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Applications of Multivalent Functions Associated with Generalized Fractional Integral Operator

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ABSTRACT

By using a method based upon the Briot-Bouquet differential subordination, we investigate some subordination properties of the generalized fractional integral operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ which was defined by Owa, Saigo and Srivastava [1]. Some interesting further consequences are also considered.

Keywords: Multivalent Functions; Subordination; Gaussian Hypergeometric Function; Fractional Integral Operator

1. Introduction

Let $\mathcal{A}_n(p)$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}z^{p+k},\left(p,n\in\mathbb{N}:=\{1,2,3,\cdots\}\right), \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{U}=\{z:z\in\mathbb{C}\text{ and }|z|<1\}$. Also let *f* and *g* be analytic in $\mathbb{U}$ with $f(0)=g(0)$. Then we say that *f* is subordinate to g in $\mathbb{U}$, written $f\prec g$ or $f(z)\prec g(z)$, if there exists the Schwarz function *w*, analytic in $\mathbb{U}$ such that $w(0)=0$, $|w(z)|<1$ and $f(z)=g(w(z))(z\in\mathbb{U})$. We also observe that

$$f(z)\prec g(z)\text{ in }\mathbb{U}$$

if and only if

$$f(0)=g(0)\text{ and }f(\mathbb{U})\subset g(\mathbb{U})$$

whenever g is univalent in $\mathbb{U}$.

Let *a*, *b* and *c* be complex numbers with $c\neq 0,-1,-2,\cdots$. Then the *Gaussian/classical hypergeometric function* ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \quad (1.2)$$

where $(\eta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\eta)_k=\frac{\Gamma(\eta+k)}{\Gamma(\eta)}=\begin{cases}1, & (k=0)\\ \eta(\eta+1)\cdots(\eta+k-1), & (k\in\mathbb{N}).\end{cases} \quad (1.3)$$

The hypergeometric function ${}_2F_1(a,b;c;z)$ is analytic in $\mathbb{U}$ and if *a* or *b* is a negative integer, then it reduces to a polynomial.

For each *A* and *B* such that $-1\le B<A\le 1$, let us define the function

$$h(A,B;z)=\frac{1+Az}{1+Bz},(z\in\mathbb{U}). \quad (1.4)$$

It is well known that $h(A,B;z)$, for $-1\le B\le 1$, is the conformal map of the unit disk onto the disk symmetrical respect to the real axis having the center $(1-AB)/(1-B^2)$ and the radius $(A-B)/(1-B^2)$. The boundary circle cuts the real axis at the points $(1-A)/(1-B)$ and $(1+A)/(1+B)$.

Many essentially equivalent definitions of fractional calculus have been given in the literature (cf., e.g. [2,3]). We state here the following definition due to Saigo [4] (see also [1,5]).

Definition 1. For $\lambda>0$, $\mu,\nu\in\mathbb{R}$, the fractional integral operator $\mathcal{I}_{0,z}^{\lambda,\mu,\nu}$ is defined by

$$\mathcal{I}_{0,z}^{\lambda,\mu,\nu}f(z)=\frac{z^{-\lambda-\mu}}{\Gamma(\lambda)}\int_0^z(z-\zeta)^{\lambda-1}{}_2F_1\left(\lambda+\mu,-\nu;\lambda;1-\frac{\zeta}{z}\right)f(\zeta)\mathrm{d}\zeta, \quad (1.5)$$

where ${}_2F_1$ is the Gaussian hypergeometric function defined by (1.2) and $f(z)$ is taken to be an analytic function in a simply-connected region of the *z*-plane containing the origin with the order

$$f(z)=\mathcal{O}\left(|z|^{\epsilon}\right)(z\to 0)$$

for $\epsilon>\max\{0,\mu-\nu\}-1$, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring that $\log(z-\zeta)$ to be real when $z-\zeta>0$.

The definition (1.5) is an interesting extension of both the Riemann-Liouville and Erdélyi-Kober fractional operators in terms of Gauss's hypergeometric functions.

