P_{\cap_{i=1}^N \mathrm{Fix}(T_i) \cap MEP(F,\varphi) \cap VI(C,B)} (\gamma f + (I-A))q,$$

which solves the following variational inequality:

$$\langle (\gamma f - A)q, p-q\rangle \le 0$$

for all

$$p \in \cap_{i=1}^{N} \mathrm{Fix}(T_i) \cap MEP(F,\varphi) \cap VI(C,B).$$

Proof Since $\alpha_n \to 0$ as $n \to \infty$ by the condition (C1), we may assume, without loss of generality, that

$$0 < \alpha_n \le (1-\beta_n)\|A\|^{-1}$$

for all *n*. We also have $0 < \alpha_n \le \|A\|^{-1}$ for all *n*. By using Lemma 2.2, we have

$$\|I - \alpha_n A\| \le 1 - \alpha_n \bar{\gamma}.$$

Since *A* is a strongly positive linear bounded operator on a Hilbert space *H*, we have

$$\langle Ax, x \rangle \ge \bar{\gamma}\|x\|^2$$

and

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}&\langle ((1-\beta_n)I - \alpha_n A)x, x\rangle \\ &= 1 - \beta_n - \alpha_n \langle Ax, x\rangle \ge 1 - \beta_n - \alpha_n\|A\| \ge 0, \ \forall x \in H.\end{aligned}$$

This shows that $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}&\|(1-\beta_n)I - \alpha_n A\| \\ &= \sup\{|\langle ((1-\beta_n)I - \alpha_n A)x, x\rangle| : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \le 1 - \beta_n - \alpha_n\bar{\gamma}.\end{aligned}$$

Next, we will assume that $\|I - A\| \le 1 - \bar{\gamma}$. First, we show $I - s_n B$ is nonexpansive. Indeed, from the relaxed (u,v)-cocoercive and μ-Lipschitzian definition on *B* and condition (C5), we have which implies the mapping $I - s_n B$ is nonexpansive.

$$\begin{aligned}\|(I - s_n B)x - (I - s_n B)y\|^2 &= \|(x-y) - s_n(Bx - By)\|^2 = \|x-y\|^2 - 2s_n\langle x-y, Bx - By\rangle + s_n^2\|Bx - By\|^2 \\ &\le \|x-y\|^2 - 2s_n\left(-u\|Bx-By\|^2 + v\|x-y\|^2\right) + s_n^2\|Bx-By\|^2 \\ &\le \|x-y\|^2 + 2s_n\mu^2 u\|x-y\|^2 - 2s_n v\|x-y\|^2 + \mu^2 s_n^2\|x-y\|^2 \\ &= \left(1 + 2s_n\mu^2 u - 2s_n v + \mu^2 s_n^2\right)\|x-y\|^2 \le \|x-y\|^2,\end{aligned}$$

We shall divide our proof into 5 steps.

Step 1. We shall show that the sequence $\{x_n\}$ is bounded. Let

$$x^* \in \cap_{i=1}^{N} \mathrm{Fix}(T_i) \cap MEP(F,\varphi) \cap VI(C,B).$$

Since $y_n = T_{r_n} x_n \in \mathrm{dom}\varphi$, we have

$$\|y_n - x^*\| = \|T_{r_n} x_n - T_{r_n} x^*\| \le \|x_n - x^*\|. \tag{3.1}$$

Putting $z_n = P_C(I - s_n B)y_n$ for all $n \ge 0$, we have

$$\begin{aligned}\|z_n - x^*\| &= \|P_C(I - s_n B)y_n - x^*\| \\ &\le \|(I - s_n B)y_n - (I - s_n B)x^*\| \\ &\le \|y_n - x^*\|.\end{aligned}$$

Using (1.19), (3.1) and (3.2), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n\gamma f(K_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)K_n z_n - x^*\| \\ &= \|\alpha_n(\gamma f(K_n x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(K_n z_n - x^*)\| \\ &\le (1 - \beta_n - \alpha_n\bar{\gamma})\|K_n z_n - x^*\| + \beta_n\|x_n - x^*\| + \alpha_n\|\gamma f(K_n x_n) - Ax^*\| \\ &\le (1 - \beta_n - \alpha_n\bar{\gamma})\|z_n - x^*\| + \beta_n\|x_n - x^*\| + \alpha_n\|\gamma f(K_n x_n) - \gamma f(x^*) + \gamma f(x^*) - Ax^*\| \\ &\le (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - x^*\| + \beta_n\|x_n - x^*\| + \alpha_n\gamma\|f(K_n x_n) - f(x^*)\| + \alpha_n\|\gamma f(x^*) - Ax^*\| \\ &\le (1 - \alpha_n\bar{\gamma})\|x_n - x^*\| + \alpha_n\gamma\alpha\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Ax^*\| \\ &= (1 - (\bar{\gamma} - \gamma\alpha)\alpha_n)\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Ax^*\|,\end{aligned}$$

which gives that

$$\|x_n - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma} - \gamma\alpha}\right\}, n \ge 0.$$

Hence $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{z_n\}$, $\{K_n z_n\}$, $\{By_n\}$ and $\{f(K_n x_n)\}$.

Step 2. We will show that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Observing that $y_n = T_{r_n} x_n \in \mathrm{dom}\varphi$ and $y_{n+1} = T_{r_{n+1}} x_{n+1} \in \mathrm{dom}\varphi$, we have

$$F\left(y_n,\eta\right)+\varphi(\eta)-\varphi\left(y_n\right)+\frac{1}{r_n}\left\langle \eta-y_n, y_n-x_n\right\rangle \geq 0, \forall \eta \in C \tag{3.3}$$

and

$$F\left(y_{n+1},\eta\right)+\varphi(\eta)-\varphi\left(y_{n+1}\right)+\frac{1}{r_{n+1}}\left\langle \eta-y_{n+1}, y_{n+1}-x_{n+1}\right\rangle \geq 0, \forall \eta \in C. \tag{3.4}$$

