Advances in Pure Mathematics, 2013, 3, 70-77
http://dx.doi.org/10.4236/apm.2013.31009 Published Online January 2013 (http://www.scirp.org/journal/apm)

Existence of Multiple Positive Solutions for n^{th} Order Two-Point Boundary Value Problems on Time Scales

Kapula Rajendra Prasad[1], Penugurthi Murali[1], Nadakuduti Veera Venkata Satya Suryanarayana[2]
[1]Department of Applied Mathematics, Andhra University, Visakhapatnam, India
[2]Department of Mathematics, VITAM College of Engineering, Visakhapatnam, India
Email: rajendra92@rediffmail.com, murali_uoh@yahoo.co.in, suryanarayana_nvvs@yahoo.com

Received July 14, 2012; revised September 18, 2012; accepted September 26, 2012

ABSTRACT

We consider the n^{th} order nonlinear differential equation on time scales

$$y^{\Delta^n}(t)+f(t,y(t))=0,\quad t\in[a,b],$$

subject to the right focal type two-point boundary conditions

$$y^{\Delta^i}(a)=0,\quad 0\le i\le n-2$$

$$y^{\Delta^p}(\sigma^{n-p}(b))=0,\quad (1\le p\le n-1,\text{but fixed}).$$

We establish a criterion for the existence of at least one positive solution by utilizing Krasnosel'skii fixed point theorem. And then, we establish the existence of at least three positive solutions by utilizing Leggett-Williams fixed point theorem.

Keywords: Time Scale; Dynamical Equation; Positive Solution; Cone; Boundary Value Problem

1. Introduction

The study of the existence of positive solutions of boundary value problems (BVPs) for higher order differential equations on time scales has gained prominence and it is a rapidly growing field, since it arises, especially for higher order differential equations on time scales arise naturally in technical applications. Meyer [1], strictly speaking, boundary value problems for higher order differential equation on time scales are a particular class of interface problems. One example in which this is exhibited is given by Keener [2] in determining the speed of a flagellate protozoan in a viscous fluid. Another particular case of a boundary value problem for a higher order differential equation on time scales arising as an interface problem is given by Wayner, *et al*. [3] in dealing with a study of perfectly wetting liquids. In these applied settings, only positive solutions are meaningful. By a time scale we mean a nonempty closed subset of $\mathbb{R}$. For the time scale calculus and notation for delta differentiation, integration, as well as concepts for dynamic equation on time scales we refer to the introductory book on time scales by Bohner and Peterson [4], and denote the time scales by the symbol $\mathbb{T}$.

By an interval we mean the intersection of the real interval with a given time scale. The existence of positive solutions for BVPs has been studied by many authors, first for differential equations, then finite difference equations, and recently, unifying results for dynamic equations. We list some papers, Erbe and Wang [5], and Eloe and Henderson [6,7], Atici and Guseinor [8], and Anderson and Avery [9], and Avery and Peterson [10], Agarwal, Regan and Wang [11], Deimling [12], Gregus [13] Guo and Lakshmikantham [14], Henderson and Ntouyas [15], Hopkins [16] and Li [17]. Recently, in 2008, Moustafa Shehed [18] obtained at least one positive solution to the boundary value problem

$$y^{(n)}+\lambda a(t)f(y(t))=0,0<t<1,$$

$$y(0)=y''(0)=\cdots=y^{(n-1)}=y'(1)=0.$$

$$y(0)=y'(0)=\cdots=y^{(n-2)}=y'(1)=0.$$

$$y(0)=y'(0)=\cdots=y^{(n-2)}=y''(1)=0.$$

This paper considers the existence of positive solutions to n^{th} order nonlinear differential equation on time scales

$$y^{\Delta^n}(t)+f(t,y(t))=0, t\in[a,b], a<b \quad (1)$$

subject to the right focal type boundary conditions

$$y^{\Delta^i}(a)=0, 0\le i\le n-2 \quad (2)$$

$$y^{\Delta^p}\left(\sigma^{n-p}(b)\right)=0, (1\le p\le n-1, \text{but fixed}). \quad (3)$$

These boundary conditions include different types of right focal boundary conditions.

We make the following assumptions throughout:

(A1) $f:\left[a,\sigma^n(b)\right]\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous with respect to y, where $\mathbb{R}^+$ is nonnegative real numbers,

(A2) The point t in $\left[a,\sigma^n(b)\right]$ is not left dense and right scattered at the same time.

Define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by

$$f_0=\lim_{y\to0^+}\lim_{t\in\left[a,\sigma^n(b)\right]}\frac{f(t,y)}{y},$$

$$f^0=\lim_{y\to0^+}\max_{t\in\left[a,\sigma^n(b)\right]}\frac{f(t,y)}{y},$$

$$f_\infty=\lim_{y\to\infty}\lim_{t\in\left[a,\sigma^n(b)\right]}\frac{f(t,y)}{y},$$

and

$$f^\infty=\lim_{y\to\infty}\max_{t\in\left[a,\sigma^n(b)\right]}\frac{f(t,y)}{y}.$$

This paper is organized as follows; In Section 2, we estimate the bounds for the Greens function which are needed for later discussions. In Section 3, we establish a criteria for the existence of at least one positive solution for the BVP by using Krasnosel'skii fixed point theorem. In Section 4, we establish the existence of at least three positive solutions for the BVP by using Leggett-Williams fixed point theorem. Finally, as an application, we give some examples to demonstrate our result.

