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The Primary Radical of a Submodule

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ABSTRACT

In this paper we introduced a definition for the primary radical of a submodule with some of its basic properties. We also define the P-radical submodule and review some results about it. We find a method to characterize the primary radical of a finitely generated submodule of a free module.

Keywords: Primary Submodule; Prime Radical of a Submodule; Radical Submodule; Free Module; Noetherian Module; Finitely Generated Submodule

1. Introduction

The prime radical of a submodule N of an *R*-module M, denoted by $rad_M(N)$ is defined as the intersection of all prime submodules of M which contain N, if there exists no prime submodule of M containing N, we put $rad_M(N) = M$ [1].

We naturally seek a counterpart in the primary radical of a submodule of module.

Firstly we introduced a definition for the primary radical of a submodule with some of its basic properties. We also define the *P*-radical submodule and review some results about it.

Finally, we find a method to characterize the primary radical of a finitely generated submodule of a free module.

2. Some Basic Properties of the Primary Radical

In this section we introduce the concept of the primary radical and give some useful properties about it.

2.1. Definition

The primary radical of a submodule N of an *R*-module M, denoted by $prad_M(N)$ is defined as the intersection of all primary submodules of M which contain N. If there exists no primary submodule of M containing N, we put $prad_M(N) = M$.

If $M = R$, since the primary submodules and the primary ideals are the same, so if I is an ideal of R, $prad_R(I)$ is the intersection of all primary ideals of R, which contain I. Now, we give useful properties of the primary radical of a submodule.

2.2. Proposition

Let N and L be submodules of an *R*-module M. Then

1) $N \subseteq prad_M(N)$

2) $prad_M(N \cap L) \subseteq prad_M(N) \cap prad_M(L)$

3) $prad_M(prad_M(N)) = prad_M(N)$

Proof.

1) It is clear.

2) Let H be primary submodule of M containing *L*, since $N \cap L \subseteq L \subseteq H$ so $prad_M(N \cap L) \subseteq H$. Thus $prad_M(N \cap L) \subseteq prad_M(L)$. By the same way $prad_M(N \cap L) \subseteq prad_M(N)$. It follows $prad_M(N \cap L) \subseteq prad_M(N) \cap prad_M(L)$.

3) By 1) we have $prad_M(N) \subseteq prad_M(prad_M(N))$. Now $prad_M(N) = \cap L$ where the intersection is over all primary submodules L of M with $L \supseteq N$.

$$prad_M(prad_M(N)) = prad_M(\cap L) \subseteq \cap prad_M(L) = \cap L$$

In the following two propositions, we give a condition under which the other inclusion of 2) holds, that is; $prad_M(N \cap L) = prad_M(N) \cap prad_M(L)$ provided that every primary submodule of M which contains $N \cap L$ is completely irreducible submodule. Where a submodule K of an R-module M is called Completely Irreducible if whenever $N \cap L \subseteq K$, then either $N \subseteq K$ or $L \subseteq K$ where N and L are submodules of M.

2.3. Proposition

Let N and L be submodules of an R-module M. If every primary submodule of M which contains $N \cap L$

is completely irreducible submodule, then:

$$prad_M(N\cap L)=prad_M(N)\cap prad_M(L).$$

Proof. By proposition (2.2, (2)) $prad_M(N\cap L)\subseteq prad_M(N)\cap prad_M(L)$. If $prad_M(N\cap L)=M$, clearly $prad_M(N)=prad_M(L)=M$. If $prad_M(N\cap L)\neq M$, there exists a primary submodule K of M such that, $N\cap L\subseteq K$ by hypothesis either $N\subseteq K$ or $L\subseteq K$ so that either $prad_M(N)\subseteq K$ or $prad_M(L)\subseteq K$, because every primary submodule containing $N\cap L$, so either $prad_M(N)\subseteq prad_M(N\cap L)$ or $prad_M(L)\subseteq prad_M(N\cap L)$ therefore $prad_M(N)\cap prad_M(L)\subseteq prad_M(N\cap L)$.

