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G-Design of Complete Multipartite Graph Where *G* Is Five Points-Six Edges

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ABSTRACT

In this paper, we construct *G*-designs of complete multipartite graph, where *G* is five points-six edges.

Keywords: Complete Multipartite Graph; Graph Design; Latin Square

1. Introduction

Let K_v be a complete graph with v vertices, and G be a simple graph with no isolated vertex. A G-design (or G-decomposition) is a pair (X, B), where X is the vertex set of K_v and B is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any edge of K_v occurs in exactly a blocks of B. For simplicity, such a G-design is denoted by G-$GD(v)$. Obviously, the necessary conditions for the existence of a G-$GD(v)$ are

$$\begin{aligned} &v \geq |V(G)| \\ &v(v-1) \equiv 0 \bmod 2|E(G)|, \\ &v-1 \equiv 0 \bmod d \end{aligned} \tag{1}$$

where d is the greatest common divisor of the degrees of the vertices in $V(G)$.

Let $K_{n_1,n_2,\cdots,n_m}$ be a complete multipartite graph with vertex set $X = \bigcup_{i=1}^{m} X_i$, where these X_i are disjoint and $|X_i| = n_i$, $1 \leq i \leq m$. For a given graph G, a holey G-design, denoted by $(X, \mathcal{G}, \mathcal{B})$, where X is the vertex set of $K_{n_1,n_2,\cdots,n_m}$, $\mathcal{G} = \{X_1, X_2, \cdots, X_m\}$ (X_i called hole) and $\mathcal{B}$ is a collection of subgraphs of $K_{n_1,n_2,\cdots,n_m}$ called blocks, such that each block is isomorphic to G and any edge of $K_{n_1,n_2,\cdots,n_m}$ occurs in exactly a blocks of $\mathcal{B}$. When the multipartite graph has k_i partite of size n_i $1 \leq i \leq r$, the holey G-design is denoted by $G-HD\left(n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}\right)$. When $n_1 = n_2 = \cdots = n_m = n$, the holey G-design is denoted by $G-HD\left(n^m\right)$ (also known as G-decomposition of complete multipartite graph $K_n(t)$).

On the G-design of existence has a lot of research. Let k be the vertex number of G, When $k \leq 4$, J. C. Bermond proved that condition (1) is also sufficient in [1]; When $k = 5$, J. C. Bermond gives a complete solution in [2]. When $G = S_k$, P_k and C_k, K. Ushio investigated the existence of G-design of complete multipartite graph in [3]. In this paper, G-designs of complete multipartite graph, where G is five points-six edges is studied. Necessary and sufficient conditions are given for the G-designs of complete multipartite graph $K_n(t)$. For graph theoretical term, see [4].

2. Fundamental Theorem and Some Direct Construction

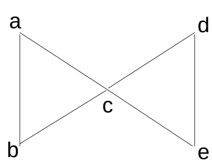
Let G be a simple graph with five points-six edges (see **Graph 1**). G is denoted by (a, b, c)-(c, d, e).

The lexicographic product $G_1 \otimes G_2$ of the graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and an edge joining (u_1, u_2) to (v_1, v_2) if and only if either u_1 is adjacent to v_1 in G_1 or $u_1 = v_1$ and u_2 and v_2 are adjacent in G_2. We are only concered with a particular kind of lexicographic product, $G \times \bar{K}_n$ ($\bar{K}_n$ be a empty graph with n vertices). Observe that

$$K_n(lt) = K_n(t) \otimes \bar{K}_l .$$

Lemma 2.1. If there exists a G-$HD(t^n)$, then there exists a G-$HD((lt)^n)$ for any integer l.

Proof. Let $V(\bar{K}_l) = \{1, 2, \cdots, l\}$, Take any $l \times l$ latin square and consider each element in the form (α, β, γ) where α denotes the row, β the column and γ the entry, with $1 \leq \alpha, \beta, \gamma \leq l$. We can construct l^2 graphs G.



Graph 1. Let *G* be a simple graph with five points-six edges.

$$((a,i),(b,j),(c,k))—((c,k),(d,i),(e,j))(1 \le i,j,k \le l)$$

Let *K* be a subset of positive integers. A *pairwise balanced design* (*PBD*(*v*, *K*)) of order *v* with block sizes from *K* is a pair ($\mathcal{Y}$, $\mathcal{B}$),where $\mathcal{Y}$ is a finite set (*the point set*) of cardinality *v* and $\mathcal{B}$ is a family of subsets (*blocks*) of $\mathcal{Y}$ which satisfy the properties:

1) If $B \in \mathcal{B}$, then $|B| \in K$.

2) Every pair of distinct elements of $\mathcal{Y}$ occurs in exactly a blocks of $\mathcal{B}$.

Let *K* be a set of positive integers and

$$B(K) = \{v \in N | \exists PBD(v,K)\},$$

then $B(K)$ is the *PBD-closure* of *K*.

