Advances in Pure Mathematics, 2012, 2, 367-370
http://dx.doi.org/10.4236/apm.2012.26054 Published Online November 2012 (http://www.SciRP.org/journal/apm)

A Note on Nilpotent Operators

Abhay K. Gaur
Department of Mathematics, Duquesne University, Pittsburgh, USA
Email: gaura@duq.edu

Received July 16, 2012; revised September 11, 2012; accepted September 19, 2012

ABSTRACT

We find that a bounded linear operator T on a complex Hilbert space H satisfies the norm relation $\left\||T|^n a\right\| = 2q$, $n = 1, 2, \cdots$, for any vector a in H such that $q \le \left(\|Ta\| - 4^{-1}\|Ta\|^2\right) \le 1$. A partial converse to Theorem 1 by Haagerup and Harpe in [1] is suggested. We establish an upper bound for the numerical radius of nilpotent operators.

Keywords: Numerical Range; Numerical Radius; Nilpotent Operator Weighted Shift; Eigenvalues

1. Introduction

The motivation for this note is provided by the results obtained in [1-4]. Let T be a bounded linear operator on a complex Hilbert space H. The numerical range of T, denoted by $W(T)$, is the subset of the complex plane and

$$W(T) = \{(Tx, x) : x \in H, \|x\| = 1\}.$$

The numerical radius of T is defined as,

$$w(T) = \sup\{|z| : z \in W(T)\}.$$

The following lemma is known and is an easy consequence of the definitions involved.

Lemma 1.1. *$W(T) = \sup\{\|zT + \bar{z}T^*\| : |z| = 1\}$, where T^* is the adjoint operator of T and $\bar{z}$ is the complex conjugate of z.*

Berger and Stampfli in [2] have proved that if $w(T) \le 1$ and $\|T^n x\| = 2\|x\|$, for some n, then $T^{n+1}x = 0$. Also, they gave an example of an operator T and an element $x \in H$ such that $w(T)=1$ implies that $\|T^n x\| \le k\|x\|$ and $k \ge \sqrt{2}$. In Theorem 2.1, we present a different proof of their result in [2] and show that $\sqrt{2}$ is indeed the best constant.

Theorem 2.1 also generalizes the result in [4] and provides a partial converse to Theorem 1 in [1, p. 372].

Our next main result in Theorem 2.3 gives an alternative and shorter proof of Theorem 1 in [1].

Applying Lemma 2 and Proposition 2 of [1], a new result on the numerical range of nilpotent operators on H is obtained in Theorem 2.4. This gives a restricted version of Theorem 1 in [3].

Finally, two examples are discussed. Example 3.1 deals with the operator T_q, where 1 is not the eigenvalue of T_q if $1 < q \le \sqrt{2}$. Example 3.3 justifies why $w(T_q)$ fails to increase until and unless $q \to \sqrt{2}$.

2. Main Results

Theorem 2.1. *The following statements are true for a bounded linear operator T on a Hilbert space H with $w(T) = 1$.*

1) $\|T^n a\|^2 = 2q, a \in H, n = 1, 2, \cdots,$ such that $q \le \|Ta\|^2 - 4^{-1}$, $\|Ta\|^4 \le 1$.

2) If $\|T^n a\| = 2$ for some integer n, then $\|T^{n-1}a\|^2 = \cdots \|Ta\|^2 = 2$ and $T^{n+1}a = 0$.

3) The set $\{a, Ta, \cdots, T^n a\}$ forms a nontrivial subspace of T so that its orthogonal complement is invariant.

Proof. 1) For each real number α and a postive integer, n, let $b = \alpha_0 a + \cdots + \alpha_n T^n a$. Then the inner product relation $|(Tb, b)| \le (b, b)$ implies that

$$\left|\alpha_0\alpha_1\|T_a\|^2 + \cdots + \sum_{j,k=0; j\ne k-1}\alpha_j\alpha_k\left(T^{j+1}a, T^k a\right)\right|$$
$$\le \alpha_0^2 + \cdots + \sum\alpha_j\alpha_k\left(T^j a, T^k a\right)$$

That is,

$$\alpha_0\alpha_1\int_0^{2\pi}\left(e^{i\theta}Ta, e^{i\theta}Ta\right) + \cdots$$
$$+\sum\alpha_j\alpha_k\left(\int_0^{2\pi}\left(e^{(j+1)i\theta}T^{(j+1)}a, e^{ki\theta}T^k a\right)\right)$$
$$\le \int_0^{2\pi}\alpha_0^2 + \cdots + \sum\alpha_j\alpha_k\left(\int_0^{2\pi}\left(e^{ji\theta}T^j a, e^{ki\theta}T^k a\right)\right)$$