2.4. Proposition

Let N and L be submodules of an R-module M such that $\sqrt{[N:M]}+\sqrt{[L:M]}=R$, then $prad_M(N\cap L)=prad_M(N)\cap prad_M(L)$.

Proof. If K is a primary submodule containing $N\cap L$, then $\sqrt{[N\cap L:M]}\subseteq\sqrt{[K:M]}$. So $\sqrt{[N:M]}\cap\sqrt{[L:M]}\subseteq\sqrt{[N\cap L:M]}\subseteq\sqrt{[K:M]}$. Since $\sqrt{[K:M]}$ is a prime ideal, either $\sqrt{[L:M]}\subseteq\sqrt{[K:M]}$ or $\sqrt{[N:M]}\subseteq\sqrt{[K:M]}$. If $\sqrt{[N:M]}\subseteq\sqrt{[K:M]}$, then $\sqrt{[L:M]}\not\subset\sqrt{[K:M]}$ for otherwise $R=\sqrt{[N:M]}+\sqrt{[L:M]}\subseteq\sqrt{[K:M]}$ which is a contradiction. Therefore $N\subseteq K$. Now, applying proposition (2.3), we can conclude that $prad_M(N\cap L)=prad_M(N)\cap prad_M(L)$. We conclude the same result if $\sqrt{[L:M]}\subseteq\sqrt{[K:M]}$.

Let N be a proper submodule of an R-module M. Let P be a prime ideal of R. For each positive integer n, we shall denote by $K(N,P^n)$ the following subset of

$$K(N,P^n)=\{m\in M\mid cm\in N+P^nM \text{ for some } c\notin P\}$$

2.5. Proposition

Let N be a submodules of an R-module M and P be a prime ideal of R. For each positive integer n: $K(N,P^n)=M$ or $K(N,P^n)$ is a P-primary submodule of M.

Proof. Let n be any positive integer, it is clear that $K(N,P^n)$ is a submodule of M. Assume $K(N,P^n)\neq M$. To show $K(N,P^n)$ is P-primary, $P^nM\subseteq K(N,P^n)$ that is $P^n\subseteq[K(N,P^n):M]$. Now, let L be a submodule of M properly containing $K(N,P^n)$, let $r\in[K(N,P^n):L]$, $rL\subseteq K(N,P^n)$. Since $K(N,P^n)\subsetneq L$, let $l\in L$, but $l\notin K(N,P^n)$ thus $rl\in K(N,P^n)$, there exists $c\notin P$ such that $c(rl)\in N+P^nM$. If $r\notin P$, then $cr\notin P$ and this implies $l\in K(N,P^n)$, which is a contradiction. It follows $r\in P$, therefore $[K(N,P^n):L]\subseteq P$.

So $K(N,P^n)$ is a primary submodule $P=\sqrt{[K(N,P^n):M]}$, we have proved above that $P^n\subseteq[K(N,P^n):M]$, that is $P\subseteq\sqrt{[K(N,P^n):M]}$.

Let $r\subseteq\sqrt{[K(N,P^n):M]}$, $r^tM\subseteq K(N,P^n)$ for some $t\in Z^+$, thus $c(r^tM)\in N+P^nM$ for some $c\notin P$. If $r\notin P$ then $cr^t\notin P$ this implies $M\subseteq K(N,P^n)$, which is a contradiction. Therefore $r\in P$ thus $\sqrt{[K(N,P^n):M]}\subseteq P$.

The following theorem gives a description of the primary radical of a submodule.

2.6. Theorem

Let N be a submodule of a module M over a Noetherian ring R. Then

$$prad_M(N)=\cap\{K(N,P^n)\mid P \text{ is a prime ideal of } R, n\in Z^+\}$$

Proof. By proposition (2.2), for each positive integer n and any prime ideal P we have $K(N,P^n)$ is a P-primary submodule containing N. Hence

$$prad_M(N)\subseteq\cap\{K(N,P^n)\mid P \text{ is a prime ideal of } R,\ n\in Z^+\}$$

For every primary submodule H containing N with $P=\sqrt{[H:M]}$ there exists a positive integer r such that $K(N,P^r)\subseteq H$. So

$$\cap\{K(N,P^n)\mid P \text{ is a prime ideal of } R, n\in Z^+\}\subseteq K(N,P^r)\subseteq H.$$

Thus

$$\cap\{K(N,P^n)\mid P \text{ is a prime ideal of } R, n\in Z^+\}\subseteq prad_M(N).$$

We will give the following definition.