2. Green's Function and Bounds

In this section, first we state a Lemma to compute delta derivatives for t^n, next, construct a Green's function for homogeneous two point BVP $-y^{\Delta^n}=0$ with (2), (3) and estimate the bounds to the Green's function.

Lemma 2.1. *Let* $n\in\mathbb{N}$, *define a function* $f:T\to\mathbb{R}$ *by* $f(t)=t^n$, *if we assume that the conditions* (*A*2) *and* (*A*3) *are satisfied*, *then*

$$f^{\Delta^m}(t)=\frac{n!}{(n-m)!}\left(\sum_{r=0}^{n-m}\omega_t\right) \quad (4)$$

$$\omega_t=t^{n-m-r}\sum_{n_1,n_2,\cdots,n_m\in\mathbb{N}\cup\{0\}}^{n_1,n_2,\cdots,n_m=r}\left[\prod_{i=1}^{m}\left(\sigma^i(t)\right)^{n_i}\right],\forall t\in\mathbb{T}^{k^m}$$

holds for all $m\le n\in\mathbb{N}$, *where* $\sum_{n_1+n_2+\cdots+n_m\in\mathbb{N}\cup\{0\}}^{n_1+n_2+\cdots+n_m=r}$ *is the set of all distinct combinations of* $\{n_1,n_2,\cdots,n_m\}$ *such that the sum is equal to given* r.

Proof see [19].

We denote

$$\omega(t,s)=\sum_{r=0}^{n-p-1}\left[(t-s)^{n-p-1-r}\,\bar{\omega}\right]$$

$$\bar{\omega}=\sum_{n_1,n_2,\cdots,n_p\in\mathbb{N}\cup\{0\}}^{n_1,n_2,\cdots,n_p=r}\left[\prod_{i=1}^{p}\left(\sigma^i(t-s)\right)^{n_i}\right]$$

Theorem 2.2. *Green's function for the homogeneous BVP*

$$-y^{\Delta^n}=0,$$

with the boundary conditions (2), (3) *is given by*

$$G(t,s)=\begin{cases}G_1(t,s), & t\le s\\ G_1(t,s)-\dfrac{1}{(n-1)!}\displaystyle\prod_{i=1}^{n-1}\left(t-\sigma^i(s)\right), & \sigma(s)\le t\end{cases}$$

where

$$G_1(t,s)=\frac{\omega\left(\sigma^{n-p}(b),\sigma^i(s)\right)\prod_{i=1}^{n-1}\left(t-\sigma^{i-1}(a)\right)}{(n-1)!\,\omega\left(\sigma^{n-p}(b),\sigma^{i-1}a\right)}$$

for all $(t,s)\in\left[a,\sigma^n(b)\right]\times[a,b]$.

Proof: It is easy to check that the BVP $-y^{\Delta^n}=0$ with the boundary conditions (2) and (3) has only trivial solution. Let $y(t,s)$ be the Cauchy function for $-y^{\Delta^n}=0$, and is given by

$$y(t,s)=\frac{-1}{(n-1)!}\prod_{i=1}^{n-1}\left(t-\sigma^i(s)\right).$$

For each fixed $s\in[a,b]$, let $u(\cdot,s)$ be the unique solution of the BVP

$$-u^{\Delta^n}(\cdot,s)=0, u^{\Delta^i}(a,s)=0, 0\le i\le n-2$$

and

$$u^{\Delta^p}\left(\sigma^{n-p}(b),s\right)=-y^{\Delta^p}\left(\sigma^{n-p}(b),s\right).$$

$$y^{\Delta^p}(t,s)\Big|_{t=\sigma^{n-p}(b)}=\frac{-1}{(n-p-1)!}\omega\left(\sigma^{n-p}(b),\sigma^i(s)\right).$$

Since

$$y_1(t)=1, y_2(t)=\int_a^t\Delta\tau,\cdots,y_n(t)=\underbrace{\int_a^t\cdots\int_{\sigma^{n-2}(a)}^t}_{(n-1)\text{times}}\Delta\tau,\cdots,\Delta\tau$$

are the solutions of $-u^{\Delta^n}=0$,

$$u(t,s)$$

$$=\alpha_1(s)\cdot1+\alpha_2(s)\cdot\int_a^t\Delta\tau+\cdots+\alpha_n(s)\underbrace{\int_a^t\cdots\int_{\sigma^{n-2}(a)}^t}_{(n-1)\text{times}}\Delta\tau\cdots\Delta\tau$$

By using boundary conditions, $u^{\Delta^i}(a)=0,\ 0\le i\le n-2,$ we have $\alpha_1=\alpha_2=\cdots=\alpha_{n-1}=0$. Therefore

$$u(t,s)=\alpha_n\underbrace{\int_a^t\cdots\int_{\sigma^{n-2}(a)}^t}_{(n-1)\text{times}}\Delta\tau\cdots\Delta\tau=\alpha_n\prod_{i=1}^{n-1}\left(t-\sigma^{i-1}(a)\right).$$

