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Some Properties on the Error-Sum Function of Alternating Sylvester Series

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ABSTRACT

The error-sum function of alternating Sylvester series is introduced. Some elementary properties of this function are studied. Also, the hausdorff dimension of the graph of such function is determined.

Keywords: Alternating Sylvester Series; Error-Sum Function; Hausdorff Dimension

1. Introduction

For any $x \in (0,1]$, let $d_1 := d_1(x) \in N$ and $T := T(x) \in (0,1]$ be defined as

$$d_1(x) = \left[\frac{1}{x}\right], (x) := \frac{1}{d_1(x)} - x, T(0) := 0. \tag{1}$$

where $[\ ]$ denote the integer part. And we define the sequence $\{d_n(x), n \ge 2\}$ as follows:

$$d_n(x) = d_1\left(T^{n-1}(x)\right), \tag{2}$$

where T^n denotes the *n*th iterate of $T\left(T^0 = Id_{(0,1]}\right)$.

It is well known that from the algorithm (1), all $x \in (0,1]$ can be developped uniquely into an infinite or finite series

$$x = \sum_{i \ge 1} (-1)^{i-1} \frac{1}{d_i(x)}, \tag{3}$$

where $_{i+1}(x) \ge d_i(x)\left(d_i(x)+1\right)$.

In the literature [2], (3) is called the Alternating Balkema-Oppenheim expansion of x and denoted by $x = \left[d_1(x), \cdots, d_n(x), \cdots\right]$ for short. From the algorithm, one can see that T maps irrational element into irrational element, and the series is infinite. While for rational numbers, in fact, we have $x \in (0,1]$ is rational if and only if its sequence of digits $d_1(x), \cdots,$ is terminate or periodic, see [1-3].

For any $x \in (0,1]$ and $n \ge 1$, define

$$\frac{p_n(x)}{q_n(x)} = \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{d_i(x)}.$$

From the algorithm of (1), it is clear that

$$x = \frac{p_n(x)}{q_n(x)} + (-1)^n T^n(x). \tag{4}$$

For any $x \in (0,1]$, let $x = \left[d_1(x), \cdots, d_n(x), \cdots\right]$ be its Alternating Sylvester expansion, then we have $d_{j+1}(x) \ge d_j(x)\left(d_j(x)+1\right)$ for any $j \ge 1$. On the other hand, any $\{d_j, j \ge 1\}$ of integer sequence satisfying $d_{j+1}(x) \ge d_j(x)\left(d_j(x)+1\right)$ for all $j \ge 1$ is a Sylvester admissible sequence, that is, there exists a unique $x \in (0,1]$ such that $d_j(x) = d_j$ for all $j \ge 1$, see [9].

The behaviors of the sequence $d_n(x)$ are of interest and the metric and ergodic properties of the sequence $\{d_n(x), n \ge 1\}$ and T have been investigated by a number of authors, see [1-3].

For any $x \in (0,1]$, define

$$S(x) := \sum_{n=1}^{+\infty} \left(x - \frac{p_n(x)}{q_n(x)}\right), \tag{5}$$

and we call $S(x)$ the error-sum function of Alternating Sylvester series. By (4), since $d_{n+1}(x) \ge d_n(x)\left(d_n(x)+1\right)$ for all $n \ge 1$, then $|S(x)| \le 1$ and $S(x)$ is well defined. In this paper, we shall discuss some basic nature of $S(x)$, also the Hausdorff dimension of the graph of $S(x)$ is determined.

2. Some Basic Properties of $S(x)$

In what follows, we shall often make use of the symbolic space.

