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Dynamic Systems of Shifts in the Space of Piece-Wise Continuous Functions

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ABSTRACT

In this paper we embark on the study of Dynamic Systems of Shifts in the space of piece-wise continuous functions analogue to the known Bebutov system. We give a formal definition of a topological dynamic system in the space of piece-wise continuous functions and show, by way of an example, stability in the sense of Poisson discontinuous function. We prove that a fixed discontinuous function, *f*, is discontinuous for all its shifts, τ, whereas the trajectory of discontinuous function is not a compact set.

Keywords: Dynamic System; Shifts; Piecewise Continuous Functions; Compact Sets; Poisson Discontinuous Function

1. Introduction

The interest in the study of Differential Equations with Impulse is increasing. Attempt to extend this study Dontwi [1] to known topological methods of the Theory of Dynamic Systems (DS) (see Sibiriskii [2], Levitan and Zhikov [3], Shcherbakov [4,5], Cheban [6,7]) brings into fore the necessity of studying DS of shifts in the space of piece-wise-continuous functions which are solutions of these equations.

In this paper we extend the study of Dynamical Systems of shifts in the space of piece-wise-continuous functions analogue to Bebutov Systems. We give a formal definition of a topological dynamic system in the space of piece-wise continuous functions and show, by way of an example, stability in the sense of poisson discontinueous function. We prove that a fixed discontinuous function, *f*, is discontinuous for all its shifts, τ, whereas the trajectory of discontinuous function is not a compact set. These should prepare the way for the introduction and application of notions of Recurrence motions of dynamic systems (Bronshtein [8], Pliss [9], Sacker and Sell [10], Sell [11]) to various trajectories of Differential Equations with Impulse (Distributions) (Hale [12], Cheban [13,14], and Dontwi [15]).

Ergodic dynamical system on the finite measure space and its kronecker factor were considered in Assani [16]. Pointwise convergence of ergodic averages along cubes was proved in Assani [17]. In Assani, Buczolich and Mauldin [18], negative solution to counting problem for measure preserving transformation was carried out. Full measures were treated in Assani [19]. A question of H. Furstenburg on the pointwise convergence of the averages was answered in Assani [20]. The pointwise convergence of some weighted averages linked to averages along cubes was studied in Assani [21]. Two questions related to the strongly continuous semigroup were answered in Assani and Lin [22]. Characteristics for certain nonconventional averages were studied in Assani and Presser [23]. Differentiable or smooth instead of topological gives a description of Differentiable Dynamics by Vries in [24].

Concepts such as metric spaces, normed spaces, convergence and homeomorphisms, compactness, and the Heine-Borel Theorem are considered to be known. In terms of discussing shifts this stems from several important applications of symbolic dynamics in the field of dynamical systems. It goes without telling that symbolic dynamics is a strong and formidable tool used in the study of dynamical system in Peyam [25]. The advantages that are gleaned from it are that the technique reduces a complicated system into a set of sequences. Mention should be made in the following passing: invariants, the Zeta function, Markov partition, and Homoclinic orbits.

2. Notions and Preliminaries

Let **R** and **N** be the set of real numbers and the set of natural numbers respectively, $f(t_0-0)$, $f(t_0+0)$ be the left and right sided limits of the function $f(t)$ at the point $t=t_0$.

We consider **PC[R]**—the space of piece-wise-continuous real-valued functions defined on the number line

R with the following properties:

1) The set of points of discontinuity of every function $f \in$ **PC[R]**, represented as $\mathbf{D}(f)$, is either empty or has points of discontinuity of the first kind;

2) The point of discontinuity of every function, if it is more than one, is distinct from each other at a distance not less than some fixed positive number for a given function.

The jump or discontinuity of the function $f(t)$ at the point $t = t_0$ is the number

$$h = \max\left\{\left|f(t_0 - 0) - f(t_0)\right|, \left|f(t_0 + 0) - f(t_0)\right|, \left|f(t_0 + 0) - f(t_0 - 0)\right|\right\}$$

In **PC[R]**, (or simply **PC**), we consider countable partitions of family of semi-norms

$$P_n(f) = \sup_{|t| \le n} \left|f(t)\right| \quad (n \in N)$$

defined for every function $f \in$ **PC** and induces metrizable topology in this space. Further we shall represent this metrizable space by **PC**.

Definition 1.

An alphabet ***A*** is a set of symbols. A common example is $[7] := \{0,1,2,3,\cdots,5,6\}$ and in general $[n] := \{0,1,2,3,\cdots,n-1\}$.

Definition 2.

Given an alphabet ***A***, the full shift space is $M = A^Z$ (*i.e.* the space of sequence is from *Z* into ***A***)

Definition 3.

A homeomorphism, ρ, from (N,ϕ) to (R,ψ) is a continuous function $\varphi : N \to R$ such that $\psi \circ \varphi = \varphi \circ \phi$.