Putting $\eta = y_{n+1}$ in (3.3) and $\eta = y_n$ in (3.4), we have

$$F\left(y_n,y_{n+1}\right)+\varphi\left(y_{n+1}\right)-\varphi\left(y_n\right)$$
$$+\frac{1}{r_n}\left\langle y_{n+1}-y_n, y_n-x_n\right\rangle \geq 0$$

and

$$F\left(y_{n+1},y_n\right)+\varphi\left(y_n\right)-\varphi\left(y_{n+1}\right)$$
$$+\frac{1}{r_{n+1}}\left\langle y_n-y_{n+1}, y_{n+1}-x_{n+1}\right\rangle \geq 0.$$

Summing up the last two inequalities and using Lemma 2.6 (A2), we obtain

$$\left\langle y_{n+1}-y_n, \frac{y_n-x_n}{r_n}-\frac{y_{n+1}-x_{n+1}}{r_{n+1}}\right\rangle \geq 0.$$

That is,

$$\left\langle y_{n+1}-y_n, y_n-y_{n+1}+y_{n+1}-x_n-\frac{r_n}{r_{n+1}}\left(y_{n+1}-x_{n+1}\right)\right\rangle \geq 0.$$

It then follows that

$$\left\|y_{n+1}-y_n\right\|^2$$
$$\leq \left\|y_{n+1}-y_n\right\|\left(\left\|x_{n+1}-x_n\right\|+\left|1-\frac{r_n}{r_{n+1}}\right|\left\|y_{n+1}-x_{n+1}\right\|\right).$$

This implies that

$$\left\|y_{n+1}-y_n\right\| \leq \left\|x_{n+1}-x_n\right\|+\left|1-\frac{r_n}{r_{n+1}}\right|\left\|y_{n+1}-x_{n+1}\right\|$$
$$\leq \left\|x_{n+1}-x_n\right\|+M_1\left|1-\frac{r_n}{r_{n+1}}\right|,$$

where M_1 is an appropriate constant such that

$$M_1=\sup_{n\geq 1}\left\|y_{n+1}-x_{n+1}\right\|.$$

Since $I-s_nB$ is nonexpansive and $z_n=P_C\left(I-s_nB\right)y_n$, using (3.5), we also have

$$\begin{aligned}
\left\|z_{n+1}-z_n\right\| &= \left\|P_C\left(I-s_{n+1}B\right)y_{n+1}-P_C\left(I-s_nB\right)y_n\right\| \leq \left\|\left(I-s_{n+1}B\right)y_{n+1}-\left(I-s_nB\right)y_n\right\| \\
&= \left\|\left(I-s_{n+1}B\right)y_{n+1}-\left(I-s_{n+1}B\right)y_n+\left(s_n-s_{n+1}\right)By_n\right\| \leq \left\|y_{n+1}-y_n\right\|+\left|s_n-s_{n+1}\right|\left\|By_n\right\| \\
&\leq \left\|x_{n+1}-x_n\right\|+M_1\left|1-\frac{r_n}{r_{n+1}}\right|+\left|s_n-s_{n+1}\right|\left\|By_n\right\| \leq \left\|x_{n+1}-x_n\right\|+M_1\left|1-\frac{r_n}{r_{n+1}}\right|+M_2\left|s_n-s_{n+1}\right|,
\end{aligned}$$

where M_2 is an appropriate constant such that

$$M_2=\sup_{n\geq 1}\left\|By_n\right\|.$$

Define

$$u_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$$

for all $n\geq 0$ so that

$$x_{n+1}=\left(1-\beta_n\right)u_n+\beta_n x_n.$$

It follows that

$$\begin{aligned}
u_{n+1}-u_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}\gamma f\left(K_{n+1}x_{n+1}\right)+\left(\left(1-\beta_{n+1}\right)I-\alpha_{n+1}A\right)K_{n+1}z_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f\left(K_nx_n\right)+\left(\left(1-\beta_n\right)I-\alpha_nA\right)K_nz_n}{1-\beta_n} \\
&= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)\gamma f\left(K_{n+1}x_{n+1}\right)-\left(\frac{\alpha_n}{1-\beta_n}\right)\gamma f\left(K_nx_n\right)+K_{n+1}z_{n+1}-K_nz_n+\left(\frac{\alpha_n}{1-\beta_n}\right)AK_nz_n-\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)AK_{n+1}z_{n+1} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\gamma f\left(K_{n+1}x_{n+1}\right)-AK_{n+1}z_{n+1}\right)+\frac{\alpha_n}{1-\beta_n}\left(AK_nz_n-\gamma f\left(K_nx_n\right)\right)+K_{n+1}z_{n+1}-K_{n+1}z_n+K_{n+1}z_n-K_nz_n.
\end{aligned}$$

Observe that $u_n = \dfrac{x_{n+1} - \beta_n x_n}{1-\beta_n}$ from (3.6), we obtain

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \left\| \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left(\gamma f\left(K_{n+1}x_{n+1}\right) - AK_{n+1}z_{n+1}\right) + \frac{\alpha_n}{1-\beta_n}\left(AK_n z_n - \gamma f\left(K_n x_n\right)\right) + K_{n+1}z_{n+1} - K_{n+1}z_n + K_{n+1}z_n - K_n z_n \right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f\left(K_{n+1}x_{n+1}\right)\right\| + \left\|AK_{n+1}z_{n+1}\right\|\right) + \frac{\alpha_n}{1-\beta_n}\left(\left\|AK_n z_n\right\| + \left\|\gamma f\left(K_n x_n\right)\right\|\right) \\
&\quad + \left\|K_{n+1}z_{n+1} - K_{n+1}z_n\right\| + \left\|K_{n+1}z_n - K_n z_n\right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f\left(K_{n+1}x_{n+1}\right)\right\| + \left\|AK_{n+1}z_{n+1}\right\|\right) + \frac{\alpha_n}{1-\beta_n}\left(\left\|AK_n z_n\right\| + \left\|\gamma f\left(K_n x_n\right)\right\|\right) + \left\|z_{n+1} - z_n\right\| + \left\|K_{n+1}z_n - K_n z_n\right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f\left(K_{n+1}x_{n+1}\right)\right\| + \left\|AK_{n+1}z_{n+1}\right\|\right) + \frac{\alpha_n}{1-\beta_n}\left(\left\|AK_n z_n\right\| + \left\|\gamma f\left(K_n x_n\right)\right\|\right) + \left\|x_{n+1} - x_n\right\| \\
&\quad + M_1\left|1 - \frac{r_n}{r_{n+1}}\right| + M_2\left|s_n - s_{n+1}\right| + \left\|K_{n+1}z_n - K_n z_n\right\|.
\end{aligned} \tag{3.7}$$

Next we estimate $\left\|K_{n+1}z_n - K_n z_n\right\|$.