For any $n \ge 1$, let

$$D_n = \left\{(\sigma_1, \sigma_2, \cdots, \sigma_n) \in N^n : \sigma_{k+1} \ge \sigma_k(\sigma_k + 1) \text{ for all } 1 \le k \le n\right\}.$$

Define

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$$D=\bigcup_{n=0}^{\infty} D_n,\left(D_0:=\varnothing\right).$$

For any $\sigma=\left(\sigma_1,\sigma_2,\cdots,\sigma_n\right)\in D_n$, write

$$A_\sigma=\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots+\left(-1\right)^{n-1}\frac{1}{\sigma_n}, \tag{6}$$

$$B_\sigma=\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots+\left(-1\right)^{n-1}\frac{1}{\sigma_n+1}. \tag{7}$$

We use J_σ to denote the following subset of (0,1],

$$J_\sigma=\left\{x\in(0,1]: d_1(x)=\sigma_1, d_2(x)=\sigma_2,\cdots,d_n(x)=\sigma_n\right\}. \tag{8}$$

From theorem 4.14 of [8], we have $J_\sigma=\left(A_\sigma,B_\sigma\right]$ when n is even, and $J_\sigma=\left(B_\sigma,A_\sigma\right]$ when n is odd. Finally, define

$$I=\left\{A_\sigma, B_\sigma, \sigma\in D_n, n\ge 1.\right\} \tag{9}$$

Lemma 1. *For any* $n\ge 1$ and $x\in(0,1]$,

1) $\lim_{x\to 0^+} S(x)=0;$ (10)

2) $-\frac{17}{30}\le S(x)\le 0;$ (11)

3) $S(x)=\sum_{i=1}^{n}\left(x-\frac{p_i(x)}{q_i(x)}\right)+(-1)^n S\left(T^n(x)\right).$ (12)

Proof. 1) Since $d_{j+1}(x)\ge d_j(x)\left(d_j(x)+1\right)$ and $d_1(x)\ge 1$, so when $n\ge 3$, we can get

$$d_{n+1}\ge d_n^2>\cdots>d_2^{2^{n-1}},$$

accordingly

$$d_n>d_2^{2^{n-1}}\ge\left(d_1^2\left(d_1+1\right)^2\right)^{n-2}.$$

we write $a(x)=d_1(x)^2\left(d_1(x)+1\right)^2$, so $d_n>a(x)^{n-2}$.

Now $d_{n+1}(x)=\left[\frac{1}{T^n(x)}\right]$ implies

$$\frac{1}{d_{n+1}(x)+1}<T^n(x)\le\frac{1}{d_{n+1}(x)},\quad \text{for } 0<T^n(x)\le 1.$$

Thus

$$\begin{aligned} S(x)&=\sum_{n=1}^{+\infty}(-1)^n T^n(x)\ge\sum_{n=1}^{+\infty}(-1)T^{2n-1}(x)\\ &\ge\sum_{n=1}^{+\infty}(-1)\frac{1}{d_{2n}(x)}\ge-\frac{1}{d_2(x)}-\sum_{n=2}^{+\infty}\frac{1}{a(x)^{2n-2}}\\ &\ge-\frac{1}{\sqrt{a(x)}}-\frac{1}{a(x)^2-1},\end{aligned}$$

let $x\to 0^+$, we have $d_1(x)\to+\infty$ and $a(x)\to+\infty$, thus

$$S(x)\to 0$$

2) From 1) we know that

$$d_{n+1}\ge d_n^2>\cdots>d_2^{2^{n-1}},$$

from the definition of $d_i(x)$ we also know that $d_1\ge 1$, so $d_2\ge d_1\left(d_1+1\right)\ge 2,$

$$d_{n+1}>d_2^{2^{n-1}}\ge 4^{n-1},$$

thus

$$S(x)\ge\sum_{n=1}^{+\infty}(-1)\frac{1}{d_{2n}(x)}\ge-\frac{1}{2}-\sum_{n=2}^{+\infty}\frac{1}{4^{2n-2}}=-\frac{17}{30}.$$

3) Since as $n>m$,

$$\frac{p_n(x)}{q_n(x)}-\frac{p_m(x)}{q_m(x)}=(-1)^m\frac{p_{n-m}\left(T^m(x)\right)}{q_{n-m}\left(T^m(x)\right)}.$$