2.7. Definition

A proper submodule N of an *R*-module M with $prad_M(N) = N$ will be called P-Radical Submodule.

Now, we are ready to consider the relationships among the following three statements for any *R*-module M.

1) M satisfies the ascending chain condition for p-radical submodules.

2) Each p-radical submodule is an intersection of a finite number of primary submodules

3) Every p-radical submodule is the p-radical of a finitely generated submodule of it.

2.8. Proposition

Let M be an R-module. If M satisfies the ascending chain condition for p-radical submodule of M is an intersection of a finite number of primary submoules.

Proof. Let N be a p-radical submodule of M and put $N = \bigcap_{i\in I} N_i$, where N_i is a primary submodule for each $i \in I$, and the expression is reduced. Assume that I is an infinite index set. Without loss of generality we may assume that I is countable, then

$N = \bigcap_{i=1}^{\infty} N_i \subseteq \bigcap_{i=2}^{\infty} N_i \subseteq \bigcap_{i=3}^{\infty} N_i \subseteq \cdots$ is an ascending chain of p-radical submodules, since by proposition (2.2),

$$\bigcap_i N_i \subseteq prad_M\left(\bigcap_i N_i\right) \subseteq \bigcap_i \left(prad_M\left(\bigcap_i N_i\right)\right) = \bigcap_i N_i$$

By hypothesis this ascending chain must terminate, so there exists $j \in I$ such that $\bigcap_{i=j}^{\infty} N_i = \bigcap_{i=j+1}^{\infty} N_i$, whence $\bigcap_{i=j+1}^{\infty} N_i \subseteq N_j$ which contradicts that the expression $N = \bigcap_i N_i$ is a reduced. Therefore I must be finite.

2.9. Proposition

Let M be an *R*-module. If M satisfies the ascending chain condition for p-radical submodules, then every p-radical submodule is the p-radical of finitely generated submodule of it.

Proof. Assume that there exists a p-radical submodule N of M which is not the p-radical of a finitely generated submodule of it. Let $m_1 \in N$ and let $N_1 = prad_M(Rm_1)$ so $N_1 \underset{\neq}{\subset} N$, hence there exists $m_2 \in N - N_1$. Let $N_2 = prad_M(Rm_1 + Rm_2)$, then $N_1 \underset{\neq}{\subset} N_2 \underset{\neq}{\subset} N$, thus there exists $m_3 \in N - N_2$, etc. This implies an ascending chain of p-radical submodules, $N_1 \underset{\neq}{\subset} N_2 \underset{\neq}{\subset} N_3 \underset{\neq}{\subset} \cdots$ which does not terminate and this contradicts the hypothesis.

2.10. Proposition

Let M be a finitely generated *R*-module. If every primary submodule of M is the p-radical of a finitely generated submodule of it, then M satisfies the ascending chain condition for primary submodules.

Proof. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be an ascending chain of primary submodules of M. Since M is finitely generated then, $N = \bigcup_i N_i$ is a primary submodule of M.

Thus by hypothesis, N is the p-radical for some finitely generated submodule $L = R(m_1, m_2, \cdots, m_n)$, hence $L \subseteq prad_M(L) = N = \bigcup_i N_i$, then there exists $j \in I$ such that $L \subseteq N_j$ hence $N = prad_M(L) \subseteq prad_M(N_j) = N_j$. Thus $\bigcup_i N_i = N_j$ for some $j \in I$. Therefore the chain of primary submodules N_i terminates

3. The Primary Radical of Submodules of Free Modules

In this section we describe the elements of $prad_F(N)$, where N is a finitely generated submodule of the free module F. Let n be a positive integer and let F be the free R-module $R^{(n)}$.

