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Uniqueness of Radial Solutions for Elliptic Equation Involving the Pucci Operator*

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ABSTRACT

The solution of a nonlinear elliptic equation involving Pucci maximal operator and super linear nonlinearity is studied. Uniqueness results of positive radial solutions in the annulus with Dirichlet boundary condition are obtained. The main tool is Lane-Emden transformation and Koffman type analysis. This is a generalization of the corresponding classical results involving Laplace operator.

Keywords: Pucci Operator; Radial Solution; Uniqueness; Super Linear

1. Introduction

We study the nonlinear elliptic equation

$$M^{+}_{\lambda,\Lambda}\left(D^2u\right)+f\left(u\right)=0, \qquad (1)$$

where $0<\lambda<\Lambda$, $M^{+}_{\lambda,\Lambda}$ is Pucci maximal operator, the potential f is super linear with some further constraints. Using $\mu_i, i=1,\cdots,n$ to denote the eigenvalues of D^2u, then explicitly, the Pucci operator $M^{+}_{\lambda,\Lambda}$ is given by

$$M^{+}_{\lambda,\Lambda}\left(D^2u\right)=\sum_{\mu_i>0}\Lambda\mu_i+\sum_{\mu_i<0}\lambda\mu_i.$$

For more detailed discussion, see for example [1,2]. This equation has been extensively studied, see [3-5], etc. and the references therein.

Normalize λ to be 1 for simplicity. We will in this paper investigate the uniqueness of C^2 positive radial solution of (1) in the annulus

$$\Omega:=\left\{x\in\mathbb{R}^n : a<\left|x\right|<b\right\}$$

with Dirichlet boundary condition. In this case, Equation (1) reduces to

$$\Lambda\left(u''\right)u''\left(r\right)+\Lambda\left(u'\right)\frac{n-1}{r}u'\left(r\right)+f\left(u\right)=0, r\in\left(a,b\right). \qquad (2)$$

where

$$\Lambda\left(s\right)=\begin{cases}\Lambda, & \text{for } s>0,\\ 1, & \text{for } s\le 0.\end{cases}$$

Throughout the paper, we assume $n\ge 2$. Note that $\Lambda>1$. Now we could state our main results.

Theorem 1. *Suppose* $\frac{b}{a}$ *is small enough and*

$$tf'\left(t\right)>f\left(t\right)>0 \text{ for } t>0.$$

Then (2) has at most one positive solution with Dirichlet boundary condition.

If instead of the smallness of $\frac{b}{a}$ we assume further growing condition on f, then we have the following

Theorem 2. *Suppose that for* $t>0$,

$$\frac{\left(2\Lambda-1\right)n-2\Lambda+2}{\left(2\Lambda-1\right)n-2\Lambda}f\left(t\right)>tf'\left(t\right)>\mu f\left(t\right)>0,$$

where

$$\mu=\max\left\{1,\frac{-\left(2\Lambda-1\right)n+2\Lambda+2}{\left(2\Lambda-1\right)n-2\Lambda}\right\}.$$

Then (2) has at most one positive solution with Dirichlet boundary condition.

In the case $\Lambda=1$, the Pucci operator reduces to the usual Laplace operator, and the corresponding unique results are proved by Ni and Nussbaum in [6].

We also remark that the above theorems could be generalized to nonlinearities f, which also depends on r. We will not pursue this further in this paper.

2. Lane-Emden Transformation and Uniqueness of the Radial Solutions

2.1. Proof of Theorem 1

We shall perform a Lane-Emden type transformation to Equation (2). Let us introduce a new function

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$$w(s):=u(r),$$

where $s=r^{\alpha}$, with

$$\alpha:=1-\Lambda(n-1)<0.$$

Then w satisfies

$$w''(s)+ms^{-1}w'(s)+\frac{f(w)s^{\frac{2}{\alpha}-2}}{\Lambda(u''(r))\alpha^2}=0, s\in(s_1,s_2), \quad (3)$$

where we have denoted

$$m(s):=\frac{\Lambda(u''(r))(\alpha-1)+\Lambda(u'(r))(n-1)}{\Lambda(u''(r))\alpha}\geq 0,$$

and $(s_1,s_2)=(b^{\alpha},a^{\alpha})$. Note that m may not be continuous at the points where $u''(r)=0$ or $u'(r)=0$. Additionally, if $u''(r)<0$ and $u'(r)>0$, then $m(s)=0$.