Thus

$$\begin{aligned} S(x)&=\sum_{i=1}^{\infty}\left(x-\frac{p_i(x)}{q_i(x)}\right)\\ &=\sum_{i=1}^{n}\left(x-\frac{p_i(x)}{q_i(x)}\right)+\sum_{i=n+1}^{\infty}\left(x-\frac{p_n(x)}{q_n(x)}+\frac{p_n(x)}{q_n(x)}-\frac{p_i(x)}{q_i(x)}\right)\\ &=\sum_{i=1}^{n}\left(x-\frac{p_i(x)}{q_i(x)}\right)+\sum_{i=n+1}^{\infty}\left[(-1)^nT^n(x)-(-1)^n\frac{p_{i-n}\left(T^n(x)\right)}{q_{i-n}\left(T^n(x)\right)}\right]\\ &=\sum_{i=1}^{n}\left(x-\frac{p_i(x)}{q_i(x)}\right)+(-1)^n\sum_{i=1}^{\infty}\left[T^n(x)-\frac{p_i\left(T^n(x)\right)}{q_i\left(T^n(x)\right)}\right]\\ &=\sum_{i=1}^{n}\left(x-\frac{p_i(x)}{q_i(x)}\right)+(-1)^n S\left(T^n(x)\right).\end{aligned}$$

Let

$$I'=I\setminus\{1\}.$$

Proposition 2. *For any* $x\in I'$, *if* $x=\left[d_1(x),\cdots,d_{2k+1}(x)\right]$, *then* $S(x)$ *is left continuous but not right continuous. If* $x=\left[d_1(x),\cdots,d_{2k}(x)\right]$, *then* $S(x)$ *is right continuous but not left continuous.*

Proof. For any $n\ge 1$ and $\sigma\in D_n$, write $x_1=A_\sigma$, $x_2=B_\sigma$, where A_σ, B_σ are given by (6) and (7).

Case I, $n=2k+1$, then

$$x_1=\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots+\frac{1}{\sigma_{2k+1}} \tag{13}$$

$$x_2=\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots+\frac{1}{\sigma_{2k+1}+1} \tag{14}$$

and $J_\sigma=\left(B_\sigma,A_\sigma\right]$. For any $x_1'\in J_\sigma$, since when $\sigma_{2k+1}=\sigma_{2k}\left(\sigma_{2k}+1\right)$,

$$\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots-\frac{1}{\sigma_{2k}}+\frac{1}{\sigma_{2k+1}}$$
$$=\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots-\frac{1}{\sigma_{2k}}+\frac{1}{\sigma_{2k}\left(\sigma_{2k}+1\right)}$$
$$=\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots-\frac{1}{\sigma_{2k}+1}.$$

This situation is included in Case II, so we can take $\sigma_{2k+1}>\sigma_{2k}\left(\sigma_{2k}+1\right)$ and

$$x_1'=x_1-\frac{1}{\alpha}\text{ for some }\alpha\geq\sigma_{2k+1}\left(\sigma_{2k}+1\right).$$

i.e.

$$x_1'=\left[\sigma_1,\cdots,\sigma_{2k},\sigma_{2k+1},[\alpha]\cdots\right]$$

$$S\left(x_1'\right)-S\left(x_1\right)=\sum_{i=1}^{2k+1}\left(x_1'-\frac{p_i\left(x_1'\right)}{q_i\left(x_1'\right)}\right)+\left(x_1'-\frac{p_{2k+2}\left(x_1'\right)}{q_{2k+2}\left(x_1'\right)}\right)$$
$$-\sum_{i=1}^{2k+1}\left(x_1-\frac{p_i\left(x_1\right)}{q_i\left(x_1\right)}\right)+S\left(T^{2k+2}\left(x_1'\right)\right)$$
$$=-\frac{2k+1}{\alpha}+T^{2k+2}\left(x_1'\right)+S\left(T^{2k+2}\left(x_1'\right)\right)$$

By (2),

$$\frac{1}{d_{n+1}\left(x\right)+1}<T^n\left(x\right)\leq\frac{1}{d_{n+1}\left(x\right)+1},\text{ for }0<T^n\left(x\right)\leq 1,$$

which implies

$$T^{n+1}\left(x\right)=\frac{1}{d_{n+1}\left(x\right)}-T^n\left(x\right)<\frac{1}{d_{n+1}\left(x\right)}-\frac{1}{d_{n+1}\left(x\right)+1}$$
$$=\frac{1}{d_{n+1}\left(x\right)\left(d_{n+1}\left(x\right)+1\right)}$$

and

$$0<T^{2k+2}\left(x_1'\right)<\frac{1}{\alpha\left(\alpha+1\right)}.$$

Let $\alpha\to+\infty$, we get $T^{2k+2}\left(x_1'\right)\to 0$ and $S\left(T^{2k+2}\left(x_1'\right)\right)\to 0$, thus

$$\lim_{x_1'\to x_1^-}S\left(x_1'\right)=S\left(x_1\right),$$

and this implies $S\left(x\right)$ is left continuous at x_1.

