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# On Some Properties of the Heisenberg Laplacian

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## ABSTRACT

Let $IH_n$ be the $(2n+1)$-dimensional Heisenberg group and let $\mathcal{L}_\alpha$ and $T$ be the sublaplacian and central element of the Lie algebra of $H_n$ respectively. For $\alpha = 0$, denote by $\mathcal{L}_0 := \mathcal{L}$, the Heisenberg Laplacian and let $K \subset Aut(H_n)$ be a compact subgroup of Automorphism of $H_n$. In this paper, we give some properties of the Heisenberg Laplacian and prove that $\mathcal{L}$ and $T$ generate the $K$-invariant universal enveloping algebra, $\mathcal{U}(\mathfrak{h}_n)^K$ of $H_n$.

**Keywords:** Heisenberg Group; Heisenberg Laplacian; Factorization; Universal Enveloping Algebra; Solvability

## 1. Preliminaries

The Heisenberg group (of order $n$), $H_n$ is a non-commutative nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times R$ with coordinates $(z,t) = (z_1, z_2, \cdots, z_n, t)$ and group law given by

$$(z,t)(z',t') = (z+z', t+t'+2Im(z\cdot z')),$$

$$\text{where } z\cdot z' = \sum_{j=1}^{n} z_j \bar{z}_j \quad z\in\mathbb{C}^n,\ t\in IR.$$

Setting $z_j = x_j + y_j$, then $(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, t)$ forms a real coordinate system for $H_n$. In this coordinate system, we define the following vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t},\ Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t}.$$

It is clear from [1] that $\{X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n, T\}$ is a basis for the left invariant vector fields on $H_n$. These vector fields span the Lie algebra $\mathfrak{h}_n$ of $H_n$ and the following commutation relations hold:

$$[Y_j, X_k] = 4\delta_{jk}T,\quad [Y_j, Y_k] = [X_j, T] = [Y_j, T] = 0.$$

Similarly, we obtain the complex vector fields by setting

$$\left.\begin{aligned} Z_j &= \frac{1}{2}(X_j - iY_j) = \frac{\partial}{\partial z_j} + i\bar{z}\frac{\partial}{\partial t} \\ \bar{Z}_j &= \frac{1}{2}(X_j + iY_j) = \frac{\partial}{\partial \bar{z}_j} - iz\frac{\partial}{\partial t} \end{aligned}\right\}. \tag{1}$$

In the complex coordinate, we also have the commutation relations

$$[Z_j, \bar{Z}_k] = -2\delta_{jk}T,$$
$$[Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] = [Z_j, T] = [\bar{Z}, T] = 0.$$

The Haar measure on $IH_n$ is the Lebesgue measure $dzd\bar{z}dt$ on $\mathbb{C}^n \times R$ [2]. In particular, for $n=1$, we obtain the 3-dimensional Heisenberg group $H_1 \cong R^3$ (since $\mathbb{C}^n \cong R^{2n}$). Hence $H_n$ may also be referred to as $(2n+1)$-dimensional Heisenberg group.

One significant structure that accompanies the Heisenberg group is the family of dilations

$$\delta_{\pm\lambda}(z,t) = (\pm\lambda z, \pm\lambda^2 t),\ \lambda > 0$$

This family is an automorphism of $H_n$. Now, if $\sigma: \mathbb{C} \to \mathbb{C}$ is an automorphism, there exists an induced automorphism, $\tilde{\sigma} \in Aut(H_n)$, such that

$$\tilde{\sigma}(z,t) = (\sigma z, t).$$

For simplicity, assume that $\tilde{\sigma}$ and $\sigma$ coincide. Thus we may simply assume that if $\sigma \in Aut(H_n)$, we have $\sigma(z,t) = (\sigma z, t)$.