Let $x_i \in F\,(1 \le i \le m)$ for some $m \in Z^+$, then $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})$, $1 \le i \le m$, for some $x_{ij} \in R$, $1 \le i \le m$, $1 \le j \le n$.

We set

$$[x_1\ x_2\ \cdots\ x_m] = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \in M_{m\times n}(R)$$

Thus the *j*th row of the matrix $[x_1\ x_2\ \cdots\ x_m]$ consists of the components of the element x_j in F. Let $A = (a_{ij}) \in M_{m\times n}(R)$.

By a $t \times t$ minor of A we mean the determinant of a $t \times t$ submatrix of A, that is a determinant of the form:

$$\begin{vmatrix} a_{i(1)j(1)} & \cdots & a_{i(1)j(t)} \\ \vdots & & \vdots \\ a_{i(t)j(1)} & \cdots & a_{i(t)j(t)} \end{vmatrix}$$

where $1 \le i(1) \le \cdots \le i(t) \le m$, $1 \le j(1) \le \cdots \le j(t) \le n$. For each $1 \le t \le \min(m,n)$.

We denote by A_t the ideal of R generated by the $t \times t$ minors of A.

Note that $A_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} Ra_{ij} \supseteq A_2 \supseteq A_3 \supseteq \cdots \supseteq A_k$, where $k = \min(m,n)$.

The key to the desired result is the following two propositions.

3.1. Proposition

Let R be a ring and F be the free R-module $R^{(n)}$, for some positive integer n. Let $N = \sum_{i=1}^{m} Rx_i$ be a finitely generated submodule of F where $m < n$. If

$r \in prad_F(N)$, then $[rx_1\, x_2 \cdots x_m]_t$ in

$$\bigcap\{P \mid \text{for every maximal ideal P such that } [x_1 \cdots x_n]_t \subset P^l \text{ for some } l \in Z^+\}, 1 \le t \le m+1$$

Proof. Suppose $r = (r_1, r_2, \ldots, r_n) \in prad_F(N)$ where $r_i \in R$, $1 \le i \le n$. Let P be any maximal ideal of R and $\ell \in Z^+$ such that $[0x_1 \cdots x_m]_t \subseteq P^\ell$. By proposition (2), there exists $c \in R \setminus P$, $s_i \in R$, $(1 \le i \le m)$ and $p_i \in P^\ell$ such that $cr = s_1x_1 + s_2x_2 + \cdots + s_mx_m + p$, where $p = (p_1, p_2, \cdots, p_n)$, that is, if $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})$ where $x_{ij} \in R$ $(1 \le i \le m, 1 \le j \le n)$, then

3.1) $cr_i = s_1x_{1i} + s_2x_{2i} + \cdots + s_mx_{mi} + p_i$; $1 \le i \le n$

Suppose that $1 \le t \le m+1$, $let 1 \le i(1) \le i(2) \le \cdots \le i(t-1) \le m$, $1 \le j(1) \le \cdots \le j(t) \le n.$

Let

$$X_t = \begin{vmatrix} r_{j(1)} & \cdots & r_{j(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \ddots & \vdots \\ x_{i(t-1)j(1)} & \cdots & x_{i(t-1)j(t)} \end{vmatrix}$$

which is a $t \times t$ minor of $[r\ x_1\ x_2 \cdots x_m]$. Then by (3.1)

$$cX_t = \begin{vmatrix} cr_{j(1)} & \cdots & cr_{j(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \ddots & \vdots \\ x_{i(t-1)j(1)} & \cdots & x_{i(t-1)j(t)} \end{vmatrix} = \sum_{h=1}^{m} s_h \begin{vmatrix} x_{h\,j(1)} & \cdots & x_{h\,j(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ x_{i(t-1)j(1)} & \cdots & x_{i(t-1)j(t)} \end{vmatrix} + \begin{vmatrix} p_{j(1)} & \cdots & p_{j(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ x_{i(t-1)j(1)} & \cdots & x_{i(t-1)j(t)} \end{vmatrix} \in P^l.$$

which is primary with $c \notin P$ (note that, here $p_{j(1)}, \cdots, p_{j(t)} \in P^\ell$) hence $X_t \in P^\ell \subseteq P$ or $X_t \in P$. It follows $[r\ x_1\ x_2 \cdots x_m]_t \in P$ for every maximal ideal P with $[0\ x_1\ x_2\ \cdots x_m]_t \in P^\ell$ for some $\ell \in Z^+$ and $1 \le t \le m+1$.

