Advances in Pure Mathematics, 2012, 2, 323-329
http://dx.doi.org/10.4236/apm.2012.25045 Published Online September 2012 (http://www.SciRP.org/journal/apm)

Differential Sandwich Theorems for Analytic Functions Defined by an Extended Multiplier Transformation

Amnah Shammaky
Department of Mathematics, Faculty of Science, Jazan University, Jazan, KSA
Email: a-e-sh27@hotmail.com

Received March 26, 2012; revised April 28, 2012; accepted May 9, 2012

ABSTRACT

In this investigation, we obtain some applications of first order differential subordination and superordination results involving an extended multiplier transformation and other linear operators for certain normalized analytic functions. Some of our results improve previous results.

Keywords: Differential Sandwich Theorems; Analytic Functions; Multiplier Transformation

1. Introduction

Let $H(U)$ be the class of functions analytic in the open unit disk $U=\{z:|z|<1\}$. Let $H[a,k]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z)=a+a_n z^n+a_{n+1}z^{n+1}+\cdots\ (a\in c) \tag{1.1}$$

For simplicity, let $H[a]=H[a,1]$. Also, let A be the subclass of $H(U)$ consisting of functions of the form:

$$f(z)=z+a_2z^2+\cdots \tag{1.2}$$

If $f,g\in H(U)$, we say that f is subordinate to g written $f(z)\prec g(z)$ if there exists Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ for all $z\in U$, such that $f(z)=g(w(z)), z\in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence , (cf., e.g. [1,2]; see also [3]):

We denote this subordination by

$$f(z)\prec g(z)(z\in U)\Leftrightarrow f(0)=g(0)$$

and $f(U)\subset g(U)$.

Let $p,h\in H(U)$ and let $\phi(r,s,t;z):C^3\times U\to C$. If p and $\phi(p(z),zp'(z),z^2p''(z);z)$ are univalent and if p satisfies the second-order superordination

$$h(z)\prec\phi(p(z),zp'(z),z^2p''(z);z), \tag{1.3}$$

then p is a solution of the differential superordination (1.3). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a subordinant if $q(z)\prec p(z)$ for all p satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q\prec\tilde{q}$ for all subordinants of (1.3) is called the best subordinant. Recently Miller and Mocanu [4] obtained conditions on the functions h,q and ϕ for which the following implication holds:

$$h(z)\prec\varphi(p(z),zp'(z),z^2p''(z);z)\Rightarrow q(z)\prec p(z). \tag{1.4}$$

Using the results of Miller and Mocanu [4], Bulboaca [5] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [6]. Ali *et al.* [7] have used the results of Bulboaca [5] and obtained sufficient conditions for normalized analytic functions f to satisfy:

$$q_1(z)\prec\frac{zf'(z)}{f(z)}\prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U. Also, Tuneski [8] obtained a sufficient condition for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{\{f'(z)\}^2}$.

Recently Shanmugam *et al.* [9] obtained sufficient conditions for a normalized analytic functions f to satisfy

$$q_1(z)\prec\frac{f(z)}{zf'(z)}\prec q_2(z),$$

and

$$q_1(z)\prec\frac{z^2f'(z)}{\{f(z)\}^2}\prec q_2(z).$$

Many essentially equivalent definitions of multiplier transformation have been given in literature (see [10-12].

In [13] Catas defined the operator $I^m(\lambda,\ell)$ as follows:

Definition 1.1. [13] Let the function $f(z)\in A$. For $m\in N_0=N\cup\{0\}$, where $N=\{1,2,\cdots\}$, $\lambda\ge 0, \ell\ge 0$. The extended multiplier transformation $I^m(\lambda,\ell)$ on A is defined by the following infinite series:

$$I^m(\lambda,\ell)f(z)=z+\sum_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m a_k z^k . \quad (1.5)$$