Lemma 3. *Let* w *be a positive solution of* (3) *with* $w(s_1)=w(s_2)=0$. *Then there exists* $\bar{s}\in(s_1,s_2)$ *such that* $w'(\bar{s})=0$, *and*

$$w'(s)>0, \text{in } (s_1,\bar{s}),$$

$$w'(s)<0, \text{in } (s_1,\bar{s}),$$

Proof. If $w'(\xi)=0$ for some $\xi\in(s_1,s_2)$, then

$$w''(\xi)=-ms^{-1}w'(\xi)-\frac{f(w)s^{\frac{2}{ga}-2}}{\Lambda(u''(r))\alpha^2}<0.$$

The conclusion of the lemma follows immediately from this inequality. ■

Given $d>0$, the solution of (3) with $w(s_1)=0$, and $w'(s_1)=d$ will be denoted by $w(s,d)$. Let

$$\varphi(s,d)=\partial_d w(s,d).$$

By standard argument, we know that positive solution of (3) with Dirichlet boundary condition is unique if we could show that

$$\varphi(s_2,d_0)<0,$$

whenever $w(s,d_0)$ is a positive solution to (3) with $w(s_1,d_0)=w(s_2,d_0)=0$.

The functions φ and w satisfy the following equations:

$$\left(s^m w'\right)'+\frac{f(w)s^{m+\frac{2}{\alpha}-2}}{\Lambda(u''(r))\alpha^2}=0,$$

$$\left(s^m \varphi'\right)'+\frac{f(w)s^{m+\frac{2}{\alpha}-2}}{\Lambda(u''(r))\alpha^2}\varphi=0,$$

The initial condition satisfied by φ is: $\varphi(s_1)=0$, $\varphi'(s_1)=1$.

Now let d_0 be a positive constant such that $w(s,d_0)$ is a positive solution to (3) with $w(s_1,d_0)=w(s_2,d_0)=0$. To show that $\varphi(s_2,d_0)<0$, let us first prove that $\varphi(\cdot,d_0)$ must vanish at some point in the interval (s_1,s_2). In the following, we write $\varphi(s,d_0)$ simply as $\varphi(s)$.

Lemma 4. *There exists* $\xi\in(s_1,s_2)$ *such that* $\varphi(\xi)=0$.

Proof. Let us consider the function

$$\eta_1(s):=s^m w'(s)\varphi(s)-s^m\varphi'(s)w(s).$$

We have

$$\eta_1'(s)=\left(s^m w'\right)'\varphi-\left(s^m\varphi'\right)'w$$
$$=\frac{s^{m+\frac{2}{\alpha}-2}}{\Lambda(u''(r))\alpha^2}\varphi\left[f'(w)w-f(w)\right].$$

We remark that η_1 is indeed not everywhere differentiable, since m is not continuous. It however could be shown that the jump points of m are isolated. Here by η_1', we mean the derivative of η_1 at the point where it is differentiable. The same remark applies to the functions η_2 and η_3 below.