3.2. Proposition

Let R be a ring and F be the free R-module $R^{(n)}$, for some positive integer n. Let $N = \sum_{i=1}^{m} Rx_i$ be a finitely generated submodule of F where $m < n$. If $[r\ x_1\ x_2 \cdots x_m]_t$ in

$$\bigcap\{P^l \mid \text{for every prime ideal } P \text{ such that } [0x_1 \cdots x_m]_t \subset P^l \text{ for some } l \in Z^+\}$$

$1 \le t \le m+1$, then $r \in prad_F(N)$.

Proof. Suppose

$$[r\ x_1 \cdots x_m]_t \in \{P^l \mid \text{for every prime ideal } P \text{ such that } [0x_1 \cdots x_m]_t \subset P^l \text{ for some } l \in Z^+\}$$

and $1 \le t \le m+1$. Let P be any prime ideal of R and k any positive integer. It is enough to show that $r \in K(N, P^k)$ for all $k \in Z^+$.

If $[0\ x_1\ x_2 \cdots x_m]_1 \in P^k$, then $r_i \in [r\ x_1\ x_2 \cdots x_m]_1 \in P^k$, hence $r = (r_1, r_2, \cdots, r_n) \in P^k F \subseteq K(N, P^k)$. Suppose SSS $[0\ x_1\ x_2 \cdots x_m]_1 \not\subseteq P$.

Note that $[0\ x_1\ x_2 \cdots x_m]_{m+1} = 0 \in P^k$ for all k.

Thus there exists $1 \le t \le m$ such that $[0\ x_1\ x_2 \cdots x_m]_t \not\subseteq P$, but $[0\ x_1\ x_2 \cdots x_m]_{t+1}$ is a subset of P^k, there exists $1 \le i(1) \le \cdots \le i(t) \le m, 1 \le j(1) \le \cdots \le j(t) \le n$, such that

$$d = \begin{vmatrix} x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ x_{i(t)j(1)} & \cdots & x_{i(t)j(t)} \end{vmatrix} \notin P$$

By hypothesis, for each $1 \le j \le n$

$$\begin{vmatrix} r_j & r_{j(1)} & \cdots & r_{j(t)} \\ x_{i(1)j} & x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i(t)j} & x_{i(t)j(1)} & \cdots & x_{i(t)j(t)} \end{vmatrix} \in P^k$$

Expanding this determinant by first column we find that $dr_j + a_{i(1)}x_{i(1)j} + \cdots + a_{i(t)}x_{i(t)j} \in P^k$ where

$$a_{i(h)} = (-1)^h \begin{vmatrix} r_{j(1)} & \cdots & r_{j(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ x_{i(h\text{-}1)j(1)} & \cdots & x_{i(h\text{-}1)j(t)} \\ x_{i(h+1)j(1)} & \cdots & x_{i(h+1)j(t)} \\ x_{i(t)j(1)} & \cdots & x_{i(t)j(t)} \end{vmatrix}$$

For each $1 \le h \le t$.

Note that d and $a_{i(h)}(1 \le h \le t)$ are independent of j. Thus

$$dr_j + ai_{(1)}xi_{(1)j} + \cdots + a_{i(t)}x_{i(t)j} \to P^k \quad 1 \le j \le n.$$

i.e. $dr \in Rx_1 + Rx_2 + \cdots + Rx_m + P^kF = N + P^kF$ with $d \notin P$, hence $r \in K(N, P^k)$. Thus $r \in prad_F(N)$.

3.3. Proposition

Let M_1 and M_2 be R-modules and

$$M = M_1 \oplus M_2 = \{(m_1, m_2) | m_i \in M_i, i = 1, 2\}$$

Let N be a proper submodule of M_1, then $x \in prad_{M_1}(N)$ if and only if $(x, 0) \in prad_M(N \oplus 0)$.