Lemma 2.2 [5] If *K* = {3, 4, 5, 6, 8}, then

$$B(K) = \{n \in N | n \ge 3\}.$$

Lemma 2.3 [5] If *K* = {3, 4, 6}, then

$$B(K) = \{v \in N | n > 3, n \equiv 0,1 (\bmod 3)\}$$

Lemma 2.4 [5] If *K* = {5, 9, 13, 17, 29, 33}, then

$$B(K) = \{v \in N | n > 4, n \equiv 1 (\bmod 4)\}$$

Lemma 2.5 If there exists a *G-HD*(t^k) where $k \in$ {3, 4, 5, 6, 8}, then there exists a *G-HD*(t^n) where $n \ge 3$.

Proof. Let *X* be $n(n \ge 3)$ element set and Z_t be a modulo *t* residual additive group. For *K* = {3, 4, 5, 6, 8}, take $Y = X \times Z_t$, by applying Lemma 2.2, we assume that (*X*, $\mathcal{A}$) be a *PBD*(*n*, *k*). In the *A*, we take a block *A*, for $|A| = k \in K$, as there exists a *G-HD*(t^k), let $A \times Z_t$ be the vertex set of *G-HD*(t^k) and block set be $\mathcal{B}_{\mathcal{A}}$. be a $\mathcal{B} = \bigcup \mathcal{B}_{\mathcal{A}} (A \in \mathcal{A})$, so (*Y*, $\mathcal{B}$) be a *G-HD*(*tn*).

Similar to the proof of lemma 2.5, We have the following conclusions.

Lemma 2.6 If there exists a *G-HD*(t^k) for $k \in$ {3, 4, 6}, then there exists a *G-HD*(t^n) for n ≡ 0, 1 (mod 3) and $n > 3$.

Lemma 2.7 If there exists a *G-HD*(t^k) for $k \in$ {5, 9, 13, 17, 29, 33}, then there exists a *G-HD*(t^n) for n ≡ 1 (mod 4) and $n > 4$.

Lemma 2.8 [2] For $n \equiv 1, 9 \pmod{12}$, there exists a (*n*, *G*, 1)-*GD*.

Lemma 2.9 There exists a *G-HD*(2^3).

Proof. Take *X* = {*a*, *b*, *c*, *d*, *e*, *f*} and *G* = {{*a*, *b*}, {*c*, *d*}, {*e*, *f*}}, we list vertex set and blocks below.

$$\mathcal{B}: (a,d,f)\text{-}(f,c,b)(c,a,e)\text{-}(e,b,d)$$

Lemma 2.10 There exists a *G-HD*(2^4).

Proof. Take $X = Z_8$ and *G* = {{0, 4}, {1, 5}, {2, 6}, {3, 7}}, we list vertex set and blocks below

$$\mathcal{B}: (3,0,1)\text{-}(1,2,4) + 2 \bmod 4$$

Lemma 2.11 There exists a *G-HD*(2^6).

Proof. Take $X = Z_{12}$, and *G* = {{0, 5}, {1, 6}, {2, 7}, {3, 8}, {4, 9}, {∞_1, ∞_2}}, we list vertex set and blocks below

$$\mathcal{B}: (1+i,3+i,\ i)\text{-}(i,4+i,\infty_1)(i = 0,1,2,3,4)$$
$$(1+i,3+i,\ i)\text{-}(i,4+i,\infty_2)(i = 5,6,7,8,9)$$

Lemma 2.12 There exists a *G-HD*(3^5).

Proof. Take *X* = {a_i, b_i, c_i, d_i, e_i}, X_1 = {a_i} X_2 = {b_i} X_3 = {c_i} X_4 = {d_i} X_5 = {e_i} (*i* = 1, 2, 3), and *G* ={X_1, X_2, X_3, X_4, X_5}, we list vertex set and blocks below

$$\mathcal{B}: (b_1,c_1,a_1)\text{-}(a_1,b_3,c_3)$$
$$(a_1,c_2,b_2)\text{-}(b_2,a_2,e_2)$$
$$(b_1,e_1,a_2)\text{-}(a_2,b_3,e_3)$$
$$(e_2,d_2,a_1)\text{-}(a_1,e_3,d_3)$$
$$(c_2,e_2,a_3)\text{-}(a_3,c_3,e_3)$$
$$(a_3,c_1,e_1)\text{-}(e_1,a_1,d_1)$$
$$(b_1,d_1,a_3)\text{-}(a_3,b_2,d_2)$$
$$(c_1,d_1,a_2)\text{-}(a_2,c_2,d_2)$$
$$(a_2,c_3,d_3)\text{-}(d_3,a_3,b_3)$$
$$(e_3,d_2,b_1)\text{-}(b_1,c_2,d_3)$$
$$(e_1,d_3,b_2)\text{-}(b_2,c_3,d_1)$$
$$(e_2,d_1,b_3)\text{-}(b_3,c_1,d_2)$$
$$(d_3,e_2,c_1)\text{-}(c_1,e_3,b_2)$$
$$(d_1,e_3,c_2)\text{-}(c_2,e_1,b_3)$$
$$(d_2,e_1,c_3)\text{-}(c_3,e_2,b_1)$$

Lemma 2.13 There exists a *G-HD*(3^8).