With the aid of the above definition, Owa, Saigo and Srivastava [1] defined a modification of the fractional integral operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ by

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(\lambda+p+1+\nu)}{\Gamma(p+1)\Gamma(p+1-\mu+\nu)} z^{\mu}\mathcal{I}_{0,z}^{\lambda,\mu,\nu} f(z) \tag{1.6}$$

for $f(z)\in\mathcal{A}_n(p)$ and $\mu-\nu-p<1$. Then it is observed that $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ also maps $\mathcal{A}_n(p)$ onto itself as follows:

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = z^p+\sum_{k=n}^{\infty}\frac{(p+1)_k(p+1-\mu+\nu)_k}{(p+1-\mu)_k(\lambda+p+1+\nu)_k}a_{p+k}z^{p+k}, \tag{1.7}$$

$$\left(\lambda>1;\mu-\nu-p<1;f\in\mathcal{A}_n(p)\right).$$

We note that $\mathcal{J}_{0,z}^{\alpha,0,\beta-1} f(z)=\mathcal{O}_{\beta}^{\alpha} f(z),(\alpha\ge 0;\beta>-1)$, where the operator $\mathcal{O}_{\beta}^{\alpha}$ was introduced and studied by Jung, Kim and Srivastava [6] (see also [7]).

It is easily verified from (1.7) that

$$z\left(\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)\right)' = (\lambda+\nu+p)\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} f(z)-(\lambda+\nu)\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z). \tag{1.8}$$

The identity (1.8) plays an important and significant role in obtaining our results.

Recently, by using the general theory of differential subordination, several authors (see, e.g. [7-9]) considered some interesting properties of multivalent functions associated with various integral operators. In this manuscript, we shall derive some subordination properties of the fractional integral operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ by using the technique of differential subordination.

2. Main Results

In order to establish our results, we shall need the following lemma due to Miller and Mocanu [10].

Lemma 1. *Let* $h(t)$ *be analytic and convex univalent in* $\mathbb{U}$ *with* $h(0)=1$, *and let* $g(z)=1+b_nz^n+b_{n+1}z^{n+1}+\cdots$ *be analytic in* $\mathbb{U}$. *If*

$$g(z)+\frac{1}{c}zg'(z)\prec h(z), \tag{2.1}$$

then for $c\ne 0$ *and* $\operatorname{Re}c\ge 0$,

$$g(z)\prec\frac{c}{n}z^{-c/n}\int_0^z t^{c/n-1}h(t)\,\mathrm{d}t. \tag{2.2}$$

We begin by proving the following theorem.

Theorem 1. *Let* $-1\le B<A\le 1$, $\lambda>1$, $\lambda+\nu>-p$, $\mu-\nu-p<1$, $\mu-1<p$ *and* $0<\alpha<1$, *and let* $f(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}z^{p+k}\in\mathcal{A}_n(p)$. *Suppose that*

$$\sum_{k=n}^{\infty}c_k\left|a_{p+k}\right|\le 1, \tag{2.3}$$

where

$$c_k=\frac{1-B}{A-B}\frac{\left[\lambda+p+\nu+k(1-\alpha)\right](p+1)_k(p+1-\mu+\nu)_k}{(\lambda+p+\nu)(p+1-\mu)_k(\lambda+p+1+\nu)_k} \tag{2.4}$$

and $(\eta)_k$ *is given by* (1.3).

1) *If* $-1\le B<0$, *then*

$$(1-\alpha)\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} f(z)}{z^p}+\alpha\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{z^p}\prec h(A,B;z). \tag{2.5}$$

2) *If* $-1\le B<0$ *and* $\gamma\ge 1$, *then*

$$\operatorname{Re}\left\{\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{z^p}\right)^{1/\gamma}\right\} > \left\{\frac{\lambda+\nu+p}{n(1-\alpha)}\int_0^1 u^{\frac{\lambda+\nu+p}{n(1-\alpha)}-1}\left(\frac{1-Au}{1-Bu}\right)\mathrm{d}u\right\}^{1/\gamma},(z\in U). \tag{2.6}$$

The result is sharp.