Since,

$$u^{\Delta^p}\left(\sigma^{n-p}(b),s\right)=-y^{\Delta^p}\left(\sigma^{n-p}(b),s\right),$$

It follows that

$$\alpha_n=\frac{\omega\left(\sigma^{n-p}(b),\sigma^i(s)\right)}{(n-1)!\omega\left(\sigma^{n-p}(b),\sigma^{i-1}a\right)}.$$

Hence $G(t,s)$ has the form for $t\le s$,

$$G(t,s)=\frac{\omega\left(\sigma^{n-p}(b),\sigma^i(s)\right)\prod_{i=1}^{n-1}\left(t-\sigma^{i-1}(a)\right)}{(n-1)!\omega\left(\sigma^{n-p}(b),\sigma^{i-1}a\right)}.$$

And for $t\ge\sigma(s)$, $G(t,s)=y(t,s)+u(t,s)$. It follows that

$$G(t,s)=G_1(t,s)-\frac{1}{(n-1)!}\prod_{i=1}^{n-1}\left(t-\sigma^i(s)\right),$$

where

$$G_1(t,s)=\frac{\omega\left(\sigma^{n-p}(b),\sigma^i(s)\right)\prod_{i=1}^{n-1}\left(t-\sigma^{i-1}(a)\right)}{(n-1)!\omega\left(\sigma^{n-p}(b),\sigma^{i-1}a\right)}$$

□

Lemma 2.3. *For* $(t,s)\in\left[a,\sigma^n(b)\right]\times\left[a,\sigma(b)\right]$, *we have*

$$G(t,s)\le G\left(\sigma^n(b),s\right). \tag{5}$$

Proof: For $a\le t\le s\le\sigma^n(b)$, we have

$$G(t,s)=G_1(t,s)\le G_1\left(\sigma^n(b),s\right)=G\left(\sigma^n(b),s\right).$$

Similarly, for $a\le\sigma(s)\le t\le\sigma^n(b)$, we have $G(t,s)\le G\left(\sigma^n(b),s\right)$. Thus, we have

$$G(t,s)\le G\left(\sigma^n(b),s\right),$$

for all $(t,s)\in\left[a,\sigma^n(b)\right]\times\left[a,\sigma(b)\right]$. □

Lemma 2.4. *Let* $I=\left[\frac{3a+\sigma^n(b)}{4},\frac{a+3\sigma^n(b)}{4}\right]$. *For* $(t,s)\in I\times\left[a,\sigma(b)\right]$, *we have*

$$G(t,s)\ge\frac{1}{p\cdot 4^{n-1}}G\left(\sigma^n(b),s\right). \tag{6}$$

Proof: The Green's function $G(t,s)$ for the homogeneous BVP corresponding to (1)-(3) is positive on $\left(a,\sigma^n(b)\right)\times\left(a,\sigma(b)\right)$.

For $a\le t\le s<\sigma^n(b)$ and $t\in I$, we have

$$\frac{G(t,s)}{G\left(\sigma^n(b),s\right)}=\prod_{i=1}^{n-1}\frac{\left(t-\sigma^{i-1}(a)\right)}{\left(\sigma^n(b)-\sigma^{i-1}(a)\right)}\ge\frac{1}{4^{n-1}}.$$

Similarly, for $a\le\sigma(s)\le t<\sigma^n(b)$ and $t\in I$ we have

$$\begin{aligned}&\frac{G(t,s)}{G\left(\sigma^n(b),s\right)}\\&=\frac{G_1(t,s)-\frac{1}{(n-1)!}\prod_{i=1}^{n-1}\left(t-\sigma^i(s)\right)}{G_1\left(\sigma^n(b)t,s\right)-\frac{1}{(n-1)!}\prod_{i=1}^{n-1}\left(\sigma^n(b)-\sigma^i(s)\right)}\\&\ge\frac{1}{p}\prod_{i=1}^{n-2}\frac{\left(t-\sigma^{i-1}(a)\right)}{\left(\sigma^n(b)-\sigma^{i-1}(a)\right)}\\&\ge\frac{1}{p}\prod_{i=1}^{n-1}\frac{\left(t-\sigma^{i-1}(a)\right)}{\left(\sigma^n(b)-\sigma^{i-1}(a)\right)}\ge\frac{1}{p\cdot 4^{n-1}}.\end{aligned}$$

□

3. Existence of at Least One Positive Solution

In this section, we establish a criteria for the existence of at least one positive solution of the BVP (1)-(3). Let $y(t)$ be the solution of the BVP (1)-(3), and is given by

$$y(t)=\int_a^{\sigma(b)}G(t,s)f\left(s,y(s)\right)\Delta s, \tag{7}$$

for all $t\in\left[a,\sigma^n(b)\right]$.

