

# Some Criteria for the Asymptotic Behavior of a Certain Second Order Nonlinear Perturbed Differential Equation

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## ABSTRACT

In this paper we give sufficient conditions so that for every nonoscillatory  $u(t)$  solution of

$(r(t)\psi(u)u')' + Q(t,u) = P(t,u,u')$ , we have  $\liminf_{t \rightarrow \infty} |u(t)| = 0$ . Our results contain the some known results in the literature as particular cases.

**Keywords:** Perturbed Differential Equations; Nonoscillatory Solution; Asymptotic Behavior

## 1. Introduction

This paper is concerned with the problem of asymptotic behavior of the second order nonlinear perturbed differential equation

$$(r(t)\psi(u)u')' + Q(t,u) = P(t,u,u') \quad (1)$$

where

$$r \in C(I, R^+), \psi \in C(R, R^+), R = (-\infty, \infty), I = [t_0, \infty).$$

Throughout the paper according to the results we shall impose the following conditions:

(H<sub>1</sub>) Let  $f \in C^1(R, R)$  and there exists a constant  $k > 0$  such that  $\frac{f'(x)}{\psi(x)} \geq k$  and  $f(x)x > 0$  for

$x \neq 0$ ,

(H<sub>2</sub>)  $Q \in C(I \times R, R)$  and there exists a continuous function  $q(t)$  such that  $\frac{Q(t,x)}{f(x)} \geq q(t)$  for  $x \neq 0$ ,

(H<sub>3</sub>)  $P \in C(I \times R \times R, R)$  and there exists a continuous function  $p(t)$  such that  $\frac{P(t,x,y)}{f(x)} \leq p(t)$  for

$x \neq 0, y \neq 0$

(H<sub>4</sub>)  $P \in C(I \times R \times R, R)$  and there exists a continuous function  $p(t)$  such that  $\frac{P(t,x,y)}{f(x)} \leq \frac{p(t)y}{f(x)}$  for

$x \neq 0, y \neq 0$

(H<sub>5</sub>)  $\int_{\varepsilon}^{\infty} \left(\frac{\psi(s)}{f(s)}\right) ds < \infty$  and  $\int_{-\varepsilon}^{-\infty} \left(\frac{\psi(s)}{f(s)}\right) ds < \infty$  for every

$\varepsilon > 0$ .

We shall also restrict our attention only to the solution of the differential Equation (1.1) which exist on some ray of the form  $[t_0, \infty)$ .

The oscillatory behavior of the solution of second order ordinary differential equations including the existence of oscillatory and nonoscillatory solutions has been the subject of intensive investigation. This problem has received the attention of many others. See for example, [1-5]. Since the publication of Hammet's paper in [6] in 1971, the asymptotic behavior of the solution of the ordinary and functional differential equations has been widely discussed in the literature [2,4,7-9].

In this paper we give sufficient conditions so that for every nonoscillatory  $u(t)$  solution of (1), we have

$\liminf_{t \rightarrow \infty} |u(t)| = 0$ . Our results contain the results in [10] as particular cases.

## 2. Main Results

In this section we prove our main results.

**Theorem 2.1.** Let conditions (H<sub>1</sub>), (H<sub>2</sub>), (H<sub>3</sub>) and (H<sub>5</sub>) hold. If there exists a differentiable function  $\rho: I \rightarrow R^+$  such that

$$\rho'(t) \geq 0, \quad (2)$$

$$\int \frac{dt}{\rho(t)r(t)} < \infty \quad (3)$$

and

$$\lim_{t \rightarrow \infty} \int_{t_0}^t \frac{1}{\rho(s)r(s)} \left( \int_{t_0}^s [A(\tau)] d\tau \right) ds = \infty \quad (4)$$

where

$$A(t) = \rho(t) \left[ q(t) - p(t) - \frac{v(t)}{4k} \left( \frac{\rho'(t)}{\rho(t)} \right)^2 \right].$$

Then for every nonoscillatory solution  $u$  of Equation (1), we have  $\liminf_{t \rightarrow \infty} |u(t)| = 0$ .