Definition 4.

Two dynamic systems (N,ϕ) and (R,ψ) are topological conjugates if there exist a homeomorphism φ between them that is also a homeomorphism. This confirms that the subject of dynamical systems studies how a given system behaves throughout time which can be discrete or continuous iterates.

Definition 5.

An infinite subset *T* of *A* is compact if and only if every infinite subset of *T* has a limit point in *T* [26].

Definition 6.

A function $f : X \to Y$ between topological spaces is called continuous, if $f^{-1}(V)$ is open in *X* for every open $V \subseteq Y$. In Hoffman [27] the set of all continuous functions $f : X \to Y$ is often denoted by $C(X,Y)$.

Remark 1.

The sequence of functions $\{f_n(t)\}$ from **PC** is convergent if in **PC** there exist a function $f(t)$ such that $f_n(t)$ converges uniformly to $f(t)$ in every interval $[-k,k]$, where $k \in N$. We write this in the form

$$\lim_{n\to\infty} f_n = f.$$

The following hold:

Lemma 1.

If the function $f(t)$ at the point $t = t_0$ is a jump of magnitude $h > 0$ while the function $g(t)$ is continuous at this point, then for every $n \in N$ and $n > |t_0|$ the following is true:

$$P_n(f - g) \ge \frac{h}{2}.$$

Lemma 2.

Let $\lim_{n\to\infty} f_n(t) = f$. Then

1) If the function $f(t)$ is discontinuous at the point $t = t_0$ then all functions $f_n(t)$ (except, maybe, for a finite number of points) are also discontinuous at the same point. As a consequence we have the following:

2) If beginning from some number, all functions $f_n(t)$ are continuous at the point $t = t_0$ then the function $f(t)$ is also continuous at this point.

The reverse of the above statements hold.

Example 1.

Let $f(t) = 0$ and

$$f_n(t) = \begin{cases} \dfrac{1}{n} & \text{for } t > 0; \\ 0 & \text{for } t \le 0. \end{cases}$$

then $\lim_{n\to\infty} f_n = f$.

Remark 2.

The space **PC** is not complete.

For any $f \in$ **PC** and $\tau \in$ **R** we represent by the symbol f^τ the shifts of the function $f(t)$ by τ, that is $f^\tau(t) = f(t + \tau)$.

Following Bebutov dynamic systems in the space **PC** we consider the family of shifts (or translates) $\varphi : \mathbf{PC} \times \mathbf{R} \to \mathbf{PC}$, defined by the formula $\varphi(f,\tau) := f^\tau$ for all $f \in$ **PC**, $\tau \in$ **R**.

Definitions 1 to 5 serve as clues to the concepts discussed.

3. Main Results

Theorem 1.

The mapping φ defined above satisfies the following conditions:

1) $\varphi(f,0) = f$, for any $f \in$ **PC**;

2) $\varphi(\varphi(f,s)) = \varphi(f,t+s)$, for any $f \in$ **PC**; and $t,s \in$ **R**;

3) $\varphi(f,\tau)$ is continuous in *f* for any fixed τ and for a fixed *f*-continuous function, the mapping $\varphi(f,\tau)$ is continuous in τ however, if for a fixed *f* it is a discontinuous function then $\varphi(f,\tau)$ is discontinuous at all points of τ

Proof.

1) and 2) are obvious.

Continuity of $\varphi(f,\tau)$ in *f* for a fixed τ by Remark 1 implies uniform convergence of the function $f_n(t)\to f(t)$ as $n\to\infty$ in every interval $|t|\le m$, $m\in N$, which in turn implies uniform convergence of the function $f_n(t+\tau)\to f(t+\tau)$ as $n\to\infty$ in every interval $|t|\le k$, $k\in N$.

If f_0 is a continuous function, then $\varphi(f,\tau)$ is continuous in τ by the known property of Bebutov Dynamic System (Sell [11]) and is uniformly continuous in Obeng-Denteh [28].

The motion corresponding to the continuous function *f* is continuous, and if it is discontinuous it will be discontinuous at every point.

Theorem 2.

For any arbitrary discontinuous function *f* from **PC**, its trajectory is not a compact set.

Proof.

Let *f* be any discontinuous function in **PC**. Consider $\{f^{\tau m}\}$, where $\tau_n=\frac{1}{n}$. The given sequence converges point-wise to the function *f* of points of continuity of *f*. However, no sub-sequence of the given sequence converges to *f* in **PC**, and this means, in general it does not converge in **PC**.

4. Conclusion

The topological dynamical system in the space of piecewise continuous functions has been shown by way of an example, as well as stability in the sense of Poisson discontinuous function. It has also been proved that a fixed discontinuous function, *f*, is discontinuous for all its shifts, τ, whereas the trajectory of the discontinuous function is not a compact set.

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