For $i \in \{2, \cdots, N-2\}$, we have

$$\begin{aligned}
&\left\|U_{n+1,N-i}z_n - U_{n,N-i}z_n\right\| \\
&= \left\|\lambda_{n+1,N-i}T_{N-i}U_{n+1,N-i-1}z_n + \left(1-\lambda_{n+1,N-i}\right)U_{n+1,N-i-1}z_n - \lambda_{n,N-i}T_{N-i}U_{n,N-i-1}z_n - \left(1-\lambda_{n,N-i}\right)U_{n,N-i-1}z_n\right\| \\
&= \left\|\lambda_{n+1,N-i}T_{N-i}U_{n+1,N-i-1}z_n - \lambda_{n+1,N-i}T_{N-i}U_{n,N-i-1}z_n + \lambda_{n+1,N-i}T_{N-i}U_{n,N-i-1}z_n - \lambda_{n+1,N-i}U_{n,N-i-1}z_n\right. \\
&\quad \left. + \lambda_{n+1,N-i}U_{n,N-i-1}z_n + \left(1-\lambda_{n+1,N-i}\right)U_{n+1,N-i-1}z_n - \lambda_{n,N-i}T_{N-i}U_{n,N-i-1}z_n - \left(1-\lambda_{n,N-i}\right)U_{n,N-i-1}z_n\right\| \\
&\le \lambda_{n+1,N-i}\left\|T_{N-i}U_{n+1,N-i-1}z_n - T_{N-i}U_{n,N-i-1}z_n\right\| + \left(1-\lambda_{n+1,N-i}\right)\left\|U_{n+1,N-i-1}z_n - U_{n,N-i-1}z_n\right\| \\
&\quad + \left|\lambda_{n+1,N-i} - \lambda_{n,N-i}\right|\left\|T_{N-i}U_{n,N-i-1}z_n\right\| + \left|\lambda_{n+1,N-i} - \lambda_{n,N-i}\right|\left\|U_{n,N-i-1}z_n\right\| \\
&\le \left\|U_{n+1,N-i-1}z_n - U_{n,N-i-1}z_n\right\| + M_3\left|\lambda_{n+1,N-i} - \lambda_{n,N-i}\right|
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
\left\|U_{n+1,1}z_n - U_{n,1}z_n\right\| &= \left\|\lambda_{n+1,1}T_1 z_n + \left(1-\lambda_{n+1,1}\right)z_n - \lambda_{n,1}T_1 z_n - \left(1-\lambda_{n,1}\right)z_n\right\| \\
&\le \left|\lambda_{n+1,1} - \lambda_{n,1}\right|\left\|T_1 z_n\right\| + \left|\lambda_{n+1,1} - \lambda_{n,1}\right|\left\|z_n\right\| \le \left|\lambda_{n+1,1} - \lambda_{n,1}\right| M_3,
\end{aligned} \tag{3.9}$$

where

$$M_3 = \sup\left\{\sum_{i=2}^{N}\left(\left\|T_i U_{n,i-1}z_n\right\| + \left\|U_{n,i-1}z_n\right\|\right) + \left\|T_1 z_n\right\| + \left\|z_n\right\|\right\} < \infty.$$

Using (3.8) and (3.9), we have

$$\begin{aligned}
\left\|K_{n+1}z_n - K_n z_n\right\| &= \left\|U_{n+1,N}z_n - U_{n,N}z_n\right\| \\
&\le \left\|U_{n+1,N-1}z_n - U_{n,N-1}z_n\right\| + M_3\left|\lambda_{n+1,N} - \lambda_{n,N}\right| \\
&\le \left\|U_{n+1,N-2}z_n - U_{n,N-2}z_n\right\| + M_3\left|\lambda_{n+1,N-1} - \lambda_{n,N-1}\right| + M_3\left|\lambda_{n+1,N} - \lambda_{n,N}\right| \\
&\;\;\vdots \\
&\le \left\|U_{n+1,1}z_n - U_{n,1}z_n\right\| + M_3\sum_{i=2}^{N}\left|\lambda_{n+1,i} - \lambda_{n,i}\right| \\
&\le M_3\sum_{i=1}^{N}\left|\lambda_{n+1,i} - \lambda_{n,i}\right|.
\end{aligned} \tag{3.10}$$

Substitute (3.10) into (3.7) yields that

$$\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(K_{n+1}x_{n+1})\|+\|AK_{n+1}z_{n+1}\|\right)+\frac{\alpha_n}{1-\beta_n}\left(\|AK_nz_n\|+\|\gamma f(K_nx_n)\|\right)$$
$$+M_1\left|1-\frac{r_n}{r_{n+1}}\right|+M_2|s_n-s_{n+1}|+M_3\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|,$$

which implies that (noting that (C1), (C2), (C3), (C4) and (C6))

$$\limsup_{n\to\infty}\left(\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|\right)\le 0.$$

Hence by Lemma 2.3, we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.11}$$

Using (3.11) and we have $x_{n+1}=(1-\beta_n)u_n+\beta_nx_n$,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|u_n-x_n\|=0 \tag{3.12}$$

Step 3. We shall show that

$$\lim_{n\to\infty}\|x_n-K_nz_n\|=\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|y_n-K_nz_n\|$$
$$=\lim_{n\to\infty}\|z_n-K_nz_n\|=0,$$

where $y_n=T_{r_n}x_n$.