**Proof.** Let  $u(t)$  be a nonoscillatory solution of (1). We may assume that  $u(t) \neq 0$  for  $t \geq t_1 \geq t_0$ . Define

$$w(t) = \rho(t) \frac{r(t)\psi(u(t))u'(t)}{f(u(t))}. \tag{5}$$

Differentiating (5) and making use of (1) and from hypothesis (H<sub>1</sub>), (H<sub>2</sub>) and (H<sub>3</sub>), it follows that

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} |w(t)| - p(t)(q(t) - p(t)) - k \frac{|w(t)|^2}{\rho(t)r(t)} \tag{6}$$

By using the inequality

$$DX - EX^2 \leq \frac{1}{4} D^2 E^{-1}, D \geq 0, E > 0, X \geq 0 \tag{7}$$

we get

$$w'(t) \leq -\rho(t) \left[ q(t) - p(t) - \frac{1}{4k} r(t) \left( \frac{\rho'(t)}{\rho(t)} \right)^2 \right] \tag{8}$$

Integrating this inequality, from  $t_1$  to  $t$ , we get

$$\rho(t) \frac{r(t)\psi(u(t))u'(t)}{f(u(t))} \leq w(t_1) - \int_{t_1}^t A(s) ds, t \geq t_1 \geq t_0 \tag{9}$$

Dividing (9) by  $\rho(t)r(t)$  and hence integrating from  $t_1$  to  $t$  we obtain

$$\int_{u(t_1)}^{u(t)} \frac{\psi(s)}{f(s)} ds \leq w(t_1) \int_{t_1}^t \frac{ds}{\rho(s)r(s)} - \int_{t_1}^t \frac{ds}{\rho(s)r(s)} \left( \int_{t_1}^s A(\tau) d\tau \right) ds \tag{10}$$

Using (4) we get

$$\liminf_{t \rightarrow \infty} \int_{u(t_1)}^{u(t)} \frac{\psi(s)}{f(s)} ds = -\infty \tag{11}$$

If  $\liminf_{t \rightarrow \infty} u(t) > 0$ , then there exists a positive constant  $c$  such that  $u(t) \geq c$  for all  $t \geq t_1$  and consequently, by (H<sub>5</sub>)

$$\left| \int_{u(t_1)}^{u(t)} \frac{\psi(s)}{f(s)} ds \right| \leq \int_c^\infty \frac{\psi(s)}{f(s)} ds < \infty$$

which contradicts (11). Thus we must have

$\liminf_{t \rightarrow \infty} u(t) = 0$ . The proof for the case  $\liminf_{t \rightarrow \infty} u(t) < 0$  for  $t \geq t_1$  is similar and hence is omitted.

We note that when  $\rho(t) \equiv 1$ , (H<sub>1</sub>) condition can be weakened. Indeed from the proof of Theorem 2.1, the following result can obtain easily.

**Theorem 2.2.** Let conditions (H<sub>1</sub>), (H<sub>2</sub>), (H<sub>3</sub>) and (H<sub>5</sub>) hold. Suppose that

$$\int_{t_0}^\infty \frac{dt}{r(t)} < \infty, \psi(x) > 0 \text{ and } \frac{f'(x)}{\psi(x)} \geq 0 \text{ for } x \neq 0,$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \frac{1}{r(s)} \left( \int_{t_0}^s (q(\tau) - p(\tau)) d\tau \right) ds = \infty.$$

Then for every solution  $u(t)$  of Equation (1), we have  $\liminf_{t \rightarrow \infty} u(t) = 0$ .