Proof: Suppose first that $x \in prad_{M_1}(N)$. Let K be any primary submodule of M such that $N \oplus 0 \subseteq K$. Let $K' = \{m \in M_1 | (m, 0) \in K\}$. K' is a submodule of M_1 and if $K' \neq M_1$ then K' is a primary submodule of M_1 since, if $rm \in K'$ where $r \in R$ and $m \in M_1$, then $(rm, 0) \in K$, so $r(m, 0) \in K$, which is primary submodule of M, hence either $(m, 0) \in K$ thus $m \in K'$ or $r^n M \subseteq K$ for some $n \in Z^+$ that is,
$r^n(M_1 \oplus M_2) \subseteq K$, so
$r^n(M_1 \oplus 0) \subseteq r^n(M_1 \oplus M_2) \subseteq K$, therefore
$(r^n M_1 \oplus 0)) = r^n(M_1 \oplus 0) \subseteq K$, thus $r^n M_1 \subseteq K'$ for some $n \in Z^+$, that is $r \in \sqrt{[K' : M_1]}$. Hence K' is a primary submodule of M_1 containing N. Thus $x \in K'$, so $(x, 0) \in K$. It follows $(x, 0) \in prad_M(N \oplus 0)$. Conversely, suppose that $(x, 0) \in prad_M(N \oplus 0)$. Let Q be a primary submodule of M_1 such that $N \subseteq Q$. Then $Q \oplus M_2$ is a primary submodule of M containing $N \oplus 0$. Hence $(x, 0) \in Q \oplus M_2$, that is
$x \in Q$ so $x \in prad_{M_1}(N)$.

Now, we have the main result of this section.

3.4. Theorem

Let R be a ring and F be the free R-module $R^{(n)}$, for some positive integer n. Let $N = \sum_{i=1}^{m} Rx_i$ be a finitely generated submodule of F where $m < n$. If $r \in prad_F(N)$, then $[r\ x_1\ x_2\ \cdots\ x_m]_t$ in

$$\bigcap\{P \mid \text{for every maximal ideal } P \text{ such that } [0x_1 \cdots x_n]_t \subset P^l \text{ for some } l \in Z^+\};$$

$1 \leq t \leq \min(m+1, n)$.

Proof. Let $k = \min(m+1, n)$. Suppose first $k = m+1$, that is $m < n$, by proposition (3.1), if $r \in prad_F(N)$, then $[r\ x_1\ x_2\ \cdots\ x_m]_t$ in

$$\bigcap\{P \mid \text{for every maximal ideal } P \text{ such that } [0x_1 \cdots x_n]_t \subset P^l \text{ for some } l \in Z^+\};$$

$1 \leq t \leq \min(m+1, n)$.

Now suppose $k = n$, *i.e.* $n \leq m+1$. Let $G = R^{(m+1)}$, $r = (r_1, r_2, \cdots, r_n)$, $xi = (x_{i1}, x_{i2}, \cdots, x_{in})$ for some $r_j \in R$ and $x_{ij} \in R$, $(1 \leq i \leq m, 1 \leq j \leq n)$. By proposition (3.3), $r \in prad_F(N)$ if and only if $(r_1, r_2, \cdots, r_n, 0, 0, \cdots, 0)$ in $prad_G(N')$. Where

$$N' = \sum_{i=1}^{m} R(x_{i1}, x_{i2}, \cdots, x_{in}, 0, 0, \cdots, 0).$$

Now apply proposition (3.1) to obtain the result.

The following example will illustrate application of the proposition (3.2).

3.5. Example

Let $R = Z$, $F = Z^3$ and N be the submodule $Z(1,3,5) + Z2(1,1,1)$ of F. Then $(r_1, r_2, r_3) \in prad_F(N)$ if $3r_1 - r_2$, $5r_1 - r_3$, $5r_2 - r_3$ in $2Z$ and $(r_1 - 2r_2 + r_3) = 0$.

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