It follows form (1.5) that $I^m(\lambda,\ell)f(z)=f(z)$,

$$\begin{aligned}&\lambda z\left(I^m(\lambda,\ell)f(z)\right)' \\ &=(1+\ell)I^{m+1}(\lambda,\ell)f(z) \\ &\quad-(1-\lambda+\ell)I^m(\lambda,\ell)f(z)\ (\lambda>0)\end{aligned} \quad (1.6)$$

and

$$\begin{aligned}&I^{m_1}(\lambda,\ell)\left(I^{m_2}(\lambda,\ell)f(z)\right) \\ &=I^{m_1+m_2}(\lambda,\ell)f(z) \\ &=I^{m_2}(\lambda,\ell)\left(I^{m_1}(\lambda,\ell)f(z)\right).\end{aligned} \quad (1.7)$$

for all integers m_1 and m_2. We note that:

1) $I^m(\lambda,0)f(z)=D_\lambda^m f(z)$ (see [14]);
2) $I^m(1,0)f(z)=D^m f(z)$ (see [15]);
3) $I^m(1,\ell)f(z)=I^m(\ell)f(z)$ (see [10,11]);
4) $I^m(1,1)f(z)=I^m f(z)$ (see [12]).

Also if $f(z)\in A$, then we can write

$$I^m(\lambda,\ell)f(z)=\left(f*\varphi_{\lambda,\ell}^m\right)(z) ,$$

where

$$\varphi_{\lambda,\ell}^m(z)=z+\sum_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m z^k .$$

In this paper, we obtain sufficient conditions for the normalized analytic function f defined by using an extended multiplier transformation $I^m(\lambda,\ell)$ to satisfy:

$$q_1(z)\prec\frac{I^m(\lambda,\ell)f(z)}{I^{m+1}(\lambda,\ell)f(z)}\prec q_2(z)$$

and

$$q_1(z)\prec\frac{zI^{m+1}(\lambda,\ell)f(z)}{\left\{I^m(\lambda,\ell)f(z)\right\}^2}\prec q_2(z)$$

and q_1 and q_2 are given univalent functions in U.

2. Definitions and Preliminaries

In order to prove our results, we shall make use of the following known results.

Definition 2.1. [4]

Denote by Q the set of all functions f that are analytic and injective on $\bar{U}-E(f)$ where

$$E(f)=\left\{\xi\in\partial U:\lim_{z\to\xi}f(z)=\infty\right\}$$

and are such that $f'(\varsigma)\ne 0$ for $\xi\in\partial U-E(f)$.

Lemma 2.1. [4]

Let the function q be univalent in the open unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w)\ne 0$ when $w\in q(U)$. Set

$$\psi(z)=zq'(z)\phi(q(z)),\ h(z)=\theta(q(z))+\psi(z) . \quad (2.1)$$

Suppose that

1) $\psi(z)$ is starlike univalent in U,
2) $\mathrm{Re}\left(\frac{zh'(z)}{\psi(z)}\right)>0$ for $z\in U$.

If p is analytic with $p(0)=q(0)$, $p(U)\subseteq D$ and

$$\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(p(z)), \quad (2.2)$$

then $p\prec q$ and q is the best dominant. Taking $\theta(w)=\alpha w$ and $\varphi(w)=\gamma$ in lemma 1, Shanmugam *et al.* [9] obtained the following lemma.

Lemma 2.2. [2]

Let q be univalent in U with $q(0)=1$. Let $\alpha\in C$; $\gamma\in C^*=C\setminus\{0\}$, further assume that

$$\mathrm{Re}\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\{0,-\mathrm{Re}(\alpha/\gamma)\} .$$

If p is analytic in U, and

$$\alpha p(z)+\gamma zp'(z)\prec\alpha q(z)+\gamma zq'(z),$$

then $p\prec q$ and q is the best dominant.

Lemma 2.3. [5]

Let the function q be univalent in the open unit disc U and ϑ and ϕ be analytic in a domain D containing $q(U)$ Suppose that

1) $\mathrm{Re}\left(\frac{\vartheta'(q(z))}{\phi(q(z))}\right)>0$ for $z\in U$ and
2) $\psi(z)=zq'(z)\phi(q(z))$ is starlike univalent in U.