Note that

$$\|x_n-K_nz_n\|=\|x_n-x_{n+1}+x_{n+1}-K_nz_n\|\le\|x_{n+1}-x_n\|+\|x_{n+1}-K_nz_n\|$$
$$\le\|x_{n+1}-x_n\|+\|\alpha_n\gamma f(K_nx_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)K_nz_n-K_nz_n\|$$
$$=\|x_{n+1}-x_n\|+\|\alpha_n(\gamma f(K_nx_n)-AK_nz_n)+((1-\beta_n)I-\alpha_nA)(K_nz_n-K_nz_n)+\beta_n(x_n-K_nz_n)\|$$
$$\le\|x_{n+1}-x_n\|+\alpha_n\|\gamma f(K_nx_n)-AK_nz_n\|+\beta_n\|x_n-K_nz_n\|.$$

This implies

$$(1-\beta_n)\|x_n-K_nz_n\|$$
$$\le\|x_{n+1}-x_n\|+\alpha_n\|\gamma f(K_nx_n)-AK_nz_n\|.$$

From condition (C1), (C4) and (3.12), we have

$$\lim_{n\to\infty}\|x_n-K_nz_n\|=0. \tag{3.13}$$

Next we prove that

$$\|x_n-y_n\|\to 0$$

as $n\to\infty$.

Indeed, picking

$$x^*\in\cap_{i=1}^{N}\operatorname{Fix}(T_i)\cap MEP(F,\varphi)\cap VI(C,B).$$

Since $y_n=T_{r_n}x_n\in\operatorname{dom}\varphi$ and T_r is firmly nonexpansive, we obtain and hence

$$\|x^*-y_n\|^2=\|T_{r_n}x^*-T_{r_n}x_n\|^2\le\langle T_{r_n}x_n-T_{r_n}x^*,x_n-x^*\rangle$$
$$=\langle y_n-x^*,x_n-x^*\rangle \tag{3.14}$$
$$=\frac{1}{2}\left(\|y_n-x^*\|^2+\|x_n-x^*\|^2-\|x_n-y_n\|^2\right)$$

Set $\rho_n=\gamma f(K_nx_n)-AK_nz_n$ and let $\lambda>0$ be an appropriate constant such that

$$\lambda\ge\sup_{n,k}\{\|\rho_n\|,\|x_k-x^*\|\}.$$

Therefore, from the convexity of $\|\cdot\|^2$, using (3.2), (3.14) and Lemma 2.8 we have

$$\|x_{n+1}-x^*\|^2=\|\alpha_n\gamma f(K_nx_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)K_nz_n-x^*\|^2$$
$$=\|((1-\beta_n)(K_nz_n-x^*)+\beta_n(x_n-x^*))+\alpha_n(\gamma f(K_nx_n)-AK_nz_n)\|^2$$
$$\le\|(1-\beta_n)(K_nz_n-x^*)+\beta_n(x_n-x^*)\|^2+2\alpha_n\langle\rho_n,x_{n+1}-x^*\rangle$$
$$\le(1-\beta_n)\|K_nz_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+2\lambda^2\alpha_n$$
$$\le(1-\beta_n)\|z_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+2\lambda^2\alpha_n\le(1-\beta_n)\|y_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+2\lambda^2\alpha_n$$
$$\le(1-\beta_n)\left(\|x_n-x^*\|^2-\|x_n-y_n\|^2\right)+\beta_n\|x_n-x^*\|^2+2\lambda^2\alpha_n$$
$$=\|x_n-x^*\|^2-(1-\beta_n)\|x_n-y_n\|^2+2\lambda^2\alpha_n.$$

It follows that

$$\left(1-\beta_n\right)\left\|x_n - y_n\right\|^2 \le \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + 2\lambda^2\alpha_n$$
$$= \left\|x_n - x_{n+1}\right\|^2 + 2\left\langle x_n - x_{n+1}, x_{n+1} - x^*\right\rangle + 2\lambda^2\alpha_n \le \left\|x_n - x_{n+1}\right\|^2 + 2\lambda\left\|x_n - x_{n+1}\right\| + 2\lambda^2\alpha_n.$$

By using condition (C1), (C4) and (3.12), we have

$$\lim_{n\to\infty}\left\|x_n - y_n\right\| = 0. \tag{3.15}$$

From (3.13) and (3.15), we obtain

$$\left\|y_n - K_n z_n\right\| \le \left\|y_n - x_n\right\| + \left\|x_n - K_n z_n\right\| \to 0 \quad (\text{as } n\to\infty). \tag{3.16}$$

From (3.11) and (3.13), we also obtain

$$\lim_{n\to\infty}\left\|z_n - K_n z_n\right\| = 0. \tag{3.17}$$

Step 4. We shall show that

$$\limsup_{n\to\infty}\left\langle\left(\gamma f - A\right)q, x_n - q\right\rangle \le 0,$$

where q is the unique solution of the variational inequality $\left\langle\left(\gamma f - A\right)q, p - q\right\rangle \le 0,$

$$\forall p \in \cap_{i=1}^{N}\mathrm{Fix}\left(T_i\right)\cap MEP\left(F,\varphi\right)\cap VI\left(C,B\right).$$

Let $Q = P_{\cap_{i=1}^{N}\mathrm{Fix}(T_i)\cap MEP(F,\varphi)\cap VI(C,B)}$. Observe that $Q\left(\gamma f + \left(I - A\right)\right)$ is a contraction. Indeed, for all x, $y \in H$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$, we have

$$\begin{aligned}&\left\|Q\left(\gamma f + \left(I-A\right)\right)x - Q\left(\gamma f + \left(I-A\right)\right)y\right\|\\ &\le \left\|\left(\gamma f + \left(I-A\right)\right)x - \left(\gamma f + \left(I-A\right)\right)y\right\|\\ &\le \gamma\left\|f\left(x\right) - f\left(y\right)\right\| + \left\|I-A\right\|\left\|x-y\right\|\\ &\le \gamma\alpha\left\|x-y\right\| + \left(1-\overline{\gamma}\right)\left\|x-y\right\|\\ &= \left(\gamma\alpha + 1 - \overline{\gamma}\right)\left\|x-y\right\| \le \left\|x-y\right\|.\end{aligned}$$

Banach's Contraction Mapping Principle guarantees that $Q\left(\gamma f + \left(I-A\right)\right)$ has a unique fixed point, say $q \in H$. That is,

$$\begin{aligned}q &= Q\left(\gamma f + \left(I-A\right)\right)q\\ &= P_{\cap_{i=1}^{N}\mathrm{Fix}(T_i)\cap MEP(F,\varphi)\cap VI(C,B)}\left(\gamma f + \left(I-A\right)\right)q,\end{aligned}$$

by (1.1) we obtain that $\left\langle\left(\gamma f - A\right)q, p - q\right\rangle \le 0$ for all

$$p \in \cap_{i=1}^{N}\mathrm{Fix}\left(T_i\right)\cap MEP\left(F,\varphi\right)\cap VI\left(C,B\right).$$