Proof. Take $X = Z_{24}$ and *G* = {{*i*, *i* + 8, *i* + 16}, $i \in Z_8$}, we list vertex set and blocks below

$$\mathcal{B}: (5,9,2)\text{-}(2,11,13) + 4 \bmod 24$$
$$(6,10,3)\text{-}(3,12,14) + 4 \bmod 24$$
$$(3,21,2)\text{-}(2,20,1) + 4 \bmod 24$$
$$(3,7,1)\text{-}(1,9,23) + 4 \bmod 24$$
$$(23,5,18)\text{-}(18,6,16) + 4 \bmod 24$$
$$(4,8,1)\text{-}(1,10,0) + 4 \bmod 24$$
$$(0,6,19)\text{-}(19,7,17) + 4 \bmod 24$$

Lemma 2.14 There exists a *G-HD*(3^9).

Proof. Take $X = Z_{27}$ and $\mathcal{G}$ = {{*i*, *i* + 9, *i* + 18}, $i \in Z_9$}, we list vertex set and blocks below

$$\mathcal{B}: (2,13,0)\text{-}(0,4,12) \bmod 27$$
$$(16,23,26)\text{-}(26,25,20) \bmod 27$$

Lemma 2.15 There exists a *G-HD*(3^{17}).

Proof. Take $X = Z_{51}$ and $\mathcal{G}$ ={{*i*, *i* + 17, *i* + 34}, *i* ∈

$Z_{17}\}$, we list vertex set and blocks below

$$\mathcal{B}:(2,16,0)\text{-}(0,23,30) \bmod 51$$
$$(3,15,0)\text{-}(0,25,29) \bmod 51$$
$$(5,11,0)\text{-}(0,24,32) \bmod 51$$
$$(1,10,0)\text{-}(0,20,33) \bmod 51$$

Lemma 2.16 There exists a *G-HD*(3^{29}).

Proof. Take $X = Z_{87}$ and $\mathcal{G} = \{\{i, i + 29, i + 58\}, i \in Z_{29}\}$, we list vertex set and blocks below

$$\mathcal{B}:(43,52,0)\text{-}(0,1,4) \bmod 87$$
$$(42,53,0)\text{-}(0,2,27) \bmod 87$$
$$(41,54,0)\text{-}(0,5,23) \bmod 87$$
$$(40,55,0)\text{-}(0,6,26) \bmod 87$$
$$(39,70,0)\text{-}(0,7,28) \bmod 87$$
$$(38,68,0)\text{-}(0,8,24) \bmod 87$$
$$(37,73,0)\text{-}(0,10,22) \bmod 87$$

Lemma 2.17 There exists a *G-HD*(6^k), for $k \in \{3, 4, 5, 6, 8\}$.

Proof. By applying Lemma 2.9, 2.10, 2.11, 2.12, 2.13 and 2.1, we can obtain the result.

Lemma 2.18 There exists a *G-HD*(3^k), for $k \in \{13, 33\}$.

Proof. By applying Lemma 2.8 and 2.1, we can obtain the result.

3. *G*-Designs of Complete Multipartite Graph

Theorem 3.1 The necessary conditions for the existence of a *G-HD*(t^n) are sufficient for the following n and t:

1) $t \equiv 0 \pmod 6$ and $n \geq 3$;
2) $t \equiv 0 \pmod 2$, $t \neq 0 \pmod 3$ and $n \equiv 0, 1 \pmod 3$,
3) $t \equiv 0 \pmod 3$, $t \neq 0 \pmod 2$ and $n \equiv 1 \pmod 4$,
4) $t \neq 0 \pmod 2$, $t \neq 0 \pmod 3$ and $n \equiv 1, 9 \pmod{12}$.

Proof. Necessary conditions are obviously, we prove the sufficient conditions.

1) For $t \equiv 0 \pmod 6$ and $n \geq 3$, by applying Lemma 2.17 and 2.5.,

2) For $t \equiv 0 \pmod 2$, $t \neq 0 \pmod 3$ and $n \equiv 0, 1 \pmod 3$ by applying Lemma 2.9, 2.10, 2.11 and 2.6.

3) For $t \equiv 0 \pmod 3$, $t \neq 0 \pmod 2$ and $n \equiv 1 \pmod 4$ by applying Lemma 2.12, 2.14, 2.15, 2.16, 2.18 and 2.7.

4) For $t \neq 0 \pmod 2$, $t \neq 0 \pmod 3$ and $n \equiv 1, 9 \pmod{12}$, by applying Lemma 2.1 and 2.8.

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