Hence,

$$\alpha_0\alpha_1\|Ta\|^2\left(\int_0^{2\pi}e^{i\theta}e^{-i\theta}d\theta\right)+\cdots$$
$$+\sum\alpha_j\alpha_k\left(T^{j+1}a,T^ka\right)\left(\int_0^{2\pi}e^{(j+1)i\theta}e^{-ki\theta}d\theta\right)$$
$$\le\alpha_0^2\int_0^{2\pi}d\theta+\cdots+\sum_{j,k=0;j\ne k-1}\alpha_j\alpha_k\left(T^ja,T^ka\right)\left(\int_0^{2\pi}e^{ji\theta}e^{-ki\theta}d\theta\right)$$

Since

$$\int_0^{2\pi}e^{mi\theta}e^{-ni\theta}d\theta=\begin{cases}0, & m\ne n\\ 2\pi, & m=n\end{cases}$$

it follows that

$$2\pi\alpha_0\alpha_1\|Ta\|^2+\cdots+2\pi\alpha_{n-1}\alpha_n\|T^na\|^2$$
$$\le 2\pi\alpha_0^2+\cdots+2\pi\alpha_n^2\|T^na\|^2$$

Dividing the above inequality by 2π, we have

$$\alpha_0\alpha_1\|Ta\|^2+\cdots+\alpha_{n-1}\alpha_n\|T^na\|^2$$
$$\le\alpha_0^2+\cdots+\alpha_n^2\|T^na\|^2$$

Let Γ be the following block-diagonal matix of order n and

$$\Gamma=\begin{pmatrix}1 & -2^{-1}\|Ta\|^2 & \cdots & 0\\ -2^{-1}\|Ta\|^2 & \|Ta\|^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & -2^{-1}\|T^na\|^2 & \|T^na\|^2\end{pmatrix}$$

If γ_n denotes the determinant of Γ such that $\gamma_0=1$ then the value of γ_n is positive because all principal minors of Γ are nonnegative. Suppose that $\gamma_n\ge 0$

$$4^{-1}\gamma_{n-2}\left(\|T^na\|^2\right)^2-\gamma_{n-1}\left(\|T^na\|^2\right)+\gamma_n=0\qquad(2.1)$$

We consider the following cases:

Case 1. If $\gamma_m>0$ for the least m then $\gamma_{m+1}+4^{-1}\gamma_{m-1}\left(\|T^{m+1}a\|^2\right)^2=0$ and $\|T^na\|$ converges to zero.

Case 2. Let $\gamma_n>0$ for all n. Then $\left(\gamma_1-2^{-1}\|T^na\|^2\right)^2\ge 0$ and by induction

$$\left(\gamma_{n-1}-2^{-1}\|T^na\|^2\gamma_{n-2}\right)^2\ge 0$$

Further, the inequality

$$\frac{\gamma_{n-1}}{\gamma_{n-2}}-\frac{\gamma_n}{\gamma_{n-1}}\ge 0$$

implies that $\frac{\gamma_n}{\gamma_{n-1}}$ converges to q as n goes to infinity for some $q\ge 0$. Therefore from Equation (2.1), $\|T^na\|^2\to 2q$ as $n\to\infty$. Thus $q\le\|T^na\|^2-4^{-1}\|Ta\|^4$. Obviously, q = 1 only if $\|Ta\|^2=\cdots=\cdots=2$.

2) By the assumption, $\|Ta\|^2=4$ for some positive integer n. Now fom Equation (2.1), we obtain:

$$4^{-1}\gamma_{n-2}(4)^2-\gamma_{n-1}(4)+\gamma_n=0$$

and $\gamma_n\ge 0$ so that $\frac{\gamma_{n-1}}{\gamma_{n-2}}\ge 1$. The equality, $1=\gamma_{n-1}=\cdots=\gamma_2=\gamma_1$ now follows from (a) and thus $\|T^{n-1}a\|^2=\cdots=\|Ta\|^2=2$. Also, $\gamma_{n+1}=0$ which gives $\|T^{n+1}a\|^2=0$ since $\gamma_n=0$.