Let

$$x_1''=x_1+\frac{1}{\alpha}\text{ for some}$$
$$\alpha\geq\left(\sigma_{2k+1}-1\right)\sigma_{2k+1}\left(\left(\sigma_{2k+1}-1\right)\sigma_{2k+1}+1\right),$$

$$i.e.\,x_1''=\left[\sigma_1,\cdots,\sigma_{2k},\sigma_{2k+1}-1,\left(\sigma_{2k+1}-1\right)\sigma_{2k+1},[\alpha],\cdots\right],$$

then

$$S\left(x_1''\right)-S\left(x_1\right)$$
$$=\sum_{i=1}^{2k}\left(x_1''-\frac{p_i\left(x_1''\right)}{q_i\left(x_1''\right)}\right)+\left(x_1''-\frac{p_{2k+1}\left(x_1''\right)}{q_{2k+1}\left(x_1''\right)}\right)$$
$$+\left(x_1''-\frac{p_{2k+2}\left(x_1''\right)}{q_{2k+2}\left(x_1''\right)}\right)+\left(x_1''-\frac{p_{2k+3}\left(x_1''\right)}{q_{2k+3}\left(x_1''\right)}\right)-S\left(T^{2k+3}\left(x_1''\right)\right)$$
$$-\left(\sum_{i=1}^{2k}\left(x_1-\frac{p_i\left(x_1\right)}{q_i\left(x_1\right)}\right)+\left(x_1-\frac{p_{2k+1}\left(x_1\right)}{q_{2k+1}\left(x_1\right)}\right)\right)$$
$$=\frac{2k+2}{\alpha}-\frac{1}{\left(\sigma_{2k+1}-1\right)\sigma_{2k+1}}-T^{2k+3}\left(x_1''\right)-S\left(T^{2k+3}\left(x_1''\right)\right).$$

Let $\alpha\to+\infty$, we have

$$\lim_{x_1''\to x_1^+}S\left(x_1''\right)=S\left(x_1\right)-\frac{1}{\left(\sigma_{2k+1}-1\right)\sigma_{2k+1}}$$

and this implies $S\left(x\right)$ is not right continuous at x_1. For

$$x_2=\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots+\frac{1}{\sigma_{2k+1}+1},\qquad(15)$$

following the same line as above, we have

$$\lim_{x_2'\to x_2^+}S\left(x_2'\right)=S\left(x_2\right)-\frac{1}{\sigma_{2k+1}\left(\sigma_{2k+1}+1\right)}.$$

Case II $n=2k$

Let

$$y_1=\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots-\frac{1}{\sigma_{2k}}\qquad(16)$$

$$y_2=\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots-\frac{1}{\sigma_{2k}+1}\qquad(17)$$

Following the same line as above, we have

$$\lim_{y_1'\to y_1^-}S\left(y_1'\right)=S\left(y_1\right)+\frac{1}{\left(\sigma_{2k}-1\right)\sigma_{2k}},$$

$$\lim_{y_2'\to y_2^-}S\left(y_2'\right)=S\left(y_1\right)+\frac{1}{\sigma_{2k}\left(\sigma_{2k}+1\right)},$$

and $S\left(y_1\right),S\left(y_2\right)$ is right continuous.

Corollary 3. *For any* $n\geq 1$ *and* $\sigma\in D_n$, *write* $\alpha_1=\max\left\{A_\sigma,B_\sigma\right\}$, $\alpha_2=\min\left\{A_\sigma,B_\sigma\right\}$. *Then for any* $x\in J_\sigma$, *if* $n=2k+1$, *then*

$$S^*\left(\alpha_2\right)<S\left(x\right)\leq S\left(\alpha_1\right),$$

where $S^*\left(\alpha_2\right)=S\left(\alpha_2\right)-\dfrac{1}{\sigma_{2k+1}\left(\sigma_{2k}+1\right)}$.