Proof. 1) If we set

$$L=(1-\alpha)\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu} f(z)}{z^p}+\alpha\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z)}{z^p},$$

then, from (1.7) we see that

$$L=1+\sum_{k=n}^{\infty}\frac{\left[\lambda+p+\nu+k(1-\alpha)\right](p+1)_k(p+1-\mu+\nu)_k}{(\lambda+p+\nu)(p+1-\mu)_k(\lambda+p+1+\nu)_k}a_{p+k}z^k. \tag{2.7}$$

For $-1\le B<0$ and $z\in\mathbb{U}$, it follows from (2.3) that

$$\left|\frac{L-1}{A-BL}\right|=\left|\frac{\sum_{k=n}^{\infty}\frac{\left[\lambda+p+\nu+k(1-\alpha)\right](p+1)_k(p+1-\mu+\nu)_k}{(\lambda+p+\nu)(p+1-\mu)_k(\lambda+p+1+\nu)_k}a_{p+k}z^k}{A-B-B\sum_{k=n}^{\infty}\frac{\left[\lambda+p+\nu+k(1-\alpha)\right](p+1)_k(p+1-\mu+\nu)_k}{(\lambda+p+\nu)(p+1-\mu)_k(\lambda+p+1+\nu)_k}a_{p+k}z^k}\right|\le\frac{\sum_{k=n}^{\infty}c_k\left|a_{p+k}\right|}{1-B+B\sum_{k=n}^{\infty}c_k\left|a_{p+k}\right|}\le 1, \tag{2.8}$$

which implies that

$$\left(1-\alpha\right)\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}f\left(z\right)}{z^{p}}+\alpha\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}{z^{p}}\prec h\left(A,B;z\right).$$

2) Let

$$g\left(z\right)=\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}{z^{p}},\left(f\in\mathcal{A}_{n}\left(p\right)\right). \tag{2.9}$$

Then the function $g\left(z\right)=1+b_{n}z^{n}+b_{n+1}z^{n+1}+\cdots$ is analytic in $\mathbb{U}$. Using (1.8) and (2.9), we have

$$\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}f\left(z\right)}{z^{p}}=g\left(z\right)+\frac{1}{\lambda+\nu+p}zg'\left(z\right). \tag{2.10}$$

From (2.5), (2.9) and (2.10) we obtain

$$\begin{aligned}&\left(1-\alpha\right)\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}f\left(z\right)}{z^{p}}+\alpha\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}{z^{p}}\\&=g\left(z\right)+\frac{1-\alpha}{\lambda+\nu+p}zg'\left(z\right)\prec h\left(A,B;z\right).\end{aligned}$$

Thus, by applying Lemma 1, we observe that

$$g\left(z\right)\prec\frac{\lambda+\nu+p}{n\left(1-\alpha\right)}z^{-\frac{\lambda+\nu+p}{n(1-\alpha)}}\int_{0}^{z}t^{\frac{\lambda+\nu+p}{n(1-\alpha)}-1}\left(\frac{1+At}{1+Bt}\right)\mathrm{d}t$$

or

$$\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}{z^{p}}=\frac{\lambda+\nu+p}{n\left(1-\alpha\right)}\int_{0}^{1}u^{\frac{\lambda+\nu+p}{n(1-\alpha)}-1}\left(\frac{1+Auw\left(z\right)}{1+Buw\left(z\right)}\right)\mathrm{d}u, \tag{2.11}$$

where $w\left(z\right)$ is analytic in $\mathbb{U}$ with $w\left(0\right)=0$ and $\left|w\left(z\right)\right|<1\left(z\in\mathbb{U}\right)$. In view of $-1\le B<A\le 1$ and $\lambda+\nu>-p$, we conclude from (2.11) that

$$\operatorname{Re}\left\{\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}{z^{p}}\right\}>\frac{\lambda+\nu+p}{n\left(1-\alpha\right)}\int_{0}^{1}u^{\frac{\lambda+\nu+p}{n(1-\alpha)}-1}\left(\frac{1-Au}{1-Bu}\right)\mathrm{d}u, \tag{2.12}$$

$\left(z\in\mathbb{U}\right)$.