3) To prove this case, we assume that if the vector v is orthogonal to the spanning set $\{a,Ta,\cdots,T^na\}$ then $(a,Tv)=\cdots=(Ta,Tv)=0$. Let $b=Ia+Ta+\cdots+T^{n-1}+T^na+\gamma v$, for $\gamma>0$. Then

$$\operatorname{Re}((Tb,b))\le(b,b)$$
$$\Rightarrow\gamma\operatorname{Re}((a+Ta+\cdots+T^na+Tv))$$
$$+\gamma^2\operatorname{Re}((v,Tv))\le\|v\|^2\cdots r^2$$
$$\Rightarrow\operatorname{Re}((a+Ta+\cdots+T^na,Tv))\le 0.$$

Hence, $(a,Tv)=\cdots=(T^na,Tv)=0$ for $T=e^{i\theta}T$ and the spanning set $\{a,Ta,\cdots,T^na\}$ is a non-trivial invariant subspace on T.

In [2, p. 1052], an example of an operator T on H and an element x in H with $w(T)=1$, is given where $\|T^nx\|=\sqrt{2}$. Theorem 2.1 above establishes that $\sqrt{2}$ is the best constant in this case.

Remark 2.2. *An operator A on H is hyponormal if* $\left(A^*A-AA^*\right)\ge 0$. *Let* $M_n=\|A^na\|^2$ *then* $M_n=(M_1)^n$, *if A is a hyponormal operator. Hence,* $M_n=M_1^{2n}$, $n=1,2,\cdots,$ *and the set of vectors* $a,Aa,\cdots,A^na$ *forms a reducing subspace of A.*

A natural connection between Feijer's inequality and the numerical radius of a nilpotent operator was estaplished by Haagerup and Harpe in [1]. They proved, using positive definite kernals, that for a bounded linear operator T on a Hilbert space H such that $T^{\alpha+1}=0$ and $\|T\|=1$, then $w(T)\le\cos\frac{\pi}{d+2}$. The external operator is shown to be a truncated shift with a suitable choice of the vector in H. The inequality is related to a result from Feijer about trigonometric polynomials of the form $\gamma(\theta)=\sum f_ke^{ik\theta}$ with $f_k\in\mathbb{C}$. Such a polynomial is positive if $\gamma(\theta)\ge 0$ for all $\theta\in\mathbb{R}$. Here, we present a simplified proof of Theorem 1 in [1].

Theorem 2.3. *For an operator N on H* with $\|N\| \le 1$ and $N^n = 0$, *we have* $w(N) \le \cos\frac{\pi}{n+1}$.

Proof. We will follow the notations of Theorem 1 in [1]. Let *S* be the operator on $\mathbb{C}^n$ and $\{e_k\}$, $k = 1, \cdots, n$ be the basis in $\mathbb{C}$. We define the operator *S* as follows:

$$S_{e_1} = 0 \text{ and } S_{e_k} = e_{k-1} \text{ for } k = 2, 3, \cdots, n$$

The matrix for *S* gives a dialation for *T*. Let *A* be the matrix for *S* and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

If U is a unitary operator on $\mathbb{C}^n$ with diagonal $A = \{1, z, \cdots, z^{n-1}\}$ then $\|S + S^*\| = \|U^*(S + S^*)U\|$. By Lemma 1, we have:

$$\|S + S^*\| = \|U^*(S + S^*)U\| = \|zS + \bar{z}S^*\|$$

This helps to define the characteristic function of a contraction.

For the operator *N* on *H*, let $\Psi = (I - N^*N)^{1/2}$ then Ψ is a positive operator and Ψ depends on *N*. Let the range of Ψ be denoted by $R(\Psi)$. Then the tensor product, $H_4 = R(4) \otimes \mathbb{C}^n$, is a Hilbert space. We define the map $F : H \to H_\Psi$ so that *F* is an isometry.

For λ, let $F(N\lambda) = \sum_{k=1}^{n} \Psi N^k \lambda \otimes e_k = (I \otimes S)F(\lambda)$ where $F(\lambda) = \sum_{k=1}^{n} \Psi N^{k-1} \lambda \otimes e_k$, I is the identity operator, and $(I \otimes S)$ is an operator on H_Ψ.

Therefore $w(N) \le w(I \otimes S)$ and $F^*(I \otimes S)F = N$.