Next, we show that

$$\limsup_{n\to\infty}\left\langle\left(\gamma f - A\right)q, y_n - q\right\rangle \le 0.$$

To see this, we choose a subsequence $\left\{y_{n_i}\right\}$ of $\left\{y_n\right\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\left\langle\left(\gamma f - Aq\right), y_n - q\right\rangle\\ &= \lim_{i\to\infty}\left\langle\left(\gamma f - A\right)q, y_{n_i} - q\right\rangle.\end{aligned}$$

Since $\left\{y_{n_i}\right\}$ is bounded, there exists $\left\{y_{n_{i_j}}\right\}$ a subsequence of $\left\{y_{n_i}\right\}$ which converges weakly to *p*. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup p$. Claim that

$$p \in \cap_{i=1}^{N}\mathrm{Fix}\left(T_i\right)\cap MEP\left(F,\varphi\right)\cap VI\left(C,B\right).$$

First, we prove $p \in MEP\left(F,\varphi\right)$.

Since $y_n = T_{r_n}x_n \in \mathrm{dom}\varphi$, we have

$$F\left(y_n,\eta\right) + \varphi\left(\eta\right) - \varphi\left(y_n\right) + \frac{1}{r_n}\left\langle\eta - y_n, y_n - x_n\right\rangle \ge 0,$$

for all $\eta \in C$. It follows from Lemma 2.6 (A2) that

$$\varphi\left(\eta\right) - \varphi\left(y_n\right) + \frac{1}{r_n}\left\langle\eta - y_n, y_n - x_n\right\rangle \ge F\left(\eta, y_n\right), \quad \forall\eta\in C$$

and hence

$$\varphi\left(\eta\right) + \varphi\left(y_n\right) - \left\langle\eta - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F\left(\eta, y_{n_i}\right), \quad \forall\eta\in C.$$

Since $\frac{y_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $y_{n_i} \rightharpoonup p$ together with the lower semicontinuity of φ and Lemma 2.6 (A4), we have $F\left(\eta, p\right) + \varphi\left(p\right) - \varphi\left(\eta\right) \le 0$ for all $\eta \in C$. For *t* with $0 < t \le 1$ and $\eta \in C$, let $\eta_t = t\eta + \left(1-t\right)p$. Since $\eta \in C$ and $p \in C$, we have $\eta_t \in C$ and hence $F\left(\eta_t, p\right) + \varphi\left(p\right) - \varphi\left(\eta_t\right) \le 0$. So, from Lemma 2.6 (A1), (A4) and the convexity of φ we have

$$\begin{aligned}0 &= F\left(\eta_t,\eta_t\right) + \varphi\left(\eta_t\right) - \varphi\left(\eta_t\right)\\ &\le tF\left(\eta_t,\eta\right) + \left(1-t\right)F\left(\eta_t, p\right)\\ &\quad + t\varphi\left(\eta\right) + \left(1-t\right)\varphi\left(p\right) - \varphi\left(\eta_t\right)\\ &\le t\left(F\left(\eta_t,\eta\right) + \varphi\left(\eta\right) - \varphi\left(\eta_t\right)\right).\end{aligned}$$

Dividing by *t*, we get $F\left(\eta_t,\eta\right) + \varphi\left(\eta\right) - \varphi\left(p\right) \ge 0.$

Letting $t \to 0$, it follows from Lemma 2.6 (A3) and the lower semicontinuity of φ that $F(p,\eta)+\varphi(\eta)-\varphi(p)\ge 0$ for all $\eta\in C$ and hence $p\in MEP(F,\varphi)$. Next, we prove $p\in \mathrm{Fix}(K)$. To see this, we observe that we may assume (by passing to a further subsequence if necessary) $\lambda_{n_i,k}\to\lambda_k\in(0,1)$ $(k=1,2,\cdots,N)$. Let K be the K-mapping generated by $T_1,T_2,\cdots,T_N$ and $\lambda_1,\lambda_2,\cdots,\lambda_N$. Then by Lemma 2.5, we have, for every $x\in C$,

$$K_{n_i}x\to Kx. \tag{3.18}$$

every $x\in C$,

Moreover, from Lemma 2.4 it follows that

$$\mathrm{Fix}(K)=\cap_{i=1}^{N}\mathrm{Fix}(T_i).$$

Suppose for contradiction $p\notin \mathrm{Fix}(K)$. Then $p\ne Kp$. Since Hilbert space are Opial's spaces and

$$p\in MEP(F,\varphi)=\mathrm{Fix}(T_{r_n}),$$

from (3.17) and (3.18), we have

$$\begin{aligned}
&\liminf_{i\to\infty}\|z_{n_i}-p\|<\liminf_{i\to\infty}\|z_{n_i}-Kp\|\\
&\le\liminf_{i\to\infty}\left(\|z_{n_i}-K_{n_i}z_{n_i}\|+\|K_{n_i}z_{n_i}-K_{n_i}p\|+\|K_{n_i}p-Kp\|\right)\\
&\le\liminf_{i\to\infty}\|K_{n_i}z_{n_i}-K_{n_i}p\|\le\liminf_{i\to\infty}\|z_{n_i}-p\|,
\end{aligned}$$

which derives a contradiction. Thus, we have $p\in\mathrm{Fix}(K)$. It follows from

$$\mathrm{Fix}(K)=\cap_{i=1}^{N}\mathrm{Fix}(T_i)$$

that

$$p\in\cap_{i=1}^{N}\mathrm{Fix}(T_i).$$

Next, we prove $p\in VI(C,B)$. Put

$$Tw_1=\begin{cases}Bw_1+N_Cw_1, & w_1\in C,\\ \varnothing, & w_1\notin C.\end{cases}$$