If $p\in H[q(0),1]\cap Q$, with $p(U)\subseteq D$, $\vartheta(p(z))+zp'(z)\phi(p(z))$, is univalent in U and

$$\vartheta(q(z))+zq'(z)\phi(q(z))\prec\vartheta(p(z))+zp'(z)\phi(p(z)) \quad (2.3)$$

then $q\prec p$ and q is the best subordinant.

Taking $\theta(w)=\alpha w$ and $\varphi(w)=\gamma$ in Lemma 2.3, Shanmugam *et al.* [9] obtained the following lemma.

Lemma 2.4. [2]

Let q be convex univalent in U, $q(0)=1$. Let

$\alpha \in C$, $\gamma \in C^* = C \setminus \{0\}$, and $\mathrm{Re}\{\alpha/\gamma\} > 0$. If $p \in H[q(0),1] \cap Q$, $\alpha p(z) + \gamma z p'(z)$ is univalent in U and $\alpha q(z) + \gamma z q'(z) \prec \alpha p(z) + \gamma z p'(z)$, then $q \prec p$ and q is the best subordinant.

3. Applications to an Extended Multiplier Transformation and Sandwich Theorems

Theorem 3.1.

Let q be convex univalent in U with $q(0)=1$, $\gamma \in C^*$. Further, assume that

$$\mathrm{Re}\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\{0, -\mathrm{Re}(1/\gamma)\}. \qquad (3.1)$$

If $f \in A$, $I^{m+1}(\lambda,\ell) f(z) \neq 0$ for $0<|z|<1$, and

$$\gamma\left(\frac{1+\ell}{\lambda}\right)+\frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)} -\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{I^{2(m+1)}(\lambda,\ell) f(z)}{\left\{I^{m+1}(\lambda,\ell) f(z)\right\}^2} \prec q(z)+\gamma z q'(z), \qquad (3.2)$$

then

$$\frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define a function p by

$$p(z)=\frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)} \quad (z \in U). \qquad (3.3)$$

Then the function p is analytic in U and $p(0)=1$. Therefore, differentiating (3.3) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we have

$$\gamma\left(\frac{1+\ell}{\lambda}\right)+\frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)} -\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{I^{2(m+1)}(\lambda,\ell) f(z)}{\left\{I^{m+1}(\lambda,\ell) f(z)\right\}^2} = p(z)+\gamma z p'(z),$$

that is,

$$p(z)+\gamma z p'(z) \prec q(z)+\gamma z q'(z)$$

and therefore, the theorem follows by applying Lemma 2.2.

Putting

$$q(z)=(1+Az)/(1+Bz) \quad (-1 \le B < A \le 1)$$

in Theorem 3.1, we have the following corollary.

Corollary 3.1.

If $f(z) \in A$ and $\gamma \in C^*$ satisfy

$$\gamma\left(\frac{1+\ell}{\lambda}\right)+\frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)} -\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{I^{2(m+1)}(\lambda,\ell) f(z)}{\left\{I^{m+1}(\lambda,\ell) f(z)\right\}^2} \prec \gamma\frac{(A-B)z}{(1+Bz)^2}+\frac{1+Az}{1+Bz},$$

then

$$\frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)} \prec \frac{1+Az}{1+Bz}.$$

Putting $A=1, B=-1$ and $q(z)=\dfrac{1+z}{1-z}$ in Corollary 3.1, we have

Corollary 3.2.

If $f(z) \in A$ and $\gamma \in C^*$ satisfy

$$\gamma\left(\frac{1+\ell}{\lambda}\right)+\frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)} -\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{I^{2(m+1)}(\lambda,\ell) f(z)}{\left\{I^{m+1}(\lambda,\ell) f(z)\right\}^2} \prec \frac{2\gamma z}{(1-z)^2}+\frac{1+z}{1-z},$$

then

$$\mathrm{Re}\left\{\frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)}\right\} > 0.$$

Taking $\ell = 0$, in Theorem 1, we have

Corollary 3.3.