Now if $\varphi(s)>0$ for $s\in(s_1,s_2)$, then

$$\eta_1'(s)>0, s\in(s_1,s_2).$$

Since $\eta_1(s_1)=0$, we infer that

$$\eta_1(s_2)>0.$$

It follows that

$$w'(s_2)\varphi(s_2)>0.$$

This is a contradiction, since $w'(s_2)\leq 0$ and $\varphi(s_2)\geq 0$. ■

With the above lemma at hand, we wish to show that in the interval $(s_1,s_2]$, φ vanishes at only one point ξ. For this purpose, let us define functions $g(s):=w'(s)$ and $h(s):=(s-s_1)w'(s)$. Put

$$\eta_2(s):=s^m g'(s)\varphi(s)-s^m\varphi'(s)g(s),$$

$$\eta_3(s):=s^m h'(s)\varphi(s)-s^m\varphi'(s)h(s),$$

and

$$\Phi(w,s):=\frac{f(w)}{\Lambda(u''(r))\alpha^2}s^{\frac{2}{\alpha}-2}.$$

Lemma 5. *We have*

$$\eta_2'(s)=s^m\left(ms^{-2}g-\Phi_s\right)\varphi, \quad (4)$$

$$\eta_3'(s)=s^m\left\{-ms_1s^{-2}g-(s-s_1)\Phi_s-2\frac{f(w)s^{\frac{2}{\alpha}-2}}{\Lambda(u''(r))\alpha^2}\right\}\varphi. \quad (5)$$

Proof. Differentiate the Equation (3) with respect to s gives us

$$g''(s)+ms^{-1}g'(s)+\frac{f'(w)}{\Lambda(u''(r))\alpha^2}s^{\frac{2}{\alpha}-2}g=ms^{-2}g-\Phi_s. \quad (6)$$

Hence

$$\eta_2'(s)=(s^m g')'\varphi-(s^m\varphi')'g=s^m(ms^{-2}g-\Phi_s)\phi.$$

As to the function h, there holds

$$\frac{(s^m h'(s))'}{s^m}=\frac{(s-l_1)(s^m g')'}{s^m}+2\frac{(s^m w')'}{s^m}-\frac{mg}{s}.$$

Combining this with (3) and (6) we get

$$\frac{(s^m h'(s))'}{s^m}=-ms_1s^{-1}g-\frac{f'(w)s^{\frac{2}{\alpha}-2}}{\Lambda(u''(r))\alpha^2}h$$
$$-(s-s_1)\Phi_s-2\frac{f(w)s^{\frac{2}{\alpha}-2}}{\Lambda(u''(r))\alpha^2}.$$

It follows that

$$\eta_3'(s)=(s^m h')'\varphi-(s^m\varphi')'h$$
$$=s^m\left[-ms_1s^{-1}g-(s-s_1)\Phi_s-2\frac{f(w)s^{\frac{2}{\alpha}-2}}{\Lambda(u''(r))\alpha^2}\right]\varphi.$$

■

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We need to show that $\varphi(s_2)<0$.

We first of all claim that the first zero ξ of φ in (s_1,s_2) must stay in the interval $[\bar{s},s_2)$, where $\bar{s}$ is given by Lemma 3. Suppose to the contrary that $\xi\in(s_1,\bar{s})$. By (5) using the fact that $m\geq 0$, we find that if $\frac{b}{a}$ is small enough, then in the interval (s_1,ξ),

$$\eta_3'(s)$$
$$=s^m\left\{-ms_1s^{-2}w'(s)-(s-s_1)\Phi_s-2\frac{f(w)s^{\frac{2}{\alpha}-2}}{\Lambda(u''(r))\alpha^2}\right\}\varphi<0.$$

Since $\eta_3(s_1)=0$, we find that

$$\eta_3(\xi)=\xi^m\left[h'(\xi)\varphi(\xi)-h(\xi)\varphi'(\xi)\right]<0.$$

Therefore

$$h(\xi)\varphi'(\xi)>0.$$

This is a contradiction, since $\varphi'(\xi)\leq 0$ and $h(\xi)=(\xi-s_1)w'(\xi)>0$.