Since B is relaxed (u,v)-cocoercive and condition (C5), we have

$$\begin{aligned}\langle Bx-By,x-y\rangle&\ge(-u)\|Bx-By\|^2+v\|x-y\|^2\\&\ge\left(v-u\mu^2\right)\|x-y\|^2\ge0,\end{aligned}$$

which yields that B is monotone. Thus T is maximal monotone. Let $(w_1,w_2)\in G(T)$. Since $w_2-Bw_1\in N_Cw_1$ and $z_n\in C$, we have

$$\langle w_1-z_n,w_2-Bw_1\rangle\ge0.$$

On the other hand, from $z_n=P_C(I-s_nB)y_n$ and (1.1), we have

$$\langle w_1-z_n,z_n-(I-s_nB)y_n\rangle\ge0$$

and hence

$$\left\langle w_1-z_n,\frac{z_n-y_n}{s_n}+By_n\right\rangle\ge0.$$

It follows that

$$\begin{aligned}
&\langle w_1-z_{n_i},w_2\rangle\ge\langle w_1-z_{n_i},Bw_1\rangle\\
&\ge\langle w_1-z_{n_i},Bw_1\rangle-\left\langle w_1-z_{n_i},\frac{z_{n_i}-y_{n_i}}{s_{n_i}}+By_{n_i}\right\rangle\\
&=\left\langle w_1-z_{n_i},Bw_1-By_{n_i}-\frac{z_{n_i}-y_{n_i}}{s_{n_i}}\right\rangle\\
&=\langle w_1-z_{n_i},Bw_1-Bz_{n_i}\rangle+\langle w_1-z_{n_i},Bz_{n_i}-By_{n_i}\rangle\\
&\quad-\left\langle w_1-z_{n_i},\frac{z_{n_i}-y_{n_i}}{s_{n_i}}\right\rangle\\
&\ge\langle w_1-z_{n_i},Bz_{n_i}-By_{n_i}\rangle-\left\langle w_1-z_{n_i},\frac{z_{n_i}-y_{n_i}}{s_{n_i}}\right\rangle,
\end{aligned}$$

which together with (3.16), (3.17) and B is Lipschitz continuous implies that $\langle w_1-p,w_2\rangle\ge0$. We have $p\in T^{-1}0$ and hence $p\in VI(C,B)$. That is,

$$p\in\cap_{i=1}^{N}\mathrm{Fix}(T_i)\cap MEP(F,\varphi)\cap VI(C,B).$$

It follows from the variational inequality $\langle(\gamma f-A)q,p-q\rangle\le0$ for all

$$p\in\cap_{i=1}^{N}\mathrm{Fix}(T_i)\cap MEP(F,\varphi)\cap VI(C,B)$$

that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle(\gamma f-A)q,y_n-q\rangle\\
&=\lim_{i\to\infty}\langle(\gamma f-A)q,y_{n_i}-q\rangle\\
&=\langle(\gamma f-A)q,p-q\rangle\le0.
\end{aligned}\tag{3.19}$$

Using (3.16) and (3.19), we have

$$\limsup_{n\to\infty}\langle(\gamma f-A)q,K_nz_n-q\rangle\le0. \tag{3.20}$$

Moreover, from (3.15) and (3.19), we have

$$\limsup_{n\to\infty}\langle(\gamma f-A)q,x_n-q\rangle\le0. \tag{3.21}$$

Step 5. Finally, we will show that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to q.