Define $\beta=\left\{y:y\in C\left[a,\sigma^n(b)\right]\right\}$ with the norm

$$\|y\|=\max_{t\in\left[a,\sigma^n(b)\right]}\left|y(t)\right|.$$

Then $(\beta,\|\cdot\|)$ is a Banach space. Define a set κ by

$$\kappa=\left\{y\in\beta:y(t)\ge 0\text{ on }\left[a,\sigma^n(b)\right]\right.\\ \left.\text{and}\min_{t\in I}y(t)\ge\frac{1}{p\cdot 4^{n-1}}\|y\|\right\} \tag{8}$$

We define the operator $T:\kappa\to\beta$ by

$$(Ty)(t)=\int_a^{\sigma(b)}G(t,s)f\left(s,y(s)\right)\Delta s, \tag{9}$$

for all $t\in\left[a,\sigma^n(b)\right]$.

Theorem 3.1. (Krasnosel'skii) *Let* β *be a Banach space,* $K\subseteq\beta$ *be a cone, and suppose that* Ω_1, Ω_2 *are open subsets of* β *with* $0\in\Omega_1$ *and* $\bar{\Omega}_1\subset\Omega_2$. *Suppose further that* $T:K\cap\left(\bar{\Omega}_2\setminus\Omega_1\right)\to K$ *is completely continuous operator such that either*

1) $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$, or

2) $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$ holds. Then T has a fixed point in $K \cap \left(\bar{\Omega}_2 \setminus \Omega_1\right)$.

Theorem 3.2. *If* $f^0 = 0$ *and* $f_\infty = \infty$, *then the BVP* (1)-(3) *has at least one positive solution that lies in* κ.

Proof: We seek a fixed point of T in κ. We prove this by showing the conditions in Theorem 3.1 hold.

First, if $y \in \kappa$, then

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s) f(s,y(s)) \Delta s \\ &\le \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) f(s,y(s)) \Delta s,\end{aligned}$$

so that

$$\|Ty\| \le \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) f(s,y(s)) \Delta s.$$

Next, if $y \in \kappa$, then

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s) f(s,y(s)) \mathrm{d}s \\ &\ge \frac{1}{p \cdot 4^{n-1}} \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) f(s,y(s)) \Delta s \\ &\ge \frac{1}{p \cdot 4^{n-1}} \|Ty\|, t \in I.\end{aligned}$$

Hence, $T : \kappa \to \kappa$. Standard argument involving the Arzela-Ascoli theorem shows that T is completely continuous operator. Since $f^0 = 0$, there exist $\eta_1 > 0$ and $H_1 > 0$ such that $\max_{t \in \left[a,\sigma^n(b)\right]} \dfrac{f(t,y)}{y} \le \eta_1$, for $0 < y \le H_1$, and $\eta_1 \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) \Delta s \le 1$. Let us choose $y \in \kappa$ with $\|y\| = H_1$. Then, we have from Lemma 2.3,

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s) f(s,y(s)) \Delta s \\ &\le \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) f(s,y(s)) \Delta s \\ &\le \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) \eta_1 y(s) \Delta s \\ &\le \eta_1 \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) \|y\| \Delta s \le \|y\|, t \in \left[a,\sigma^n(b)\right]\end{aligned}$$

Therefore, $\|Ty\| \le \|y\|$. Hence, if we set

$$\Omega_1 = \left\{y \in \beta : \|y\| < H_1\right\}.$$

Then

$$\|Ty\| \le \|y\|, \text{for } y \in \kappa \cap \partial\Omega_1. \tag{10}$$

Since $f_\infty = \infty$, there exist η_2 and $\bar{H}_2 > 0$ such that $\min_{t \in \left[a,\sigma^n(b)\right]} \dfrac{f(t,y)}{y} \ge \eta_2$, for $y \ge \bar{H}_2$ and $\eta_2 \dfrac{1}{p^2 \cdot 4^{2(n-1)}} \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) \Delta s \ge 1$. If we set

$$H_2 = \max\left\{2H_1, p \cdot 4^{n-1} \bar{H}_2\right\},$$

and define

$$\Omega_2 = \left\{y \in \beta : \|y\| < H_2\right\}.$$

If $y \in \kappa \cap \partial\Omega_2$, so that $\|y\| = H_2$, then

$$\min_{t \in I} y(t) \ge \frac{1}{p \cdot 4^{n-1}} \|y\| \ge \bar{H}_2.$$

And we have

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s) f(s,y(s)) \Delta s \\ &\ge \int_a^{\sigma(b)} \frac{1}{p \cdot 4^{n-1}} G\left(\sigma^n(b),s\right) f(s,y(s)) \Delta s \\ &\ge \frac{1}{p \cdot 4^{n-1}} \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) \eta_2 y(s) \Delta s \\ &\ge \frac{\eta_2}{p^2 \cdot 4^{2(n-1)}} \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) \|y\| \Delta s \ge \|y\|.\end{aligned}$$

Thus, $\|Ty\| \ge \|y\|$, and so

$$\|Ty\| \ge \|y\|, \text{for } y \in \kappa \cap \partial\Omega_2. \tag{11}$$

An application of Theorem 3.1 to (10) and (11) yields a fixed point of T that lies in $\kappa \cap \left(\bar{\Omega}_2 \setminus \Omega_1\right)$. This fixed point is a solution of the BVP (1)-(3). □

Theorem 3.3. *If* $f_0 = \infty$ *and* $f^\infty = 0$, *then the BVP* (1)-(3) *has at least one positive solution that lies in* κ.