Let q be convex univalent in U with $q(0)=1$, $\gamma \in C^*$. Further, assume that (3.1) holds. If $f \in A$, and

$$\frac{\gamma}{\lambda}+\frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)}-\frac{\gamma}{\lambda}\frac{D_\lambda^{2(m+1)} f(z)}{\left\{D_\lambda^{m+1} f(z)\right\}^2} \prec q(z)+\gamma z q'(z),$$

then

$$\frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} \prec q(z)$$

and q is the best dominant.

Taking $\lambda = 1, \ell = 0$, in Theorem 3.1, we have

Corollary 3.4.

Let q be convex univalent in U with $q(0)=1$, $\gamma \in C^*$. Further, assume that (3.1) holds. If $f \in A$, and

$$\gamma+\frac{D^m f(z)}{D^{m+1} f(z)}-\gamma\frac{D^{2(m+1)} f(z)}{\left\{D^{m+1} f(z)\right\}^2} \prec q(z)+\gamma z q'(z),$$

then

$$\frac{D^m f(z)}{D^{m+1} f(z)} \prec q(z)$$

and q is the best dominant.

Taking $\lambda = 1,$ in Theorem 3.1, we have

Corollary 3.5.

Let q be convex univalent in U with $q(0)=1$, $\gamma \in C^*$. Further, assume that (3.1) holds. If $f \in A$, and

$$\gamma(1+\ell) + \frac{I^m(\ell) f(z)}{I^{m+1}(\ell) f(z)} - \gamma(1+\ell)\frac{I^{2(m+1)}(\ell) f(z)}{\left\{I^{m+1}(\ell) f(z)\right\}^2} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{I^m(\ell) f(z)}{I^{m+1}(\ell) f(z)} \prec q(z)$$

and q is the best dominant.

Taking $\lambda = 1, \ell = 1,$ in Theorem 1, we have

Corollary 3.6.

Let q be convex univalent in U with $q(0)=1$, $\gamma \in C^*$. Further, assume that (3.1) holds. If $f \in A$, and

$$2\gamma + \frac{I^m f(z)}{I^{m+1} f(z)} - 2\gamma\frac{I^{2(m+1)} f(z)}{\left\{I^{m+1} f(z)\right\}^2} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{I^m f(z)}{I^{m+1} f(z)} \prec q(z)$$

and q is the best dominant.

Now, by appealing to Lemma 2.4 it can be easily prove the following theorem.

Theorem 3.2.

Let q be convex univalent in U. Let $\gamma \in C$ with $\operatorname{Re}\gamma > 0.$

If $f \in A$, $\frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)} \in H[1,1] \cap Q,$

$$\gamma\left(\frac{1+\ell}{\lambda}\right) + \frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)} - \gamma\left(\frac{1+\ell}{\lambda}\right)\frac{I^{2(m+1)}(\lambda,\ell) f(z)}{\left\{I^{m+1}(\lambda,\ell) f(z)\right\}^2}$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec \gamma\left(\frac{1+\ell}{\lambda}\right) + \frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)} - \gamma\left(\frac{1+\ell}{\lambda}\right)\frac{I^{2(m+1)}(\lambda,\ell) f(z)}{\left\{I^{m+1}(\lambda,\ell) f(z)\right\}^2}$$

then

$$q(z) \prec \frac{I^m(\lambda,\ell) f(z)}{I^{m+1}(\lambda,\ell) f(z)}$$

and q is the best subordinant.

Taking $\ell = 0$, in Theorem 3.2, we have

Corollary 3.7.

Let q be convex univalent in U. Let $\gamma \in C$ with $\operatorname{Re}\gamma > 0.$

If $f \in A$, $\frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} \in H[1,1] \cap Q,$

$$\frac{\gamma}{\lambda} + \frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} - \frac{\gamma}{\lambda}\frac{D_\lambda^{2(m+1)} f(z)}{\left\{D_\lambda^{m+1} f(z)\right\}^2}$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec \frac{\gamma}{\lambda} + \frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} - \frac{\gamma}{\lambda}\frac{D_\lambda^{2(m+1)} f(z)}{\left\{D_\lambda^{m+1} f(z)\right\}^2},$$

then

$$q(z) \prec \frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)}$$

and q is the best subordinant.