From the corollary, for any $\sigma\in D_n$

$$\sup_{x,y\in J_\sigma}\left|S\left(x\right)-S\left(y\right)\right|=\frac{n}{\sigma_n\left(\sigma_n+1\right)}=n\lambda\left(J_\sigma\right)$$

where $\lambda(J_\sigma)$ is the Lebesgue measure of J_σ.

Theorem 4. *$S(x)$ is continuous on $(0,1]\setminus I'$.*

Proof: For any $x\in(0,1]\setminus I'$ and $x\neq 1$, let $x=(d_1(x),\cdots,d_n(x),\cdots)$ be its Alternating Sylvester expansion. For any $n\geq 1$, write $\sigma^{(n)}=(d_1(x),\cdots,d_n(x))$. By (Corollary 3), for any $y\in J_{\sigma^{(n)}}$, we have

$$|S(x)-S(y)|\leq (n)\lambda\left(J_{\sigma^{(n)}}\right)\to 0,\ \text{as}\to\infty.$$

Write $I_0=\{C_\sigma\}$, where

$$C_\sigma=\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\cdots+\frac{1}{\sigma_{2k+1}}$$

Theorem 5. *If $0<a<b<1, S(a)<y<S(b)$, then there exists $c\in(a,b)\setminus\{I_0\}$, such that $S(c)=y$.*

Proof. Set $g(x)=S(x)-y$, then $g(x)$ has the same continuity as $S(x)$. Write

$$E=\{x|g(x)<0,x\in[a,b]\},\quad x_0=\sup E.$$

trivially, $a\in E$, then the set is well defined.

If $b=[\sigma_1,\sigma_2,\cdots,\sigma_{2k+1}]$, then by the left continuity of $S(b)$, we have

$$\lim_{x\to b^-}g(x)=g(b)>0,$$

As a result, there exists a $\delta_1>0$ such that for any $x\in(b-\delta_1,b), g(x)>0$.

If $b=[\sigma_1,\sigma_2,\cdots,\sigma_{2k}]$, since $g(b)$ is not left continuous, then $\exists\delta_2>0$ such that for any $x\in(b-\delta_2,b)$, $g(x)>0$, that is $x_0\neq b$.

Following the same line as above, we can prove $x_0>a$.

Now we shall prove that $g(x_0)\leq 0$. We can choose $x_n\in E$ such that $x_n\to x_0^-$, if $x_0=[\sigma_1,\sigma_2,\cdots,\sigma_{2k+1}]$, then

$$g(x_0)=\lim_{x_n\to x_0^-}g(x_n)\leq 0,$$

if $x_0=[\sigma_1,\sigma_2,\cdots,\sigma_{2k}]$, then

$$g(x_0)+\frac{1}{(\sigma_{2k}-1)\sigma_{2k}}=\lim_{x_n\to x_0^-}g(x_n)\leq 0$$

In both case $g(x_0)\leq 0$. Following the same line as above, we can prove $g(x_0)=0$, and $x_0\neq[\sigma_1,\sigma_2,\cdots,\sigma_{2k+1}]$.

Therefore, there exists $c\in(a,b)\setminus\{I_0\}$, such that $S(c)=y$.

Theorem 6. $\int_0^1 S(x)\mathrm{d}x+\sum_{k=1}^{+\infty}\int_0^{\frac{1}{k(k+1)}}S(x)\mathrm{d}x=\frac{9-\pi^2}{6}$,

and $\int_0^1 S(x)\mathrm{d}x=-0.1250$.