Proof: Let T be the cone preserving, completely continuous operator defined as in (9). Since $f_0 = \infty$, there exist $\bar{\eta}_1 > 0$ and $J_1 > 0$ such that $\min_{t \in \left[a,\sigma^n(b)\right]} \dfrac{f(t,y)}{y} \ge \bar{\eta}_1$, for $0 < y \le J_1$, and $\bar{\eta}_1 \dfrac{1}{p^2 \cdot 4^{2(n-1)}} \int_a^{\sigma(b)} G\left(\sigma^n(b),s\right) \Delta s \ge 1$. In this case, define $\Omega_1 = \left\{y \in \beta : \|y\| < J_1\right\}$. Then, for $y \in \kappa \cap \partial\Omega_1$, we have $f(s,y(s)) \ge \bar{\eta}_1 y(s)$, $s \in I$ and moreover,

$y(t) \geq \frac{1}{p \cdot 4^{n-1}} \|y\|, \ t \in I$. Thus

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s) f(s, y(s)) \Delta s \\ &\geq \int_a^{\sigma(b)} \frac{1}{p \cdot 4^{n-1}} G(\sigma^n(b), s) f(s, y(s)) \Delta s \\ &\geq \frac{1}{p \cdot 4^{n-1}} \int_a^{\sigma(b)} G(\sigma^n(b), s) \overline{\eta}_1 y(s) \Delta s \\ &\geq \frac{\overline{\eta}_1}{p^2 \cdot 4^{2(n-1)}} \int_a^{\sigma(b)} G(\sigma^n(b), s) \|y\| \Delta s \\ &\geq \|y\|.\end{aligned}$$

From which we have

$$\|Ty\| \geq \|y\|, \text{for } y \in \kappa \cap \partial\Omega_1. \tag{12}$$

It remains for us to consider $f^{\infty} = 0$, in this case, there exist $\overline{\eta}_2 > 0$ and $\overline{J}_2 > 0$ such that $\max_{t \in [a, \sigma^n(b)]} \frac{f(t,y)}{y} \leq \overline{\eta}_2$, for $y \geq \overline{J}_2$, and $\overline{\eta}_2 \int_a^{\sigma(b)} G(\sigma^n(b), s) \Delta s \leq 1$. There are two subcases.

Case (i) f is bounded. Suppose $L > 0$ is such that $f(t,y) \leq L$, for all $0 < y < \infty$.

Let $J_2 = \max\left\{2J_1, L \int_a^{\sigma(b)} G(\sigma^n(b), s) \Delta s\right\}$, and let

$$\Omega_2 = \{y \in \beta : \|y\| < J_2\}.$$

Then, for $y \in \kappa \cap \partial\Omega_2$, we have

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s) f(s, y(s)) \Delta s \\ &\leq \int_a^{\sigma(b)} G(\sigma^n(b), s) f(s, y(s)) \Delta s \\ &\leq L \int_a^{\sigma(b)} G(\sigma^n(b), s) \Delta s \\ &\leq \|y\|, t \in [a, \sigma^n(b)]\end{aligned}$$

and so

$$\|Ty\| \leq \|y\|, \text{for } y \in \kappa \cap \partial\Omega_2. \tag{13}$$

Case (ii) f is unbounded. Let $J_2 > \max\{2J_1, \overline{J}_2\}$ be such that $f(t,y) \leq f(t, J_2)$ for $0 < y \leq J_2$. Let

$$\Omega_2 = \{y \in \beta : \|y\| < J_2\}.$$

Choosing $y \in \kappa \cap \partial\Omega_2$,

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s) f(s, y(s)) \Delta s \\ &\leq \int_a^{\sigma(b)} G(\sigma^n(b), s) f(s, y(s)) \Delta s \\ &\leq \int_a^{\sigma(b)} G(\sigma^n(b), s) f(s, J_2) \Delta s \\ &\leq \int_a^{\sigma(b)} G(\sigma^n(b), s) \overline{\eta}_2 \|J_2\| \Delta s \\ &\leq J_2 = \|y\|, t \in [a, \sigma^n(b)]\end{aligned}$$

And so

$$\|Ty\| \leq \|y\|, \text{for } y \in \kappa \cap \partial\Omega_2. \tag{14}$$

An application of Theorem 3.1, to (12), (13) and (14) yields a fixed point of T that lies in $\kappa \cap (\overline{\Omega}_2 \setminus \Omega_1)$. This fixed point is a solution of the BVP (1)-(3). □

4. Existence of Multiple Positive Solutions

In this section, we establish the existence of at least three positive solutions to the BVP (1)-(3).

Let E be a real Banach space with cone P. A map $S : P \to [0, \infty)$ is said to be a nonnegative continuous concave functional on P, if S is continuous and

$$S(\lambda x + (1-\lambda) y) \geq \lambda S(x) + (1-\lambda) S(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Let a' and b' be two real numbers such that $0 < a' < b'$ and S be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_{a'} = \{y \in P : \|y\| < a'\},$$

$$P(S, a', b') = \{y \in P : a' \leq S(y), \|y\| \leq b'\}.$$

We now state the famous Leggett-Williams fixed point theorem.