Taking $\lambda = 1, \ell = 0$, in Theorem 3.2, we have

Corollary 3.8.

Let q be convex univalent in U. Let $\gamma \in C$ with $\operatorname{Re}\gamma > 0.$

If $f \in A$, $\frac{D^m f(z)}{D^{m+1} f(z)} \in H[1,1] \cap Q,$

$$\gamma + \frac{D^m f(z)}{D^{m+1} f(z)} - \gamma\frac{D^{2(m+1)} f(z)}{\left\{D^{m+1} f(z)\right\}^2}$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec \gamma + \frac{D^m f(z)}{D^{m+1} f(z)} - \gamma\frac{D^{2(m+1)} f(z)}{\left\{D^{m+1} f(z)\right\}^2},$$

then

$$q(z) \prec \frac{D^m f(z)}{D^{m+1} f(z)}$$

and q is the best subordinant.

Taking $\lambda = 1,$ in Theorem 3.2, we have

Corollary 3.9.

Let q be convex univalent in U. Let $\gamma \in C$ with $\operatorname{Re}\gamma > 0$.

If $f \in A$, $\dfrac{I^m(\ell)f(z)}{I^{m+1}(\ell)f(z)} \in H[1,1] \cap Q$,

$$\gamma(1+\ell) + \frac{I^m(\ell)f(z)}{I^{m+1}(\ell)f(z)} - \gamma(1+\ell)\frac{I^{2(m+1)}(\ell)f(z)}{\left\{I^{m+1}(\ell)f(z)\right\}^2}$$

is univalent in U, and

$$q(z) + \gamma z q(z) \prec \gamma(1+\ell) + \frac{I^m(\ell)f(z)}{I^{m+1}(\ell)f(z)} - \gamma(1+\ell)\frac{I^{2(m+1)}(\ell)f(z)}{\left\{I^{m+1}(\ell)f(z)\right\}^2},$$

then

$$q(z) \prec \frac{I^m(\ell)f(z)}{I^{m+1}(\ell)f(z)}$$

and q is the best subordinant.

Taking $\lambda = 1, \ell = 1,$ in Theorem 3.2, we have

Corollary 3.10.

Let q be convex univalent in U. Let $\gamma \in C$ with $\operatorname{Re}\gamma > 0$.

If $f \in A$, $\dfrac{I^m f(z)}{I^{m+1}f(z)} \in H[1,1] \cap Q$,

$$2\gamma + \frac{I^m f(z)}{I^{m+1}f(z)} - 2\gamma\frac{I^{2(m+1)}f(z)}{\left\{I^{m+1}f(z)\right\}^2}$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec 2\gamma + \frac{I^m f(z)}{I^{m+1}f(z)} - 2\gamma\frac{I^{2(m+1)}f(z)}{\left\{I^{m+1}f(z)\right\}^2},$$

then

$$q(z) \prec \frac{I^m f(z)}{I^{m+1}f(z)}$$

and q is the best subordinant.

Combining Theorems 3.1 and 3.2, we get the following sandwich theorem.

Theorem 3.3.

Let q_1 be convex univalent in U, $\gamma \in C$ with $\operatorname{Re}\gamma > 0$, q_2 be univalent in U, $q_2(0) = 1$ and satisfies (3.1). If $f \in A$,

$$\frac{I^m(\lambda,\ell)f(z)}{I^{m+1}(\lambda,\ell)f(z)} \in H[1,1] \cap Q,$$

$$\gamma\left(\frac{1+\ell}{\lambda}\right) + \frac{I^m(\lambda,\ell)f(z)}{I^{m+1}(\lambda,\ell)f(z)} - \gamma\left(\frac{1+\ell}{\lambda}\right)\frac{I^{2(m+1)}(\lambda,\ell)f(z)}{\left\{I^{m+1}(\lambda,\ell)f(z)\right\}^2}$$