## 2. Heisenberg Laplacian

An operator that occurs as an analogue (for the Heisenberg group) of the Laplacian

$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ on $R^n$ is denoted by $\mathcal{L}_\alpha$ where $\alpha$ is a parameter and defined by

 

$$\mathcal{L}_\alpha = -\frac{1}{2}\sum_{j=1}^{n}\left(Z_j\bar{Z}_j + \bar{Z}_j Z_j\right) + i\alpha T,$$

where $\bar{Z}_j$ and $Z_j$ are as defined in (1) so that $\mathcal{L}_\alpha$ can be written as

$$\mathcal{L}_\alpha = \frac{1}{4}\sum_{j=1}^{n}\left(X_j^2 + Y_j^2\right) + i\alpha T. \tag{2}$$

$\mathcal{L}_\alpha$ is called the *sublaplacian*. $\mathcal{L}_\alpha$ satisfies symmetry properties analogous to those of $\Delta$ on $R^n$. Indeed, we have that $\mathcal{L}_\alpha$

1) is left-invariant on $H_n$;

2) has degree 2 with respect to the dilation automorphism of $H_n$ and

3) is invariant under unitary rotations.

Several methods for the determination of solutions, fundamental solutions of (2) and conditions for local solvability are well known [3-5].

The Heisenberg-Laplacian is a subelliptic differential operator defined for $\alpha = 0$ as $\Delta_{H_n}$ on $H_n$ and denoted by $\mathcal{L}$. It is obtained from the usual vector fields as

$$\begin{aligned}\mathcal{L} := \Delta_{H_n} &:= \sum_{j=1}^{n} X_j \circ X_j + Y_j \circ Y_j \\ &= \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j\frac{\partial^2}{\partial x_j \partial t} - 4x_j\frac{\partial^2}{\partial y_j \partial t} \\ &\quad + 4\left(x_j^2 + y_j^2\right)\frac{\partial^2}{\partial t^2}.\end{aligned} \tag{3}$$

By a technique in [6], the operator $\mathcal{L}$ is factorized into two quasi-linear first order operators on $H_n$ as:

$$A = \left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j} + \left(2y_j - 2ix_j\right)\frac{\partial}{\partial t}\right)$$

and

$$A^\dagger = \left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j} + \left(2y_j + 2ix_j\right)\frac{\partial}{\partial t}\right)$$

so that

$$\mathcal{L} = \left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j} + \left(2y_j - 2ix_j\right)\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j} + \left(2y_j + 2ix_j\right)\frac{\partial}{\partial t}\right).$$

Introducing the Lie algebra structure, we have

$$\left[A, A^\dagger\right] = -4x\delta_{ij}T, \quad \text{where } T = \frac{\partial}{\partial t},$$

indicating that the Heisenberg algebra is noncommutative and $\mathcal{L}$ is hypoelliptic [4]. We thus obtain an operator (which is a homogeneous element of $\mathcal{U}(\mathfrak{h}_n)$, the universal enveloping algebra of the Heisenberg group when $\mathfrak{h}_n$ is the Heisenberg algebra) [5] consistent with that of Hans Lewy [7]. In [2], it has been shown that none of the factors of $\mathcal{L}$, $A$ or $A^\dagger$ is solvable and as such, $\mathcal{L}$ is not solvable.

In this paper, we shall prove that $\mathcal{L}$ only possesses a trivial group-invariant solution and for $K \subset Aut(H_n)$ a compact subgroup of $Aut(H_n)$, we have that $\mathcal{U}(\mathfrak{h}_n)^K$ the $K$-invariant universal enveloping algebra of the Heisenberg group is generated by $T$ and $\mathcal{L}$.

Now, by a solution of a factor $A^\dagger$ say, we shall mean that if $x, y, t$ are independent real variables, and $\psi \in C^1$, such that $A^\dagger$ has a solution $u(x, y, t)$ in the neighbourhood $N_{t_0}$ of the point $(0, 0, t_0)$, with $A^\dagger = \psi(x, y, t)$ then $\psi$ is analytic at $t = t_0$.

