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A $\wp_x$- and Open C_D^*-Filters Process of Compactifications and Any Hausdorff Compactification

Hueytzen J. Wu[1], Wan-Hong Wu[2]
[1]Department of Mathematics, Texas A & M University-Kingsville, Kingsville, USA
[2]University of Texas at San Antonio, One UTSA Circle, San Antonio, USA
Email: hueytzen.wu@tamuk.edu, dd1273@yahoo.com

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ABSTRACT

By means of a characterization of compact spaces in terms of open C_D^*-filters induced by a $\boldsymbol{D} \subseteq \mathbf{C}^*(\mathbf{Y})$, a $\wp_x$- and open C_D^*-filters process of compactifications of *an arbitrary topological space* **Y** is obtained in Sec. 3 by embedding **Y** as a dense subspace of $(\mathbf{Y}_S^W, \Im_{\mathcal{B}})$ or $(\mathbf{Y}_T^W, \Im_{\mathcal{B}})$, where $\mathbf{Y}_S^W = \mathbf{Y}_E \cup \mathbf{Y}_S$, $\mathbf{Y}_T^W = \mathbf{Y}_E \cup \mathbf{Y}_T$, $\mathbf{Y}_E = \{\mathbf{N}_x | \mathbf{N}_x$ is a $\wp_x$-filter, $x \in \mathbf{Y}\}$, $\mathbf{Y}_S = \{\mathcal{E} | \mathcal{E}$ is an open C_D^*-filter that does not converge in $\mathbf{Y}\}$, $\mathbf{Y}_T = \{\mathbf{Å} | \mathbf{Å}$ is a basic open C_D^*-filter that does not converge in $\mathbf{Y}\}$, $\Im_{\mathcal{B}}$ is the topology induced by the base $\mathcal{B} = \{\mathbf{U}^* | \mathbf{U}$ is open in $\mathbf{Y}$, $\mathbf{U} \neq \phi\}$ and $\mathbf{U}^* = \{\mathcal{F} \in \mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$)$| \mathbf{U} \in \mathcal{F}\}$. Furthermore, an *arbitrary Hausdorff compactification* (**Z**, h) of a Tychonoff space **X** can be obtained from a $\boldsymbol{D} \subseteq \mathbf{C}^*(\mathbf{X})$ by the similar process in Sec. 3.

Keywords: Net; Open Filter; Open C_D^*-Filter Base; Basic Open C_D^*-Filter; Open C_D^*-Filter; $\wp$-Filter; $\wp_x$-Filter; Tychonoff Space; Normal $\mathbf{T}_1$-Space; Compact Space; Compactifications; Stone-Čech Compactification; Wallman Compactification

1. Introduction

Throughout this paper, $[\mathbf{T}]^{<\omega}$ denotes the collection of all finite subsets of the set **T**. For the other notations and terminologies in General Topology which are not explicitly defined in this paper, the readers will be referred to the Ref. [1].