Since $\operatorname{Re}\left(w^{1/\gamma}\right)\ge\left(\operatorname{Re}w\right)^{1/\gamma}$ for $\operatorname{Re}w>0$ and $\gamma\ge 1$, from (2.12) we see that the inequality (2.6) holds.

To prove sharpness, we take $f\left(z\right)\in\mathcal{A}_{n}\left(p\right)$ defined by

$$\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}{z^{p}}=\frac{\lambda+\nu+p}{n\left(1-\alpha\right)}\int_{0}^{1}u^{\frac{\lambda+\nu+p}{n(1-\alpha)}-1}\left(\frac{1+Auz^{n}}{1+Buz^{n}}\right)\mathrm{d}u.$$

For this function we find that

$$\left(1-\alpha\right)\frac{\mathcal{J}_{0,z}^{\lambda-1,\mu,\nu}f\left(z\right)}{z^{p}}+\alpha\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}{z^{p}}=\frac{1+Az^{n}}{1+Bz^{n}}$$

and

$$\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}{z^{p}}\to\frac{\lambda+\nu+p}{n\left(1-\alpha\right)}\int_{0}^{1}u^{\frac{\lambda+\nu+p}{n(1-\alpha)}-1}\frac{1-Au}{1-Bu}\mathrm{d}u\text{ as }z\to\mathrm{e}^{i\pi/n}.$$

Hence the proof of Theorem 1 is evidently completed.

Theorem 2. *Let* $-1\le B<A\le 1$, $\lambda>1$, $\lambda+\nu>-p$, $\mu-\nu-p<1$, $\mu-1<p$ *and* $0<\alpha<1$. *Suppose that* $f\left(z\right)=z^{p}+\sum_{k=n}^{\infty}a_{p+k}z^{p+k}\in\mathcal{A}_{n}\left(p\right)$, $s_{1}\left(z\right)=z^{p}$ *and* $s_{m}\left(z\right)=z^{p}+\sum_{k=n}^{n+m-2}a_{p+k}z^{p+k}\left(m\ge 2\right)$. *If the sequence* $\left\{c_{k}\right\}$ *is nondecreasing with*

$$c_{k}\ge\frac{\left(1-B\right)\left[\lambda+p+\nu+k\left(1-\alpha\right)\right]}{\left(A-B\right)\left(\lambda+p+\nu\right)}\left(k\ge n\right), \tag{2.13}$$

where c_{k} *is given by* $\left(2.4\right)$ *and satisfies the condition* $\left(2.3\right)$, *then*

$$\operatorname{Re}\left\{\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}{s_{m}\left(z\right)}\right\}>0 \tag{2.14}$$

and

$$\operatorname{Re}\left\{\frac{s_{m}\left(z\right)}{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}\right\}>0. \tag{2.15}$$

Each of the bounds in (2.14) *and* (2.15) *is best possible for* $m\in\mathbb{N}$.

Proof. We prove the bound in (2.14). The bound in (2.15) is immediately obtained from (2.14) and will be omitted. Let

$$h\left(z\right)=\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f\left(z\right)}{s_{m}\left(z\right)}\left(f\in\mathcal{A}_{n}\left(p\right);z\in\mathbb{U}\right).$$

Then, from (1.7) we observe that

$$h\left(z\right)=1+\frac{\sum_{k=n}^{n+m-2}\left(\delta_{k}-1\right)a_{p+k}z^{k}+\sum_{k=n+m-1}^{\infty}\delta_{k}a_{p+k}z^{k}}{1+\sum_{k=n}^{n+m-2}a_{p+k}z^{k}},$$

where, for convenience,

$$\delta_{k}=\frac{\left(p+1\right)_{k}\left(p+1-\mu+\nu\right)_{k}}{\left(p+1-\mu\right)_{k}\left(\lambda+p+1+\nu\right)_{k}}.$$

It is easily seen from (2.4) and (2.13) that $c_{k}>1$ and

$$\delta_{k}=\frac{\left(A-B\right)\left(\lambda+p+\nu\right)}{\left(1-B\right)\left[\lambda+p+\nu+k\left(1-\alpha\right)\right]}c_{k}\ge 1. \tag{2.16}$$