Theorem 4.1. *See ref.* [20] *Let* $T : \overline{P}_{c'} \to \overline{P}_{c'}$ *be completely continuous and S be a nonnegative continuous concave functional on P such that* $S(y) \leq \|y\|$ *for all* $y \in \overline{P}_{c'}$. *Suppose that there exist* a', b', c', *and* d' *with* $0 < d' < a' < b' \leq c'$ *such that*

1) $\{y \in P(S, a', b') : S(y) > a'\} \neq \varnothing$ *and* $S(Ty) > a$ for $y \in P(S, a', b')$,

2) $\|Ty\| < d'$ *for* $\|y\| \leq d'$,

3) $S(Ty) > a'$ *for* $y \in P(S, a', c')$ *with* $\|Ty\| > b'$.

Then T *has at least three fixed points* y_1, y_2, y_3 *in* $\overline{P}_{c'}$ *satisfying*

$$\|y_1\| < d', a' < S(y_2), \|y_3\| > d', S(y_3) < a'.$$

For convenience, we let

$$D = \max_{t\in\left[a,\sigma^n(b)\right]} \int_a^{\sigma(b)} G(t,s)\Delta s;$$

$$C = \min_{t\in I} \int_{s\in I} G(t,s)\Delta s \text{ and } \gamma = \frac{1}{p\cdot 4^{n-1}}.$$

Theorem 4.2. *Assume that there exist real numbers* d_0, d_1, and c *with* $0 < d_0 < d_1 < \frac{d_1}{\gamma} < c$ *such that*

$$f(t,y(t)) < \frac{d_0}{D}, \text{ for } t\in\left[a,\sigma^n(b)\right] \text{ and } y\in[0,d_0], \quad (15)$$

$$f(t,y(t)) > \frac{d_1}{C}, \text{ for } t\in I \text{ and } y\in\left[d_1,\frac{d_1}{\gamma}\right], \quad (16)$$

$$f(t,y(t)) < \frac{c}{D}, \text{ for } t\in\left[a,\sigma^n(b)\right] \text{ and } y\in[0,c]. \quad (17)$$

Then the BVP (1)-(3) *has at least three positive solutions.*

Proof: Let the Banach space $E = C\left[a,\sigma^n(b)\right]$ be equipped with the norm

$$\|y\| = \max_{t\in\left[a,\sigma^n(b)\right]} |y(t)|.$$

We denote

$$P = \left\{y\in E : y(t)\ge 0, t\in\left[a,\sigma^n(b)\right]\right\}.$$

Then, it is obvious that P is a cone in E. For $y\in P$, we define

$$S(y) = \min_{t\in I}|y(t)|, \text{ and } Ty(t) = \int_a^{\sigma(b)} G(t,s) f(s,y(s))\Delta s, t\in\left[a,\sigma^n(b)\right].$$

It is easy to check that S is a nonnegative continuous concave functional on P with $S(y)\le\|y\|$ for $y\in P$ and that $T: P\to P$ is completely continuous and fixed points of T are solutions of the BVP (1)-(3). First, we prove that if there exists a positive number r such that $f(t,y(t)) < \frac{r}{D}$ for $y\in[0,r]$, then $T:\overline{P}_r\to P_r$. Indeed, if $y\in\overline{P}_r$, then for $t\in\left[a,\sigma^n(b)\right]$.

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s) f(s,y(s))\Delta s \\ &< \frac{r}{D}\int_a^{\sigma(b)} G(t,s)\Delta s \\ &\le \frac{r}{D}\max_{t\in\left[a,\sigma^n(b)\right]}\int_a^{\sigma(b)} G(t,s)\Delta s = r.\end{aligned}$$

Thus, $\|Ty\| < r$, that is, $Ty\in P_r$. Hence, we have shown that if (15) and (17) hold, then T maps $\overline{P}_{d_0}$ into P_{d_0} and $\overline{P}_c$ into P_c. Next, we show that $\left\{y\in P\left(S,d_1,\frac{d_1}{\gamma}\right): S(y) > d_1\right\}\ne\varnothing$ and $S(Ty) > d_1$ for all $y\in P\left(S,d_1,\frac{d_1}{\gamma}\right)$. In fact, the constant function

$$\frac{d_1+\frac{d_1}{\gamma}}{2} \in \left\{y\in P\left(S,d_1,\frac{d_1}{\gamma}\right): S(y) > d_1\right\}.$$