Since $z_n=P_C(I-s_nB)y_n$, using (1.19), (3.1), (3.2) and Lemma 2.8, we have

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \left\|\alpha_n\gamma f(K_nx_n)+\beta_nx_n+\left((1-\beta_n)I-\alpha_nA\right)K_nz_n-q\right\|^2\\
&= \left\|\left((1-\beta_n)I-\alpha_nA\right)(K_nz_n-q)+\beta_n(x_n-q)+\alpha_n\left(\gamma f(K_nx_n)-Aq\right)\right\|^2\\
&\le \left\|\left((1-\beta_n)I-\alpha_nA\right)(K_nz_n-q)+\beta_n(x_n-q)\right\|^2+\alpha_n^2\left\|\gamma f(K_nx_n)-Aq\right\|^2\\
&\quad +2\beta_n\alpha_n\left\langle x_n-q,\gamma f(K_nx_n)-Aq\right\rangle+2\alpha_n\left\langle\left((1-\beta_n)I-\alpha_nA\right)(K_nz_n-q),\gamma f(K_nx_n)-Aq\right\rangle\\
&\le \left((1-\beta_n-\alpha_n\bar{\gamma})\|K_nz_n-q\|+\beta_n\|x_n-q\|\right)^2+\alpha_n^2\left\|\gamma f(K_nx_n)-Aq\right\|^2+2\beta_n\gamma\alpha_n\left\langle x_n-q,f(K_nx_n)-f(q)\right\rangle\\
&\quad +2\beta_n\alpha_n\left\langle x_n-q,\gamma f(q)-Aq\right\rangle+2(1-\beta_n)\gamma\alpha_n\left\langle K_nz_n-q,f(K_nx_n)-f(q)\right\rangle\\
&\quad +2(1-\beta_n)\alpha_n\left\langle K_nz_n-q,\gamma f(q)-Aq\right\rangle-2\alpha_n^2\left\langle A(K_nz_n-q),\gamma f(K_nx_n)-Aq\right\rangle\\
&\le (1-\beta_n-\alpha_n\bar{\gamma})^2\|K_nz_n-q\|^2+\beta_n^2\|x_n-q\|^2+2(1-\beta_n-\alpha_n\bar{\gamma})\beta_n\|K_nz_n-q\|\|x_n-q\|+\alpha_n^2\left\|\gamma f(K_nx_n)-Aq\right\|^2\\
&\quad +2\beta_n\gamma\alpha_n\left\langle x_n-q,f(K_nx_n)-f(q)\right\rangle+2\beta_n\alpha_n\left\langle x_n-q,\gamma f(q)-Aq\right\rangle+2(1-\beta_n)\gamma\alpha_n\left\langle K_nz_n-q,f(K_nx_n)-f(q)\right\rangle\\
&\quad +2(1-\beta_n)\alpha_n\left\langle K_nz_n-q,\gamma f(q)-Aq\right\rangle-2\alpha_n^2\left\langle A(K_nz_n-q),\gamma f(K_nx_n)-Aq\right\rangle\\
&\le \left((1-\beta_n-\alpha_n\bar{\gamma})^2+\beta_n^2+2(1-\beta_n-\alpha_n\bar{\gamma})\beta_n\right)\|x_n-q\|^2+\alpha_n^2\left\|\gamma f(K_nx_n)-Aq\right\|^2+2\beta_n\gamma\alpha_n\left\langle x_n-q,f(K_nx_n)-f(q)\right\rangle\\
&\quad +2\beta_n\alpha_n\left\langle x_n-q,\gamma f(q)-Aq\right\rangle+2(1-\beta_n)\gamma\alpha_n\left\langle K_nz_n-q,f(K_nx_n)-f(q)\right\rangle\\
&\quad +2(1-\beta_n)\alpha_n\left\langle K_nz_n-q,\gamma f(q)-Aq\right\rangle-2\alpha_n^2\left\langle A(K_nz_n-q),\gamma f(K_nx_n)-Aq\right\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &\le \left((1-\beta_n-\alpha_n\bar{\gamma})^2+\beta_n^2+2(1-\beta_n-\alpha_n\bar{\gamma})\beta_n+2\gamma\alpha_n\alpha\right)\|x_n-q\|^2+\alpha_n^2\left\|\gamma f(K_nx_n)-Aq\right\|^2\\
&\quad +2\beta_n\alpha_n\left\langle x_n-q,\gamma f(q)-Aq\right\rangle+2(1-\beta_n)\alpha_n\left\langle K_nz_n-q,\gamma f(q)-Aq\right\rangle-2\alpha_n^2\left\langle A(K_nz_n-q),\gamma f(K_nx_n)-Aq\right\rangle\\
&= \left(1-2\alpha_n\bar{\gamma}+\alpha_n^2\bar{\gamma}^2+2\alpha\gamma\alpha_n\right)\|x_n-q\|^2+\alpha_n^2\left\|\gamma f(K_nx_n)-Aq\right\|^2+2\beta_n\alpha_n\left\langle x_n-q,\gamma f(q)-Aq\right\rangle\\
&\quad +2(1-\beta_n)\alpha_n\left\langle K_nz_n-q,\gamma f(q)-Aq\right\rangle-2\alpha_n^2\left\langle A(K_nz_n-q),\gamma f(K_nx_n)-Aq\right\rangle\\
&\le \left(1-2(\bar{\gamma}-\alpha\gamma)\alpha_n\right)\|x_n-q\|^2+\alpha_n^2\bar{\gamma}^2\|x_n-q\|^2+\alpha_n^2\left\|\gamma f(K_nx_n)-Aq\right\|^2+2\beta_n\alpha_n\left\langle x_n-q,\gamma f(q)-Aq\right\rangle\\
&\quad +2(1-\beta_n)\alpha_n\left\langle K_nz_n-q,\gamma f(q)-Aq\right\rangle+2\alpha_n^2\left\|A(K_nz_n-q)\right\|\cdot\left\|\gamma f(K_nx_n)-Aq\right\|\\
&= \left(1-2(\bar{\gamma}-\alpha\gamma)\alpha_n\right)\|x_n-q\|^2+\alpha_n^2\left(\bar{\gamma}^2\|x_n-q\|^2+\left\|\gamma f(K_nx_n)-Aq\right\|^2+2\left\|A(K_nz_n-q)\right\|\cdot\left\|\gamma f(K_nx_n)-Aq\right\|\right)\\
&\quad +2\beta_n\alpha_n\left\langle x_n-q,\gamma f(q)-Aq\right\rangle+2(1-\beta_n)\alpha_n\left\langle K_nz_n-q,\gamma f(q)-Aq\right\rangle\\
&= \left(1-2(\bar{\gamma}-\alpha\gamma)\alpha_n\right)\|x_n-q\|^2+\alpha_n\left(\alpha_n\left(\bar{\gamma}^2\|x_n-q\|^2\right)+\left\|\gamma f(K_nx_n)-Aq\right\|^2\right.\\
&\quad \left.+2\left\|A(K_nz_n-q)\right\|\cdot\left\|\gamma f(K_nx_n)-Aq\right\|\right)+2\beta_n\left\langle x_n-q,\gamma f(q)-Aq\right\rangle+2(1-\beta_n)\left\langle K_nz_n-q,\gamma f(q)-Aq\right\rangle.
\end{aligned}
$$

Since $\{x_n\}$, $\{f(K_nx_n)\}$ and $\{K_nz_n\}$ are bounded, we can take a constant $\xi>0$ such that

$$
\begin{aligned}
\xi &\ge \bar{\gamma}^2\|x_n-q\|^2+\left\|\gamma f(K_nx_n)-Aq\right\|^2\\
&\quad +2\left\|A(K_nz_n-q)\right\|\cdot\left\|\gamma f(K_nx_n)-Aq\right\|
\end{aligned}
$$

for all $n\ge 0$. It then follows that

$$
\|x_{n+1}-q\|^2\le\left(1-2(\bar{\gamma}-2\gamma)\alpha_n\right)\|x_n-q\|^2+\alpha_n\zeta_n, \quad (3.22)
$$

where

$$
\begin{aligned}
\zeta_n &= 2\beta_n\left\langle x_n-q,\gamma f(q)-Aq\right\rangle\\
&\quad +2(1-\beta_n)\left\langle K_nz_n-q,\gamma f(q)-Aq\right\rangle+\alpha_n\xi.
\end{aligned}
$$

By using (3.20), (3.21) and condition (C1), we get

$$
\limsup_{n\to\infty}\zeta_n\le 0.
$$

Now applying Lemma 2.1 to (3.22) concludes that $x_n\to q$ as $n\to\infty$. Finally, noticing

$$
\|y_n-q\|=\left\|T_{r_n}x_n-T_{r_n}q\right\|\le\|x_n-q\|
$$

we also conclude that $y_n\to q$ as $n\to\infty$. This completes the proof.

4. Applications

In this section, by Theorem 3.1, we can obtain some new and interesting strong convergence theorems. Now we give some examples as follows:

Let $T_i = I$ for all $i = 1,2,\cdots,N$ and setting $\gamma = 1$, $A = I$ and $f := x$ in Theorem 3.1, we obtain the following result.