is univalent in U, and

$$q_1(z) + \gamma z q_1'(z) \prec \gamma\left(\frac{1+\ell}{\lambda}\right) + \frac{I^m(\lambda,\ell)f(z)}{I^{m+1}(\lambda,\ell)f(z)} - \gamma\left(\frac{1+\ell}{\lambda}\right)\frac{I^{2(m+1)}(\lambda,\ell)f(z)}{\left\{I^{m+1}(\lambda,\ell)f(z)\right\}^2} \prec q_2(z) + \gamma z q_2'(z),$$

Then

$$q_1(z) \prec \frac{I^m(\lambda,\ell)f(z)}{I^{m+1}(\lambda,\ell)f(z)} \prec q_2(z)$$

and q_1 and q_2 are respectively, the best subordinant and the best dominant.

4. Remarks

Combining: 1) Corollary 3.3 and Corollary 3.7; 2) Corollary 3.4 and Corollary 3.8; 3) Corollary 3.5 and Corollary 3.9; 4) Corollary 3.6 and Corollary 3.10, we obtain similar sandwich theorems for the corresponding operators.

Theorem 3.4.

Let q be convex univalent in U, $\gamma \in C^*$. Further, assume that (3.1) holds.

If $f \in A$ satisfies

$$\left[1 + \gamma\left(\frac{1+\ell}{\lambda}\right)\right]\frac{zI^{m+1}(\lambda,\ell)f(z)}{\left\{I^m(\lambda,\ell)f(z)\right\}^2} + \gamma\left(\frac{1+\ell}{\lambda}\right)\frac{zI^{m+2}(\lambda,\ell)f(z)}{\left\{I^m(\lambda,\ell)f(z)\right\}^2} - 2\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{z\left\{I^{m+1}(\lambda,\ell)\right\}^2}{\left\{I^m(\lambda,\ell)f(z)\right\}^3} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{zI^{m+1}(\lambda,\ell)f(z)}{\left\{I^m(\lambda,\ell)f(z)\right\}^2} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \frac{zI^{m+1}(\lambda,\ell)f(z)}{\left\{I^m(\lambda,\ell)f(z)\right\}^2}\ (z \in U).$$

Then, simple computations show that

$$p(z)+\gamma zp'(z)=\left[1+\gamma\left(\frac{1+\ell}{\lambda}\right)\right]\frac{zI^{m+1}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2}$$
$$+\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{zI^{m+2}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2}$$
$$-2\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{z\{I^{m+1}(\lambda,\ell)f(z)\}^2}{\{I^m(\lambda,\ell)f(z)\}^3}$$

Applying Lemma 2, the theorem follows.

Taking $\ell=0$, in Theorem 3.4, we have the following corollary.

Corollary 3.11.

Let q be convex univalent in U, $\gamma\in C^*$. Further, assume that (3.1) holds. If $f\in A$ satisfies

$$\left[1+\frac{\gamma}{\lambda}\right]\frac{zD_\lambda^{m+1}f(z)}{\{D_\lambda^m f(z)\}^2}+\frac{\gamma}{\lambda}\frac{zD_\lambda^{m+2}f(z)}{\{D_\lambda^m f(z)\}^2}$$
$$-2\frac{\gamma}{\lambda}\frac{z\{D_\lambda^{m+1}f(z)\}^2}{\{D_\lambda^m f(z)\}^3}\prec q(z)+\gamma zq'(z),$$

then

$$\frac{zD_\lambda^{m+1}f(z)}{\{D_\lambda^m f(z)\}^2}\prec q(z)$$

and q is the best dominant.

Taking $\lambda=1,\ell=0$, in Theorem 3.4, we have

Corollary 3.12.