Now, we claim that $w(S) = w(I \otimes S)$, for we hope that $2w(I \otimes S) = \sup\{\|zI \otimes S + \bar{z}I \otimes S^*\| : |z| = 1\}$ By Lemma 1.1

$$\begin{aligned} &= \sup\{\|zS + \bar{z}S^*\| : |z| = 1\} \\ &= 2w(S) \\ &\Rightarrow w(N) = w(I \otimes S) = w(S) \end{aligned}$$

That is, $w(N) \le w(S)$.

Since $\|S + S^*\| = \|zS + \bar{z}S^*\|$, we have:

$$2w(S) = \|S + S^*\|$$

and

$$2w(S) = f(S + S^*)$$

where $\rho(S + S^*)$ is the spectral radius of $(S + S^*)$. By the definition of the spectral radius, we have the characteristic polynomial *f* such that $f(x) = 0$ by [5, p. 179, Example 9], the roots of $f(x)$ are given by $-2\cos\left(\frac{k\pi}{n+1}\right)$, $k = 1, 2, \cdots, n$ and $w(N) \le w(S)$ and

$$2w(S) = \sup\left|-2\cos\left(\frac{k\pi}{n+1}\right)\right| = \cos\left(\frac{\pi}{n+1}\right).$$

Karaev in [3] has proved, using Theorem 1 in [1] and the Sz.-Nagy-Foias model in [6] that the numerical range $W(N)$ of an arbitrary nilpotent operator *N* on a complex Hilbert space *H* is an open or closed disc centered at zero with radius less than or equal to $\|N\|\cos\left(\frac{\pi}{n+1}\right)$, $n = 1, 2, \cdots$.

Using Theorem 2 and the assumption that $w(N) = \cos\left(\frac{\pi}{n+1}\right)$, $\|N\| = 1$, we have $W(N)$ as a closed or an open disc centered at zero with radius equal to $\cos\left(\frac{\pi}{n+1}\right)$. In fact, we have the following theorem.

Theorem 2.4. *For a nilpotent operator N on H with* $N^n = 0$, $n \ge 1$ *and* $w(N) = \cos\left(\frac{\pi}{n+1}\right)$, *the numerical range* $W(N)$ *is a disc centered at zero with radius* $\cos\left(\frac{\pi}{n+1}\right)$.

Proof. For any θ we must claim that $\lambda e^{i\theta} \in W(N)$, for $\lambda = (NZ, Z)$ and Z is a vector in $\mathbb{C}^n$.

From [1, p. 374, Proposition 2], we have $\alpha = \beta$. Also, for some ϕ,

$\alpha_{2_1} = (Sz_0, z_0) = \cos\left(\frac{\pi}{n+1}\right) = \left(e^{i\theta} Nz, z\right) = \beta_{2_1}$. Now by [1] [P.375, Lemma 2], we obtain:

$$Dz_0 = \sum_{k=0}^{n-1} c_k S_{z_0}^k, k = 0, 1, \cdots, n-1$$

and

$$\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} \bar{c}_k c_j \left(N^k z, N^j z\right) = 1 = \|D_{z_0}\|^2$$

Let $\mu = \sum_{k=0}^{n-1} c_k N^k z$. Then:

$$\begin{aligned} (N\mu, \mu) &= \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} \bar{c}_k c_j \left(N^{k+1} z, N^j z\right) \\ &= e^{i\theta}(Sz_0, z_0) = e^{i(\theta+\phi)}(Nz, z) \end{aligned}$$

and the theorem follows from above since θ is arbitrarily chosen.

3. An Application

An operator A is a unilateral weghted shift if there is an orthonormal basis $\{e_n : n \geq 0\}$ and a sequence of scalers $\{\alpha_n\}$ such that $Ae_n = \alpha_n e_n + 1$ for all $n \geq 0$. It is easy to see that $A = SD$ where S is the unilateral shift and D is the diagonal operator with $De_n = \alpha + ne_n$, for all n. Thus, $|A| = |D|$ and $|A| e_n = |\alpha_n| e_n$ for all n. So $\{e_n\}$ is the basis of eigenvectors for $|A|$. Also, note that A is bounded if $\{\alpha_n\}$ is bounded.

If A is a unilateral shift then $A^* e_0 = 0$ and $A^* e_n = \alpha_{n-1} e_{n-1}$ for $n \geq 1$. Consequently, for a hyponormal operator A, $\left(A^*A - AA^*\right)e_0 = \alpha_0^2 e_0 \geq 0$ and $\left(A^*A - AA^*\right)e_n = \left(\alpha_n^2 - \alpha_{n-1}^2\right)e_n$ for $n \geq 1$. A wighted shift is hyponormal if and only if its weight sequence is increasing.