Hence, by applying (2.3) and (2.16), we have

$$\begin{aligned}\left|\frac{h\left(z\right)-1}{h\left(z\right)+1}\right|&=\left|\frac{\sum_{k=n}^{n+m-2}\left(\delta_{k}-1\right)a_{p+k}z^{k}+\sum_{k=n+m-1}^{\infty}\delta_{k}a_{p+k}z^{k}}{2+\sum_{k=n}^{n+m-2}\left(\delta_{k}+1\right)a_{p+k}z^{k}+\sum_{k=n+m-1}^{\infty}\delta_{k}a_{p+k}z^{k}}\right|\\&\le\frac{\sum_{k=n}^{n+m-2}\left(\delta_{k}-1\right)\left|a_{p+k}\right|+\sum_{k=n+m-1}^{\infty}\delta_{k}\left|a_{p+k}\right|}{2-\sum_{k=n}^{n+m-2}\left(\delta_{k}+1\right)\left|a_{p+k}\right|-\sum_{k=n+m-1}^{\infty}\delta_{k}\left|a_{p+k}\right|}\le 1\left(z\in\mathbb{U}\right)\end{aligned}$$

which readily yields the inequality (2.14).

If we take $f(z)=z^p-z^{p+n+m-1}$, then

$$\frac{f(z)}{s_m(z)}=1-z^{n+m-1}\to 0 \text{ as } z\to 1^-.$$

This show that the bound in (2.14) is best possible for each m, which proves Theorem 2.

Finally, we consider the generalized Bernardi-Livera-Livingston integral operator $\mathcal{L}_\sigma(\sigma>-p)$ defined by (cf. [11-13])

$$\mathcal{L}_\sigma(f)(z):=\frac{\sigma+p}{z^\sigma}\int_0^z t^{\sigma-1}f(t)\,\mathrm{d}t\left(f\in\mathcal{A}_n(p);\sigma>-p\right). \tag{2.17}$$

Theorem 3. *Let* $-1\le B<A\le 1$, $\sigma>-p$, $\lambda>1$, $\lambda+\nu>-p$, $\mu-\nu-p<1$, $\mu-1<p$ *and* $0<\alpha<1$, *and let* $f(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}z^{p+k}\in\mathcal{A}_n(p)$. *Suppose that*

$$\sum_{k=n}^{\infty}d_k\left|a_{p+k}\right|\le 1, \tag{2.18}$$

where

$$d_k=\frac{1-B}{A-B}\frac{\left[\sigma+p+k(1-\alpha)\right](p+1)_k(p+1-\mu+\nu)_k}{(\sigma+p+k)(p+1-\mu)_k(\lambda+p+1+\nu)_k}$$

and $(\eta)_k$ *is given by* (1.3).

1) *If* $-1\le B<0$, *then*

$$(1-\alpha)\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{z^p}+\alpha\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{z^p}\prec h(A,B;z). \tag{2.19}$$

2) *If* $-1\le B<0$ *and* $\gamma\ge 1$, *then*

$$\begin{aligned}&\mathrm{Re}\left\{\left(\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{z^p}\right)^{1/\gamma}\right\}\\&>\left\{\frac{\sigma+p}{n(1-\alpha)}\int_0^1 u^{\frac{\sigma+p}{n(1-\alpha)}-1}\left(\frac{1-Au}{1-Bu}\right)\mathrm{d}u\right\}^{1/\gamma}\quad(z\in U).\end{aligned} \tag{2.20}$$

The result is sharp.

Proof. 1) If we put

$$M=(1-\alpha)\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{z^p}+\alpha\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{z^p},$$

then, from (1.7) and (2.17) we have

$$\begin{aligned}M&=1+\sum_{k=n}^{\infty}\frac{\left[\sigma+p+k(1-\alpha)\right](p+1)_k(p+1-\mu+\nu)_k}{(\sigma+p+k)(p+1-\mu)_k(\lambda+p+1+\nu)_k}\\&\quad\cdot a_{p+k}z^k.\end{aligned}$$

Therefore, by using same techniques as in the proof of Theorem 1 1), we obtain the desired result.