By taking (H<sub>4</sub>) instead of (H<sub>3</sub>) we obtain the following result which can be applied for example to the damped equation

$$(r(t)\psi(u)u')' + b(t)u' + q(t)f(u) = 0, t \geq t_1 \geq t_0.$$

**Theorem 2.3.** Let conditions (H<sub>1</sub>), (H<sub>2</sub>), (H<sub>4</sub>) and (H<sub>5</sub>) hold. Suppose that

$$\psi(x) \geq c > 0. \tag{12}$$

If there exists a differentiable function  $\rho: I \rightarrow R^+$  such that (3) holds and,

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \frac{1}{\rho(s)r(s)} \left( \int_{t_0}^s B(\tau) d\tau \right) ds = \infty \tag{13}$$

where

$$B(t) = \rho(t) \left[ q(t) - \frac{r(t)}{4k} \left( \frac{|\rho'(t)|}{\rho(t)} + \frac{|p(t)|}{cr(t)} \right)^2 \right]$$

then for every solution  $u(t)$  of Equation (1), we have  $\liminf_{t \rightarrow \infty} u(t) = 0$ .

**Proof.** Let  $u(t)$  be a nonoscillatory solution of Equation (1). We may assume that  $u(t) \neq 0$  for  $t \geq t_1 \geq t_0$ . Differentiating (5) and making use of (1) and from hypothesis (H<sub>1</sub>), (H<sub>2</sub>), (H<sub>4</sub>) and (12), as in the proof of Theorem 2.1, we can obtain easily that

$$w'(t) \leq -\rho(t) \left[ q(t) - \frac{r(t)}{4k} \left( \frac{|\rho'(t)|}{\rho(t)} + \frac{|p(t)|}{cr(t)} \right)^2 \right] \tag{14}$$

Integrating this inequality from  $t_1$  to  $t$  we get

$$\rho(t) \frac{r(t)\psi(u(t))u'(t)}{f(u(t))} \leq w(t_1) - \int_{t_1}^t B(s) ds \tag{15}$$

Dividing (15) by  $\rho(t)r(t)_1$  and hence integrating from  $t_1$  to  $t$  we obtain

$$\liminf_{t \rightarrow \infty} \int_{x(t_1)}^{x(t)} \frac{\psi(u)}{f(u)} du = -\infty$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted.

**Remark 2.1.** Grace and Lalli, consider the following equation

$$(a(t)\psi(x(t))x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0$$

in [10] and give a similar result. But if we compare Theorem 2.3 with Theorem 2 in [10], we observe that they have a condition such as  $\rho'(t)p(t) \leq 0$  for  $t \geq t_0$  which impose some restriction on  $\rho$  and  $p$ . In our result we remove this condition, so Theorem 2.3 is weaker than Theorem 2 in [10].

**Remark 2.2.** To give similar results for the equation

$$(r(t)\psi(u)|u|^{\alpha-1}u')' + Q(t,u) = P(t,u,u')$$

where  $r \in C(I, R^+)$ ,  $\psi \in C(R, R^+)$ ,  $R = (-\infty, \infty)$ ,  $I = [t_0, \infty)$  and  $\alpha$  is a positive real number, still remains as an open problem and will be interesting.

### 3. Example

Consider the differential equation of the form

$$(e^{4t}u^2u')' + (e^{6t} + \theta_1(t,u))u^5 = \theta_2(t,u,u')u^5, t \geq t_0$$

where  $\theta_1$  and  $\theta_2$  are continuous function such that  $\theta_1(t,u) \geq 0$  and  $\theta_2(t,u,u') \leq 1$ . All conditions of Theorem 2.1 are satisfied. Then every nonoscillatory solutions  $u$  of (16) we have  $\liminf_{t \rightarrow \infty} |u(t)| = 0$ . In particular

$$(e^{4t}u^2u')' + (e^{6t} + e^{-2t})u^5 = e^{-2t}u^5, t \geq 0$$

has a solution  $u(t) = e^{-t}$  and  $\liminf_{t \rightarrow \infty} |u(t)| = 0$ .

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