Let q be convex univalent in U, $\gamma\in C^*$. Further, assume that (3.1) holds. If $f\in A$ satisfies

$$[1+\gamma]\frac{zD^{m+1}f(z)}{\{D^m f(z)\}^2}+\gamma\frac{zD^{m+2}f(z)}{\{D^m f(z)\}^2}-2\gamma\frac{z\{D^{m+1}f(z)\}^2}{\{D^m f(z)\}^3}$$
$$\prec q(z)+\gamma zq'(z),$$

then

$$\frac{zD^{m+1}f(z)}{\{D^m f(z)\}^2}\prec q(z)$$

and q is the best dominant.

Taking $\lambda=1,$ in Theorem 3.4, we have

Corollary 3.13.

Let q be convex univalent in U, $\gamma\in C^*$. Further, assume that (3.1) holds. If $f\in A$ satisfies

$$[1+\gamma(1+\ell)]\frac{zI^{m+1}(\ell)f(z)}{\{I^m(\ell)f(z)\}^2}+\gamma(1+\ell)\frac{zI^{m+2}(\ell)f(z)}{\{I^m(\ell)f(z)\}^2}$$
$$-2\gamma(1+\ell)\frac{z\{I^{m+1}(\ell)f(z)\}^2}{\{I^m(\ell)f(z)\}^3}\prec q(z)+\gamma zq'(z),$$

then

$$\frac{zI^{m+1}(\ell)f(z)}{\{I^m(\ell)f(z)\}^2}\prec q(z)$$

and q is the best dominant.

Taking $\lambda=1,\ell=1$, in Theorem 3.4, we have

Corollary 3.14.

Let q be convex univalent in U, $\gamma\in C^*$. Further, assume that (3.1) holds. If $f\in A$ satisfies

$$[1+2\gamma]\frac{zI^{m+1}f(z)}{\{I^m f(z)\}^2}+2\gamma\frac{zI^{m+2}f(z)}{\{I^m f(z)\}^2}$$
$$-4\gamma\frac{z\{I^{m+1}f(z)\}^2}{\{I^m f(z)\}^3}\prec q(z)+\gamma zq'(z),$$

then

$$\frac{zI^{m+1}f(z)}{\{I^m f(z)\}^2}\prec q(z)$$

and q is the best dominant.

Theorem 3.5.

Let q be convex univalent in U. Let $\gamma\in C$ with $\operatorname{Re}\gamma>0$.

If $f\in A$, $\frac{zI^{m+1}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2}\in H[1,1]\cap Q$,

$$\left[1+\gamma\left(\frac{1+\ell}{\lambda}\right)\right]\frac{zI^{m+1}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2}$$
$$+\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{zI^{m+2}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2}$$
$$-2\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{z\{I^{m+1}(\lambda,\ell)f(z)\}^2}{\{I^m(\lambda,\ell)f(z)\}^3},$$

is univalent in U, and

$$q(z)+\gamma zq'(z)$$
$$\prec\left[1+\gamma\left(\frac{1+\ell}{\lambda}\right)\right]\frac{zI^{m+1}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2}$$
$$+\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{zI^{m+2}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2},$$
$$-2\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{z\{I^{m+1}(\lambda,\ell)f(z)\}^2}{\{I^m(\lambda,\ell)f(z)\}^3},$$

then

$$q(z) \prec \frac{zI^{m+1}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2},$$

and q is the best subordinant.

Proof. The proof follows by applying Lemma 3.4.

Combining Theorems 3.4 and 3.5, we get the following sandwich theorem.

Theorem 3.6.

Let q_1 be convex univalent in U, $\gamma \in C$ with $\operatorname{Re}\gamma > 0$, q_2 be univalent in U, $q_2(0)=1$ and satisfies (3.1). If $f \in A$, $\frac{zI^{m+1}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2} \in H[1,1] \cap Q$,

$$\left[1+\gamma\left(\frac{1+\ell}{\lambda}\right)\right]\frac{zI^{m+1}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2} +\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{zI^{m+2}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2} -2\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{z\{I^{m+1}(\lambda,\ell)f(z)\}^2}{\{I^m(\lambda,\ell)f(z)\}^3},$$

is univalent in U, and

$$q_1(z)+\gamma z q_2'(z) \prec \left[1+\gamma\left(\frac{1+\ell}{\lambda}\right)\right]\frac{zI^{m+1}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2} +\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{zI^{m+2}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2} -2\gamma\left(\frac{1+\ell}{\lambda}\right)\frac{z\{I^{m+1}(\lambda,\ell)f(z)\}^2}{\{I^m(\lambda,\ell)f(z)\}^3} \prec q_2(z)+\gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{zI^{m+1}(\lambda,\ell)f(z)}{\{I^m(\lambda,\ell)f(z)\}^2} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