2) From (2.17) we have

$$\begin{aligned}&(\sigma+p)\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)\\&=\sigma\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)+z\left(\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)\right)'.\end{aligned} \tag{2.21}$$

Let

$$g(z)=\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{z^p}(z\in\mathbb{U}). \tag{2.22}$$

Then, by virtue of (2.21), (2.22) and (2.19), we observe that

$$\begin{aligned}&(1-\gamma)\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}{z^p}+\gamma\frac{\mathcal{J}_{0,z}^{\lambda,\mu,\nu}\mathcal{L}_\sigma(f)(z)}{z^p}\\&=g(z)+\frac{1-\gamma}{\sigma+p}zg'(z)\prec h(A,B;z).\end{aligned}$$

Hence, by applying the same argument as in the proof of Theorem 1 2), we obtain (2.20), which evidently proves Theorem 3.

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REFERENCES

[1] S. Owa, M. Saigo and H. M. Srivastava, "Some Characterization Theorems for Starlike and Convex Functions Involving a Certain Fractional Integral Operator," *Journal of Mathematical Analysis and Applications*, Vol. 140, No. 2, 1989, pp. 419-426. doi:10.1016/0022-247X(89)90075-9

[2] S. G. Samko, A. A. Kilbas and O. I. Marichev, "Fractional Integral and Derivatives, Theory and Applications," Gordon and Breach, New York, Philadelphia, London, Paris, Montreux, Toronto, Melbourne, 1993.

[3] H. M. Srivastava and R. G. Buschman, "Theory and Applications of Convolution Integral Equations," Kluwer Academic Publishers, Dordrecht, Boston, London, 1992.

[4] M. Saigo, "A Remark on Integral Operators Involving the Gauss Hypergeometric Functions," *Mathematical Reports, Kyushu University*, Vol. 11, No. 2, 1977-1978, pp. 135-143.

[5] J. H. Choi, "Note on Differential Subordination Associated with Fractional Integral Operator," *Far East Journal of Mathematical Sciences*, Vol. 26, No. 2, 2007, pp. 499-511.

[6] I. B. Jung, Y. C. Kim and H. M. srivastava, "The Hardy Space of Analytic Functions Associated with Certain One-Parameter Families of Integral Operators," *Journal of Mathematical Analysis and Applications*, Vol. 176, No. 1, 1993, pp. 138-147. doi:10.1006/jmaa.1993.1204

[7] J.-L. Liu, "Notes on Jung-Kim-Srivastava Integral Operator," *Journal of Mathematical Analysis and Applications*, Vol. 294, No. 1, 2004, pp. 96-103. doi:10.1016/j.jmaa.2004.01.040

[8] R. M. EL-Ashwash and M. K. Aouf, "Some Subclasses of Multivalent Functions Involving the Extended Fractional Differintegral Operator," *Journal of Mathematical Inequalities*, Vol. 4, No. 1, 2010, pp. 77-93.

[9] J. Patel, A. K. Mishra and H. M. Srivastava, "Classes of Multinalent Analytic Functions Involving the Dziok-Srivastava Operator," *Computers and Mathematics with Applications*, Vol. 54, No. 5, 2007, pp. 599-616. doi:10.1016/j.camwa.2006.08.041

[10] S. S. Miller and P. T. Mocanu, "Differential Subordinations and Univalent Functions," *Michigan Mathematical Journal*, Vol. 28, No. 2, 1981, pp. 157-172. doi:10.1307/mmj/1029002507

[11] S. D. Bernardi, "Convex and Starlike Univalent Functions," *Transactions of the American Mathematical Society*, Vol. 135, 1969, pp. 429-446. doi:10.1090/S0002-9947-1969-0232920-2

[12] R. J. Libera, "Some Classes of Regular Univalent Functions," *Proceedings of the American Mathematical Society*, Vol. 16, No. 4, 1965, pp. 755-758. doi:10.1090/S0002-9939-1965-0178131-2

[13] H. M. Srivastava and S. Owa, Eds., "Current Topics in Analytic Function Theory," World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, 1992. doi:10.1142/1628