**Definition 2.0.** Let $\Omega$ be any open subset of $IR^n$, and $\alpha$ a number such that $0 \le \alpha \le 1$. A function $u$ on $\Omega$ satisfying

$$\sup_{x,y\in\Omega, x\ne y}\frac{|u(x) - u(y)|}{|x - y|^\alpha} < +\infty$$

is said to be uniformly Holder continuous with Holder exponent $\alpha$ if $0 \le \alpha \le 1$; when $\alpha = 1$, they are called uniformly Lipschitz continuous. When $\alpha = 0$, they are simply continuous and bounded. A function is said to be in $H^1$-space if its first partial derivatives satisfy a Holder condition with positive exponent, provided the distance of the points involved does not exceed 1.

**Theorem 2.1.** Let $\psi$ be a periodic real $C^\infty$-function which is analytic in no t-interval. Then there exists a $C^\infty$-function $F(x, y, t)$ determined by the derivative $\psi'$ of $\psi$ such that

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + 2i(x + iy)\frac{\partial}{\partial t}\right)\psi = F(x, y, t)$$

has no $H^1$-solution,(no matter what open $(x, y, t)$-set taken as domain of existence).

For Proof, see [8].

**Theorem 2.2.** The Heisenberg Laplacian, $\mathcal{L}$ defined in (3) has no non-trivial group invariant solution.

**Proof.** Let $\varphi$ be a group-invariant solution of (3). We wish to show that $\varphi \equiv 0$. To do this, let $(x, y, t) \mapsto \delta_r(x, y, t)$ be a map generated by the group of automorphisms, dilations $\{\delta_r : r > 0\}$ where $r$ determines the growth or decay rate. If $\varphi : H_n \to R^{2n+1}$ is defined by

$$\varphi_{\delta_r}(x, y, t) = \left(rx, ry, r^2 t\right),$$

then obtaining the first and second order derivatives of

 

$\varphi$ with respect to the independent variables we have

$$\frac{\partial\varphi}{\partial x}=r,\ \frac{\partial\varphi}{\partial y}=r\ \ \frac{\partial\varphi}{\partial t}=r^2,\ \frac{\partial^2\varphi}{\partial x\partial t}=0$$
$$\frac{\partial^2\varphi}{\partial y\partial t}=0,\ \frac{\partial^2\varphi}{\partial x^2}=0\ \ \frac{\partial^2\varphi}{\partial y^2}=0,\ \frac{\partial^2\varphi}{\partial t^2}=0.$$

Substituting these into (3), we obtain a trivial equation. But by Group-invariant method, we should obtain a system of ordinary differential equations of lower order (see [9] p. 185). Thus, there exists no non-trivial group-invariant solution for $\mathcal{L}$. □

**Theorem 2.3.** Let $K\subset U(n)$ be a compact subgroup of $U(n)$, then $\mathcal{U}(\mathfrak{h}_n)^K$ the $K$-invariant universal enveloping algebra of the Heisenberg group is generated by $T$ and $\mathcal{L}$.

**Proof.** Let $\mathcal{U}(\mathfrak{h}_n)^K$ be the algebra of $K$-invariant differential operators on $H_n$ and let $S(\mathfrak{h}_n)$ be the symmetric algebra generated by the set

$$\{X_1,X_2,\cdots,X_n,Y_1,Y_2,\cdots,Y_n,T\}.$$

We note that the derived action of $K$ on $\mathfrak{h}_n$ is given by

$$\exp(k\cdot X)=k\cdot\exp(X),\quad X\in\mathfrak{h}_n,k\in K$$

and $K$ acts on $\mathfrak{h}_n^*=Hom_{IR}(\mathfrak{h}_n,R)$ via

$$(k\cdot\alpha)(X)=\alpha\left(k^{-1}\cdot X\right),\ \alpha\in\mathfrak{h}_n^*$$