REFERENCES

[1] T. Bulboaca, "Differential Superordinations and Superordinations," Recent Results, House of Scientific, Cluj-Napoca.

[2] S. S. Miller and P. T. Mocanu, "Differential Subordinations and Univalent Functions," *Michigan Math Journal*, Vol. 28, No. 2, 1981, pp. 157-171. doi:10.1307/mmj/1029002507

[3] S. S. Miller and P. T. Mocanu, "Differential Subordinations: Theory and Applications," *Pure and Applied Mathematics* No. 225, Marcel Dekker, New York, 2000.

[4] S. S. Miller and P. T. Mocanu, "Subordinants of Differential Superordinations," *Complex Variables*, Vol. 48, No. 10, 2003, pp. 815-826. doi:10.1080/02781070310001599322

[5] T. Bulboaca, "Classes of First-Order Differential Superordinations," *Demonstratio Mathematica*, Vol. 35, No. 2, 2002, pp. 287-292.

[6] T. Bulboaca, "A Class of Superordination-Preserving Integral Operators," *Indagationes Mathematicae*, Vol. 13, No. 3, 2002, pp. 301-311. doi:10.1016/S0019-3577(02)80013-1

[7] R. M. Ali, V. Ravichandran, M. Hussain Khan and K. G. Subramanian, "Differential Sandwich Theorems for Certain Analytic Functions," *Far East Journal of Mathematical Sciences*, Vol. 15, No. 1, 2005, pp. 87-94.

[8] N. Tuneski, "On Certain Sufficient Conditions for Starlikeness," *International Journal of Mathematics and Mathematical Sciences*, Vol. 23, No. 8, 2000, pp. 521-527. doi:10.1155/S0161171200003574

[9] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, "Differential Sandwich Theorems for Some Subclasses of Analytic Functions," *Australian Journal of Mathematical Analysis and Applications*, Vol. 3, No. 1, 2006, Article 8, 11.

[10] N. E. Cho and H. M. Srivastava, "Argument Estimates of Certain Analytic Functions Defined by a Class of Multiplier Transformations," *Mathematical and Computer Modelling*, Vol. 37, No. 1-2, 2003, 39-49. doi:10.1016/S0895-7177(03)80004-3

[11] N. E. Cha and T. H. Kim, "Multiplier Tnsformations and Strongly Close-to-Convex Functions," *Bulletin of the Korean Mathematical Society*, Vol. 40, No. 3, 2003, 399-410. doi:10.4134/BKMS.2003.40.3.399

[12] B. A. Uralegaddi and C. Somanama, "Certain Classes of Univalent Functions," In: H. M. Srivastava and S. Owa, Eds., *Current Topics in Analytic Function Theory*, World Scientific, Publishing Company, Singapore City, 1992, pp. 371-374.

[13] A. Catas, "A Note on a Certain Subclass of Analytic Functions Defined by Multiplier Transformations," *Proceedings of the Internat, Symposium on Geometric Function Theory and Applications*, Istanbul, 20-24 August 2007.

[14] F. M. Al-Oboudi, "On Univalent Functions de Fined by a Generalized Salagean Operator Internat," *International Journal of Mathematics and Mathematical Sciences*, Vol. 27, 2004, pp. 1429-1436. doi:10.1155/S0161171204108090

[15] G. S. Salagean, "Subclasses of Univalent Functions," *Lecture Notes in Mathematics*, Vol. 1013, 1983, pp. 362-372. doi:10.1007/BFb0066543