Now the first zero ξ of φ lies in $[\bar{s},s_2)$, If $\varphi(s_2)\geq 0$ then the second zero ξ_1 of φ lies in $(\xi,s_2]$. Note that in (ξ,ξ_1), $m=0$. Therefore, by identity (4)

$$\eta_2'(s)=-\Phi_s\varphi<0.$$

This together with

$$g'(\xi)\varphi(\xi)-g(\xi)\varphi'(\xi)\leq 0$$

implies that

$$g'(\xi_1)\varphi(\xi_1)-g(\xi_1)\varphi'(\xi_1)<0$$

but this contradicts with $\varphi(\xi_1)=0$, $\varphi'(\xi_1)\geq 0$, and $g(\xi_1)<0$. This finishes the proof. ■

2.2. Proof of Theorem 2

Similar arguments as that of Theorem 1 could be used to prove Theorem 2. In this case, we shall make the following transform:

$$u(r)=r^\alpha w(s),\ s=r^\beta,$$

where

$$\alpha=1-\Lambda(n-1)<0,$$

and $\beta=2-n-2\alpha>0$. Then

$$u''(r)=\beta^2r^{\alpha-2}s^2w''(s)+\beta(2\alpha+\beta-1)r^{\alpha-2}sw'(s)$$
$$+\alpha(\alpha-1)r^{\alpha-2}w(s) \quad (7)$$

With this transformation, in the interval $(s_1,s_2)=(a^\beta,b^\beta)$, w satisfies

$$w''(s)+m_1s^{-1}w'(s)+m_2s^{-2}w+\frac{s^{\frac{2}{\beta}-1}f(s^{-1}w)}{\Lambda(u''(r))\beta^2}=0, \quad (8)$$

where

$$m_1(s):=\frac{\Lambda(u''(r))(2\alpha+\beta-1)+\Lambda(u'(r))(n-1)}{\Lambda(u''(r))\beta},$$

$$m_2(s):=\frac{\alpha\left[\Lambda(u''(r))(\alpha-1)+\Lambda(u'(r))(n-1)\right]}{\Lambda(u''(r))\beta^2}.$$

By the definition of α,β, one could verify that $m_2\geq 0$. Note that m_1 and m_2 are step functions and not continous.

Let $w(s,d)$ be the solution of (8) with $w(s_1,d)=0$ and $w_s(s_1,d)=d$. Now similar as in the proof of Theorem 1, we suppose $w=w(s,d_0)$ is a positive solution with Dirichlet boundary condition and $\varphi=\partial_d w(s,d_0)$. We have the following lemma, whose proof will be omitted.

Lemma 6. *There exists* $\bar{s}\in(s_1,s_2)$ *such that* $w'(\bar{s})=0$, *and*

$$w'(s)>0,\ \text{in}\,(s_1,\bar{s}),$$

$$w'(s)<0, \text{in}\left(\bar{s},s_2\right).$$

With this lemma at hand, we observe that by (8)

$$w''(\bar{s})+m_2\bar{s}^{-2}w(\bar{s})+\frac{\bar{s}^{\frac{2}{\beta}-1}f\left(\bar{s}^{-1}w\right)}{\Lambda\left(u''\left(\bar{s}^{\frac{1}{\beta}}\right)\right)\beta^2}=0.$$

This combined with (7) tells us that $u''\left(\bar{s}^{\frac{1}{\beta}}\right)<0.$ Then it is not difficult to show that for $s\in(s_1,\bar{s})$, $u''\left(\bar{s}^{\frac{1}{\beta}}\right)<0.$ and $m_1(s)\geq 0$; while for $s\in(\bar{s},s_2)$, $m_1(s)\leq 0.$

Recall that φ satisfies

$$\varphi''(s)+m_1s^{-1}\varphi'(s)+m_2s^{-2}\varphi+\frac{s^{\frac{2}{\beta}-1}f'\left(s^{-1}w\right)}{\Lambda\left(u''(r)\right)\beta^2}s^{-1}\varphi=0.$$

Consider the function $\eta_1=s^{m_1}w'\varphi-s^{m_1}\varphi'w,$ then

$$\eta_1'(s)=\frac{s^{m_1+\frac{2}{\beta}-1}}{\Lambda\left(u''(r)\right)\beta^2}\varphi\left[f'\left(s^{-1}w\right)s^{-1}w-f\left(s^{-1}w\right)\right].$$

From this we infer that the function φ must change sign in the interval (s_1,s_3), similar as that of Theorem 1.