Corollary 4.1. Let *H* be a real Hilbert space, *C* a nonempty closed convex subset of *H*, *F* a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies (A1)-(A4), $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ a proper lower semicontinuous and convex function and $T_1,T_2,\cdots,T_N$ a finite family of nonexpansive mappings of *C* into *H* such that the common fixed points set $MEP(F,\varphi)\cap VI(C,B)\neq\varnothing$. Assume that either (B1) or (B2) holds and x is an arbitrary point in *C*. Let $\{x_n\}$ and $\{y_n\}$, be sequences generated by $x_1 \in H$ and $\forall n \ge 1$,

$$\begin{cases} F(y_n,\eta)+\varphi(\eta)-\varphi(y_n) & \\ \qquad +\dfrac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \ge 0, & \forall \eta\in C, \\ x_{n+1} = \alpha_n x + \beta_n x_n + (1-\alpha_n-\beta_n)P_C(I - s_n B)y_n, & \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}\subset(0,1)$, $\{r_n\}\subset(0,\infty)$, $\{s_n\}\subset[0,\infty)$ satisfying the conditions (C1)-(C5) in Theorem 3.1. Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to a point

$$q\in MEP(F,\varphi)\cap VI(C,B),$$

where

$$q = P_{MEP(F,\varphi)\cap VI(C,B)}(x).$$

Setting $\gamma = 1$, $A = I$, $f := x$ and $\{s_n\} = 0$ for all *n* in Theorem 3.1, we obtain the following result.

Corollary 4.2. Let *H* be a real Hilbert space, *C* a nonempty closed convex subset of *H*, *F* a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies (A1)-(A4), $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ a proper lower semicontinuous and convex function and $T_1,T_2,\cdots,T_N$ a finite family of nonexpansive mappings of *C* into *H* such that the common fixed points set $\cap_{i=1}^N \mathrm{Fix}(T_i)\cap MEP(F,\varphi)\neq\varnothing$. Let K_n and *K* be the *K*-mappings defined by (1.16) and (1.17), respectively. Assume that either (B1) or (B2) holds and *x* is an arbitrary point in *C*. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and $\forall n \ge 1$,

$$F(y_n,\eta)+\varphi(\eta)-\varphi(y_n)+\frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \ge 0,$$
$$\forall \eta\in C,$$
$$x_{n+1} = \alpha_n x + \beta_n x_n + (1-\alpha_n-\beta_n)K_n y_n,$$

where $\lambda_1,\lambda_2,\cdots,\lambda_N$ are real numbers such that $0<\lambda_i<1$ for every $i=1,2,\cdots,N-1$ and $0<\lambda_N\le 1, \lambda_{n,i}\to\lambda_i\ (i=1,\cdots,N)$ and $\{\alpha_n\}$, $\{\beta_n\}\subset(0,1)$, $\{r_n\}\subset(0,\infty)$, $\{s_n\}\subset[0,\infty)$ satisfying the conditions (C1), (C3), (C4) and (C6) in Theorem 3.1. Then, $\{x_n\}$, and $\{y_n\}$ converge strongly to a point

$$q\in\cap_{i=1}^N \mathrm{Fix}(T_i)\cap MEP(F,\varphi),$$

where

$$q = P_{\cap_{i=1}^N \mathrm{Fix}(T_i)\cap MEP(F,\varphi)}(x).$$

Finally as applications, we will utilize the results presented in this paper to study the following optimization problem:

$$\min\nolimits_{\eta\in C}\varphi(\eta), \tag{4.1}$$

where *C* is a nonempty bounded closed convex subset of a Hilbert space and $\varphi : C\to\mathbb{R}\cup\{+\infty\}$ is a proper lower semicontinuous and convex function. We denote by $\mathrm{Argmin}(\varphi)$ the set of solutions in (4.1). Let $F(x,y) = 0$ for all $x,y\in C$ in Corollary 4.1, then

$$MEP(F,\varphi) = \mathrm{Argmin}(\varphi).$$

It follows from Corollary 4.1 that the sequence $\{x_n\}$ generated by $x_1\in H$ and $\forall n\ge 1$,

$$\begin{cases} y_n = \mathrm{argmin}_{\eta\in C}\left\{\varphi(\eta)+\dfrac{1}{2r_n}\|\eta - x_n\|^2\right\}, \\ x_{n+1} = \alpha_n x + \beta_n x_n + (1-\alpha_n-\beta_n)P_C(I - s_n B)y_n, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}\subset(0,1)$, $\{r_n\}\subset(0,\infty)$ and $\{s_n\}\subset[0,\infty)$ satisfying the conditions (C1)-(C5) in Theorem 3.1. Then the sequence $\{x_n\}$ converges strongly to a point

$$q\in\mathrm{Argmin}(\varphi)\cap VI(C,B),$$

where

$$q = P_{\mathrm{Argmin}(\varphi)\cap VI(C,B)}(x).$$

Let $T_i = I$ for all $i=1,2,\cdots,N$ and $F(x,y)=0$ for all $x,y\in C$ in Corollary 4.2, then $MEP(F,\varphi) = \mathrm{Argmin}(\varphi)$. It follows from Corollary 4.2 that the iterative sequence $\{x_n\}$ generated by $x_1\in H$ and $\forall n\ge 1$,

$$\begin{cases} y_n = \mathrm{argmin}_{\eta\in C}\left\{\varphi(\eta)+\dfrac{1}{2r_n}\|\eta - x_n\|^2\right\}, \\ x_{n+1} = \alpha_n x + \beta_n x_n + (1-\alpha_n-\beta_n)y_n, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}\subset(0,1)$ and $\{r_n\}\subset(0,\infty)$ satisfying the conditions (C1), (C3) and (C4) in Theorem 3.1. Then the sequence $\{x_n\}$ converges strongly to a point $q\in\mathrm{Argmin}(\varphi)$, where $q = P_{\mathrm{Argmin}(\varphi)}(x)$.

Remark 4.3. The algorithms (4.2) and (4.3) are variants of the proximal method for optimization problems introduced and studied by Martinet [41], Rockafellar [42], Ferris [43] and many others.

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