Proof.

$$\int_0^1 S(x)\mathrm{d}x=\sum_{d_1=1}^{+\infty}\int_{\frac{1}{d_1+1}}^{\frac{1}{d_1}}S(x)\mathrm{d}x$$
$$=\sum_{d_1=1}^{+\infty}\int_{\frac{1}{d_1+1}}^{\frac{1}{d_1}}\left(\left(x-\frac{1}{d_1}\right)-S(T(x))\right)\mathrm{d}x$$
$$=\sum_{d_1=1}^{+\infty}\int_{\frac{1}{d_1+1}}^{\frac{1}{d_1}}x\mathrm{d}x-\sum_{d_1=1}^{+\infty}\int_{\frac{1}{d_1+1}}^{\frac{1}{d_1}}\frac{1}{d_1}\mathrm{d}x-\sum_{d_1=1}^{+\infty}\int_{\frac{1}{d_1+1}}^{\frac{1}{d_1}}S(T(x))\mathrm{d}x$$

Let $Tx=u=\frac{1}{d_1(x)}-x$, then $\mathrm{d}u=-\mathrm{d}x$ thus

$$\int_0^1 S(x)\mathrm{d}x=\frac{1}{2}\sum_{d_1=1}^{+\infty}\left(\frac{1}{d_1^2}-\frac{1}{(d_1+1)^2}\right)$$
$$-\sum_{d_1=1}^{+\infty}\frac{1}{d_1^2}+\sum_{d_1=1}^{+\infty}\frac{1}{d_1(d_1+1)}-\sum_{d_1=1}^{+\infty}\int_0^{\frac{1}{d_1(d_1+1)}}S(u)\mathrm{d}u$$

thus,

$$\int_0^1 S(x)\mathrm{d}x+\sum_{k=1}^{+\infty}\int_0^{\frac{1}{k(k+1)}}S(x)\mathrm{d}x=\frac{3}{2}-\sum_{d_1=1}^{+\infty}\frac{1}{d_1^2}=\frac{9-\pi^2}{6}.$$

Through the MATLAB program we can get the definite integration

$$\int_0^1 S(x)\mathrm{d}x=-0.1250.$$

3. Hausdorff Dimension of Graph for $S(x)$

Write

$$Gr(S)=\{(x,S(x)),x\in(0,1]\}.$$

Theorem 7. $\dim_H Gr(S)=1$.

Proof. For any $n\geq 1$, $\{J_\sigma\times S(J_\sigma),\sigma\in D_n\}$ is a covering of $Gr(S)$. From (Cor 3), $J_\sigma\times S(J_\sigma)$ can be covered by n squares with side of length $\lambda(J_\sigma)$. For any $\varepsilon>0$,

$$H^{1+\varepsilon}(Gr(S))\leq\liminf_{n\to\infty}\sum_{\sigma\in D_n}n\left(\sqrt{2}\right)^{1+\varepsilon}\left(\lambda(J_\sigma)\right)^{1+\varepsilon}$$
$$\leq\liminf_{n\to\infty}n\left(\sqrt{2}\right)^{1+\varepsilon}2^{-n\varepsilon}\sum_{\sigma\in D_n}n\left(\sqrt{2}\right)^{1+\varepsilon}$$
$$=\liminf_{n\to\infty}n\left(\sqrt{2}\right)^{1+\varepsilon}2^{-n\varepsilon}=0.$$

Thus, $\dim_H Gr(S)\leq 1$

Since

$$\left|Proj(x,S(x))-Proj(y,S(y))\right|\leq d(x,S(x),(y,S(y))),$$

then

$$1=\lambda((0,1])=H^1(0,1]=H^1\left(Proj(G_r(S))\right)\leq^1(G_r(S)),$$

so $\dim_H Gr(S)=1$.

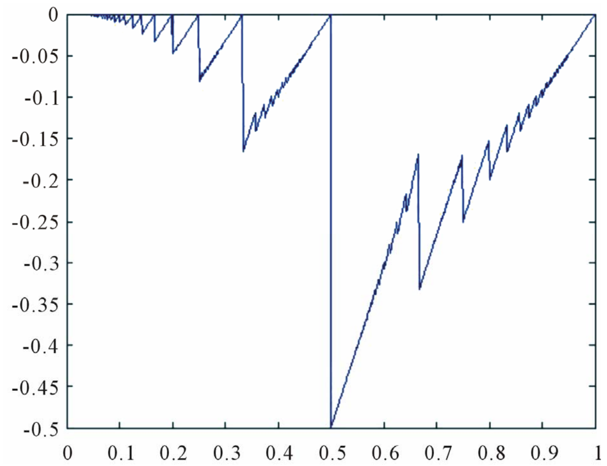

Figure 1. The graph of $S(x)$.

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