Moreover, for $y\in P\left(S,d_1,\frac{d_1}{\gamma}\right)$, we have

$$\frac{d_1}{\gamma}\ge\|y\|\ge y(t)\ge\min_{t\in I} y(t) = S(y)\ge d_1,$$

for all $t\in I$. Thus, in view of (16) we see that

$$\begin{aligned}S(Ty) &= \min_{t\in I}\int_a^{\sigma(b)} G(t,s) f(s,y(s))\Delta s \\ &\ge \min_{t\in I}\int_{s\in I} G(t,s) f(s,y(s))\Delta s \\ &> \frac{d_1}{C}\min_{t\in I}\int_{s\in I} G(t,s)\Delta s = d_1.\end{aligned}$$

as required. Finally, we show that if $y\in P(S,d_1,c)$ and $\|Ty\| > \frac{d_1}{\gamma}$, then $S(Ty) > d_1$. To see this, we suppose that $y\in P(S,d_1,c)$ and $\|Ty\| > \frac{d_1}{\gamma}$, then, by Lemma 2.4, we have

$$\begin{aligned}S(Ty) &= \min_{t\in I}\int_a^{\sigma(b)} G(t,s) f(s,y(s))\Delta s \\ &\ge \gamma\int_a^{\sigma(b)} G(\sigma^n(b),s) f(s,y(s))\Delta s \\ &\ge \gamma\max_{t\in\left[a,\sigma^n(b)\right]}\int_a^{\sigma(b)} G(t,s) f(s,y(s))\Delta s.\end{aligned}$$

for all $t\in\left[a,\sigma^n(b)\right]$. Thus

$$\begin{aligned}S(Ty) &\ge \gamma\max_{t\in\left[a,\sigma^n(b)\right]}\int_a^{\sigma(b)} G(t,s) f(s,y(s))\Delta s \\ &= \gamma\|Ty\| > \gamma\frac{d_1}{\gamma} = d_1.\end{aligned}$$

To sum up the above, all the hypotheses of Theorem 4.1 are satisfied. Hence T has at least three fixed points, that is, the BVP (1)-(3) has at least three positive solutions y_1, y_2 and y_3 such that

$\|y_1\| < d_0,\ d_1 < \min_{t\in I} y_2(t),\ \|y_3\| > d_0,\ \min_{t\in I} y_3(t) < d_1.$ □

5. Examples

Now, we give some examples to illustrate the main result.

Example 1

Consider the following boundary value problem

$$\begin{cases} y^{\Delta^{(3)}} + \left|y^2\right|\left(t^2 y + |y|^2\right) = 0, t\in[0,\pi] \\ y(0) = y^{\Delta}(0) = y^{\Delta^{(2)}}(\sigma(\pi)) = 0. \end{cases} \tag{18}$$

The Green's function for the homogeneous boundary value problem is given by

$$G(t,s) = \begin{cases} \dfrac{t^2}{2}, & t<s \\ \dfrac{t^2}{2} - \dfrac{(t-s)^2}{2}, & s<t. \end{cases}$$

It is easy to see that all the conditions of Theorem 3.2 hold. It follows from Theorem 3.2, the BVP (18) has at least one positive solution.

Example 2

Consider the following boundary value problem

$$\begin{cases} y^{\Delta^{(5)}} + \dfrac{t^5 e^{-8y^2} + 1}{y} = 0, t\in[0,1] \\ y(0) = y^{\Delta^{(2)}}(0) = y^{\Delta^{(3)}} = y^{\Delta^{(4)}}(\sigma(1)) = 0. \end{cases} \tag{19}$$

The Green's function for the homogeneous boundary value problem is given by

$$G(t,s) = \begin{cases} \dfrac{t^4}{24}, & t<s \\ \dfrac{t^4}{24} - \dfrac{(t-s)^4}{24}, & s<t. \end{cases}$$

It is easy to see that all the conditions of Theorem 3.3 hold. It follows from Theorem 3.3, the BVP (19) has at least one positive solution.

Example 3

Consider the following boundary value problem on time scale

$$\mathbb{T} = \{0\} \cup \left\{\frac{1}{2^{n+1}} : n\in\mathbb{N}\right\} \cup \left[\frac{1}{2},\frac{3}{2}\right] \begin{cases} y^{\Delta^{(6)}} + f(t,y) = 0, t\in[0,1]\cap\mathbb{T} \\ y(0) = y^{\Delta^{(1)}}(0) = y^{\Delta^{(2)}}(0) = 0 \\ y^{\Delta^{(3)}}(0) = y^{\Delta^{(4)}}(0) = y^{\Delta^{(3)}}(1) = 0. \end{cases} \tag{20}$$

where

$$f(t,y) = \begin{cases} \dfrac{100(y+1)}{16(4y^2+999)}, & y\in\left[0,\dfrac{1}{2}\right] \\ 181499.98y + 90749.981, & y\in\left[\dfrac{1}{2},1\right] \\ 90749 + y, & y\in[1,3072] \\ 21.16569y - 28800, & y\in[3072,4000] \\ \dfrac{2880000}{y+100}, & y\in[4000,\infty). \end{cases}$$

The Green's function for the homogeneous boundary value problem is given by

$$G(t,s) = \begin{cases} \dfrac{t^5}{120}, & t<s \\ \dfrac{t^5}{120} - \dfrac{(t-s)^5}{120}, & s<t. \end{cases}$$

A simple calculation shows that $C = \dfrac{13}{1179648}$, $D = \dfrac{1}{720}$ and $\gamma = \dfrac{1}{3072}$. If we choose $d_0 = \dfrac{1}{2}$, $d_1 = 1$ and $c = 4000$ then, we see that all the conditions of Theorem 4.2 hold. It follows from Theorem 4.2, the BVP (20) has at least three positive solutions.