and on $P\left(\mathfrak{h}_n^*\right)$, the $\mathbb{C}$-valued polynimial functions on $R$-vector space $\mathfrak{h}_n^*$ via

$$(x\cdot p)=p\left(k^{-1}\cdot\alpha\right).$$

Now, if we identify $P\left(\mathfrak{h}_n^*\right)$ with the complexified symmetric algebra $S(\mathfrak{h}_n)_{IC}$ then the symmetric product $X_1X_2\cdots X_n$ of $X_1,X_2,\cdots,X_n\in\mathfrak{h}_n$ becomes the polynomial $\mathfrak{h}_n^*\to\mathbb{C}$ given by

$$P_{X_1\cdots X_n}(\alpha)=\alpha(k\cdot X_1)\cdots\alpha(k\cdot X_n)=P_{(k\cdot X_1)\cdots(k\cdot X_n)}(\alpha).$$

Now, define a symmetrization map by

$$\lambda:S(\mathfrak{h}_n):=P\left(\mathfrak{h}_n^*\right)\to U(\mathfrak{h}_n),$$

with

$$\begin{aligned}&(\lambda(p)f)(z,t)\\&=p\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right)f\left((z,t)\exp\left(\sum_{j=1}^n u_jX_j+\sum_{j=1}^n v_jY_j\right)\right)\Bigg|_{u=v=0}\\&=p\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right)f\left(z+(u+iv),t+\frac{1}{2}w(z,u+iv)\right)\Bigg|_{u=v=0}\end{aligned}$$

Now since $U(n)$ acts on $S(\mathfrak{h}_n)$ and $\mathcal{U}(\mathfrak{h}_n)$ by automorphism and $\tilde{\lambda}$ defined by

$$\begin{aligned}&\tilde{\lambda}(p)f(z,t)\\&=p\left(2\frac{\partial}{\partial\xi},2\frac{\partial}{\partial\xi}\right)f\left(z+\xi,t+\frac{1}{2}w(z,t)\right)\Bigg|_{\xi=0}\end{aligned}$$

induces an algebra map on the associated graded algebras and by induction [10, p. 282] the eigenfunctions of $\mathcal{L}$ and $\frac{\partial}{\partial t}$ are eigenfunctions of any element in $\mathcal{U}(\mathfrak{h}_n)^K$ we have that the following diagram is commutative.

$$\begin{tikzcd}\mathcal{S}(\mathfrak{h}_n) \arrow[r, "\lambda"] \arrow[d, "\sigma"'] & \mathcal{U}(\mathfrak{h}_n) \arrow[d, "\sigma"] \\ \mathcal{S}(\mathfrak{h}_n) \arrow[r, "\lambda"] & \mathcal{U}(\mathfrak{h}_n)\end{tikzcd}$$

for $\sigma\in Aut(H_n)$. Since $\lambda$ is a linear isomorphism, it maps $\mathcal{S}(\mathfrak{h}_n)^K$ onto $\mathcal{U}(\mathfrak{h}_n)^K$. Since the action of $U(n)$ preserves degree on $\mathcal{S}(\mathfrak{h}_n)$, and by [11], if $\{1,u_1,\cdots,u_m\}$ generates $\mathcal{S}(\mathfrak{h}_n)^K$, then, $\{1,\lambda(u_1),\cdots,\lambda(u_m)\}$ generates $\mathcal{U}(\mathfrak{h}_n)^K$ If $u\in\mathcal{S}(\mathfrak{h}_n)^K$, then

$$u=\sum_{j=1}^n P_j\left(X_1,\cdots,X_n,Y_1,\cdots,Y_n,T_j\right)$$

where the sum is finite and each $P_j$ is a polynomial which is $U(n)$-invariant. Thus, the result follows by the fact that the eigenfunctions of $\mathcal{L}$ and $\frac{\partial}{\partial t}$ are the eigenfunctions of $\mathcal{U}(\mathfrak{h}_n)^K$ [12]. □

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