Example 3.1. Let T_q be an operator on $H = l^2$ such that $T_q e_n = e_{n+1}$ and $T_q e_1 = qe_2$ for $n > 1$ and $q \geq 0$. Here, we show that 1 is not an eigenvalue of T_q if $q \in \left(1, \sqrt{2}\right]$. We prove our claim by contradiction

Let 1 be an eigenvalue of T_q. Then, there exists $f \in H$ with $2f_1 = qf_2$ and $f_n + f_{n+2} = 2f_{n+1}$, $n = 2, 3, \cdots$. It is not hard to see that:

$$f_3 = \frac{2\left(f_2^2 - f_1^2\right)}{f_2} = \frac{f_1\left(4 - q^2\right)}{q}$$

For $1 < q \leq \sqrt{2}$, we have $f_1 \leq f_2 \leq \cdots$ and thus $f_{n+2} \geq f_{n+1}$, which shows that $f \in H$, contrary to our assumption. Thus, 1 is not an eigenvalue of T_q if $q \in \left(1, \sqrt{2}\right]$.

Remark 3.2. Following [2], if $h \leq \sqrt{2} - 1 = 0.414$ then

$$\lim_{h \to 0.414} \frac{(1+h)^2}{2\sqrt{h(h+2)}} = 1$$

Therefore, the numerical radius, $w\left(T_q\right)$ is equal to 1.

The example below shows that there exists an operator Φ such that $w\left(\Phi_q\right) \leq 1$ for $0 \leq q \leq \sqrt{2}$.

Example 3.3. Let A be a unilateral shift. If E is the orthogonal projection of $H = l^2$ onto the spanning set of vectors $e_1, e_2, \cdots, e_n$, then $A = EAE$ and A has the usual matrix representation. Let

$$B = \begin{pmatrix} 0 & \sqrt{2} & 0 & \cdots & 0 \\ \sqrt{2} & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Then the characteristic polynomial of B is given by a Chebyshev polynomial $\Psi_n(x)$ of the first kind. Let $\Psi_n(x) = \cos(n\theta)$ where $x = \cos\theta$. Then:

$$\Psi_{n+1}(x) = 2x\Psi_n(x) - \Psi_{n-1}(x), n \geq 1$$

(easily proven by trigonometric identities) and $\Psi_n(x)$ for $n = 0, 1, 2, \cdots$ is a linear combination of powers of x^k. Also, det $(B - xI) = x\Psi_n(x) - 2\Psi_{n-1}(x)$. If $\det(B - xI) = 0$ then the roots are given by the Chebyshev polynomial of the first kind. The roots can be found by finding the eigenvalues of matrix B. By [2, p. 179, Example 9], the eigenvalues of B are given by

$$\cos\left[\frac{(1+2q)\pi}{2(n+1)}\right], \text{ for } q = 0, 1, 2, \cdots, n.$$

Suppose that

$$l_n = \cos\left[\frac{(1+2q)\pi}{2(n+1)}\right]$$

then $\lim_{n \to \infty} l_n = 1$. Hence, $w\left(\Phi_q\right) \leq 1$ if $q \in \left[0, \sqrt{2}\right]$.

REFERENCES

[1] U. Haagerup and P. de la Harpe, "The Numerical Radius of a Nilpotent Operator on a Hilbert Space," *Proceedings of the American Mathematical Society*, Vol. 115, 1999, pp. 371-379.

[2] C. A. Berger and J. G. Stampi, "Mapping Theorems for the Numerical Range," *American Journal of Mathematics*, Vol. 89, 1967, pp. 1047-1055.

[3] M. T. Karev, "The Numerical Range of a Nilpotent Operator on a Hilbert Space," *Proceedings of the American Mathematical Society*, Vol. 132, 2004, pp. 2321-2326.

[4] J. P. Williams and T. Crimmins, "On the Numerical Radius of a Linear Operator," *American Mathematical Monthly*, Vol. 74, No. 7, 1967, pp. 832-833. doi:10.2307/2315808

[5] J. T. Scheick, "Linear Algebra with Applications," *International Series in Pure and Applied Mathematics*, McGraw-Hill, New York, 1997.

[6] B. Sz.-Nagy and C. Folias, "Harmonic Analysis of Operators on a Hilbert Sapce," North Holland, Amsterdam, 1970.