6. Conclusion

In this paper, we have established the existence of positive solutions for higher order boundary value problems on time scales which unifies the results on continuous intervals and discrete intervals, by using Leggett-Williams fixed point theorem. These results are rapidly arising in the field of modelling and determination of flagellate protozoan in a viscous fluid in further research.

REFERENCES

[1] G. H. Meyer, "Initial Value Methods for Boundary Value Problems," *Journal of Applied Mathematics and Computing*, Vol. 158, No. 1, 2004, pp. 345-351.

[2] J. P. Keener, "Principles of Applied Mathematics," Addison-Wesley, Redwood City, 1988.

[3] P. C. Wayner, Y. K. Kao and L. V. Lacroin, "The Interlimne Heat Transfer Coefficient of an Eveporating Wetting Film," *International Journal of Heat and Mass Transfer*, Vol. 19, No. 5, 1976, pp. 487-492. doi:10.1016/0017-9310(76)90161-7

[4] M. Bohner and A. C. Peterson, "Dynamic Equations on Time Scales, an Introduction with Applications," Birkhauser, Boston, 2001.

[5] L. H. Erbe and H. Wang, "On the Existence of Positive Solutions of Ordinary Differential Equations," *Proceedings of the American Mathematical Society*, Vol. 120, No. 3, 1994, pp. 743-748.

doi:10.1090/S0002-9939-1994-1204373-9

[6] P. W. Eloe and J. Henderson, "Positive Solutions for (n-1,1) Conjugate Boundary Value Problems," *Nonlinear Analysis*, Vol. 28, No. 10, 1997, pp. 1669-1680. doi:10.1016/0362-546X(95)00238-Q

[7] P. W. Eloe and J. Henderson, "Positive Solutions and Nonlinear (k,n-k) Conjugate Eigenvalue Problems," *Journal of Differential Equations Dynamical Systems*, Vol. 6, 1998, pp. 309-317.

[8] F. M. Atici and G. Sh. Guseinov, "Positive Periodic Solutions for Nonlinear Difference Equations with Periodic Coefficients," *Journal of Mathematical Analysis and Applications*, Vol. 232, No. 1, 1999, pp. 166-182. doi:10.1006/jmaa.1998.6257

[9] D. R. Anderson and R. I. Avery, "Multiple Positive Solutions to a Third-Order Discrete Focal Boundary Value Problem," *Journal of Computers and Mathematics with Applications*, Vol. 42, No. 3-5, 2001, pp. 333-340. doi:10.1016/S0898-1221(01)00158-4

[10] R. I. Avery and A. C. Peterson, "Multiple Positive Solutions of a Discrete Second Order Conjugate Problem," *Panamerican Mathematical Journal*, Vol. 8, No. 3, 1998, pp. 1-12.

[11] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, "Positive Solutions of Differential, Difference and Integral Equations," Kluwer Academic Publishers, Dordrecht, 1999.

[12] K. Deimling, "Nonlinear Functional Analysis," Springer, New York, 1985. doi:10.1007/978-3-662-00547-7

[13] M. Gregus, "Third Order Linear Differential Equations and Mathematical Applications," Reidel, Dordrecht, 1987. doi:10.1007/978-94-009-3715-4

[14] D. Guo and V. Lakshmikantham, "Nonlinear Problems in Abstract Cones," Academic Press, San Diego, 1988.

[15] J. Henderson and S. K. Ntouyas, "Positive Solutions for System of n^{th} Order Three-Point Nonlocal Boundary Value Problems," *Electronic Journal of Qualitative Theory of Differential Equations*, Vol. 18, No. 5, 2007, pp. 1-12.

[16] B. Hopkins and N. Kosmator, "Third Order Boundary Value Problem with Sign-Changing Solution," *Nonlinear Analysis*, Vol. 67, No. 1, 2007, pp. 126-137. doi:10.1016/j.na.2006.05.003

[17] S. Li, "Positive Solutions of Nonlinear Singular Third Order Two-Point Boundary Value Problem," *Journal of Mathematical Analysis and Applications*, Vol. 323, No. 1, 2006, pp. 413-425. doi:10.1016/j.jmaa.2005.10.037

[18] M. El-Shehed, "Positive Solutions of Boundary Value Problems for n^{th} Order Differential Equations," *Electronic Journal of Qualitative Theory of Differential Equations*, No. 1, 2008, pp. 1-9.

[19] K. R. Prasad and P. Murali, "Eigenvalue Intervals for n^{th} Order Differential Equations on Time Scales," *International Journal of Pure and Applied Mathematics*, Vol. 44, No. 5, 2008, pp. 737-753.

[20] R. W. Leggett and L. R. Williams, "Multiple Positive Fixed Points of Nonlinear Operator on Order Banach Spaces," *Indiana University Mathematics Journal*, Vol. 28, No. 4, 1979, pp. 673-688. doi:10.1512/iumj.1979.28.28046