Now let us define

$$\eta_2(s):=s^{m_1}g'\varphi-s^{m_1}g\varphi',$$

and

$$\eta_3(s)=s^{m_1}h'(s)\varphi(s)-s^{m_1}h(s)\varphi'(s),$$

where $g=w'$ and $h=(s-s_1)g$. Moreover, denote

$$\Phi(w,s):=\frac{s^{\frac{2}{\beta}-1}f\left(s^{-1}w\right)}{\Lambda\left(u''(r)\right)\beta^2}.$$

Lemma 7. *There holds*

$$\eta_2'(s)=s^{m_1}\left[m_1s^{-2}g+2m_2s^{-3}w-\Phi_s\right]\varphi,$$

$$\eta_3'(s)=s^{m_1}\left[-s_1m_1s^{-2}g-2s_1m_2s^{-3}w-(s-s_1)\Phi_s-2\frac{s^{\frac{2}{\beta}-1}f\left(s^{-1}w\right)}{\Lambda\left(u''(r)\right)\beta^2}\right]\varphi.$$

Proof. Direct calculation shows

$$g''(s)+m_1s^{-1}g'(s)+m_2s^{-2}g+\frac{s^{\frac{2}{\beta}-1}f'\left(s^{-1}w\right)s^{-1}g}{\Lambda\left(u''(r)\right)\beta^2}=m_1s^{-2}g+2m_2s^{-3}w-\Phi_s,$$

and

$$\left(s^{m_1}h'\right)'+m_2s^{m_1-2}h=-s_1m_1s^{m_1-2}g-2s_1m_2s^{m_1-3}w-(s-s_1)s^{m_1}\Phi_s-\frac{s^{m_1+\frac{2}{\beta}-1}f'\left(s^{-1}w\right)s^{-1}h}{\Lambda\left(u''(r)\right)\beta^2}-2\frac{s^{m_1+\frac{2}{\beta}-1}f\left(s^{-1}w\right)}{\Lambda\left(u''(r)\right)\beta^2}.$$

This then leads to the desired identity. ■

Now with the help of this lemma, we could prove Theorem 2.

Proof of Theorem 2. First we show the first zero ξ of φ is in the interval $[\bar{s},s_2)$ Otherwise, since

$$(s-s_1)\Phi_s+2\frac{s^{\frac{2}{\beta}-1}f\left(s^{-1}w\right)}{\Lambda\left(u''(r)\right)\beta^2}=\frac{(s-s_1)s^{\frac{2}{\beta}-2}}{\Lambda\left(u''(r)\right)\beta^2}\left[\left(\frac{2}{\beta}+1\right)f+2\frac{s_1}{s-s_1}f\left(s^{-1}w\right)-s^{-1}wf'\left(s^{-1}w\right)\right]\geq 0,$$

one could then use the fact that $m_1\geq 0$ in $(s_1,\bar{s})$ and $m_2\geq 0$ to deduce that in (s_1,ξ),

$$\eta_3'(s)<0.$$

But this contradicts with $\eta_3(s_1)=0$ and $\eta_3(\xi)\geq 0$.

Now if the second zero ξ_1 of φ is in $(\xi,s_2]$. Then since

$$\Lambda\left(u''(r)\right)\beta^2\Phi_s=s^{\frac{2}{\beta}-2}\left[\left(\frac{2}{\beta}-1\right)f\left(s^{-1}w\right)-s^{-1}wf'\left(s^{-1}w\right)\right]<0,$$

one could use $m_1\leq 0$ in $(\bar{s},s_2)$ to deduce that $\eta_2'<0$ in (ξ,ξ_1), which contradicts with $\eta_2(\xi)=0$ and

$\eta_2(\xi_1) \geq 0.$ ■

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