For an arbitrary topological space **Y**, let $\mathbf{C}^*(\mathbf{Y})$ be the set of bounded real-valued continuous functions on **Y**, $\boldsymbol{D} \subseteq \mathbf{C}^*(\mathbf{Y})$. It is shown in Sec. 2 that there exists a unique $r_f \in \mathrm{Cl}(f(\mathbf{Y}))$ for each f in $\boldsymbol{D}$ such that for any $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, $\varepsilon > 0$, $\cap_{f\in H} f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \neq \phi$. Let $\mathbf{V}_r = \{\cap_{f\in H} f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) | \cap_{f\in H} f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \neq \phi$ for any $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, $\varepsilon > 0\}$. $\mathbf{V}_r$ is called an *open* C_D^**-filter base*. An open filter $\mathcal{E}_r$ on **Y** containing an open C_D^*-filter base $\mathbf{V}_r$ is called an open C_D^*-filter. An open filter $\mathbf{Å}_r$ on **Y** generated by an open C_D^*-filter base $\mathbf{V}_r$ is called a *basic open* C_D^*-filter. By a characterization of compact spaces in Sec. 2 and the $\wp_x$- and open C_D^*-filters process of compactifications in Sec. 3, **Y** can be embedded as a dense subspace of $(\mathbf{Y}_S^W, \Im_{\mathcal{B}})$ or $(\mathbf{Y}_T^W, \Im_{\mathcal{B}})$, where $\mathbf{Y}_S^W = \mathbf{Y}_E \cup \mathbf{Y}_S$, $\mathbf{Y}_T^W = \mathbf{Y}_E \cup \mathbf{Y}_T$, $\mathbf{Y}_E = \{\mathbf{N}_x | \mathbf{N}_x$ is a $\wp_x$-filter, $x \in \mathbf{Y}\}$, $\mathbf{Y}_S = \{\mathcal{E} | \mathcal{E}$ is an open C_D^*-filter that does not converge in $\mathbf{Y}\}$, $\mathbf{Y}_T = \{\mathbf{Å} | \mathbf{Å}$ is a basic open C_D^*-filter that does not converge in $\mathbf{Y}\}$, $\Im_{\mathcal{B}}$ is the topology induced by the base $\mathcal{B} = \{\mathbf{U}^* | \mathbf{U} \neq \phi$, **U** is open in $\mathbf{Y}\}$ and $\mathbf{U}^* = \{\mathcal{F} \in \mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$)$| \mathbf{U} \in \mathcal{F}\}$. Furthmore an arbitrary Hausdorff compactification (**Z**, h) of a Tychonoff space **X** can be obtained from a $\boldsymbol{D} \subseteq \mathbf{C}^*(\mathbf{X})$ by the similar process in Sec. 3.

2. Open C_D^*-Filters and a Characterization of Compact Spaces

Let **Y**, $\mathbf{C}^*(\mathbf{Y})$ and $\boldsymbol{D}$ be the sets that are defined in Sec. 1.

Theorem 2.1 *Let* $\mathcal{F}$ *be a filter on a topological space* **Y**. *For each* $f \in \boldsymbol{D}$, *there exists a* $r_f \in \mathrm{Cl}(f(\mathbf{Y}))$ *such that* $f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \cap \mathbf{F} \neq \phi$ *for any* $\mathbf{F} \in \mathcal{F}$ *and any* $\varepsilon > 0$. (*See* Thm 2.1 *in the* Ref. [2, p. 1164].)

Proof. If the conclusion is not true, then there is an $f \in \boldsymbol{D}$ such that for each $r_t \in \mathrm{Cl}(f(\mathbf{Y}))$, there exist an $\mathbf{F}_t \in \mathcal{F}$ and an $\varepsilon_t > 0$ such that $\mathbf{F}_t \cap f^{-1}((r_t - \varepsilon_t, r_t + \varepsilon_t)) = \phi$. Since $\mathrm{Cl}(f(\mathbf{Y}))$ is compact and $\mathrm{Cl}(f(\mathbf{Y})) \subset \cup\{(r_t - \varepsilon_t, r_t + \varepsilon_t) | r_t \in \mathrm{Cl}(f(\mathbf{Y}))\}$, there exist $r_1, \cdots, r_n$ in $\mathrm{Cl}(f(\mathbf{Y}))$ such that $\mathbf{Y} = f^{-1}(\mathrm{Cl}(f(\mathbf{Y}))) = \cup\{f^{-1}((r_i - \varepsilon_i, r_i + \varepsilon_i)) | i = 1, \cdots, n\}$. Let $\mathbf{F}_o = \cap\{\mathbf{F}_i | i = 1, \cdots, n\}$, then $\mathbf{F}_o \in \mathcal{F}$ and $\mathbf{F}_o = \mathbf{F}_o \cap \mathbf{Y} \subseteq \cup\{[\mathbf{F}_i \cap f^{-1}((r_i - \varepsilon_i, r_i + \varepsilon_i))] | i = 1, \cdots, n\} = \phi$, contradicting that $\phi \notin \mathcal{F}$. □

Corollary 2.2 *Let* **Q** *be an open ultrafilter on* **Y**. *For* *each* f ∈ ***D***, *there exists a unique* $r_f \in Cl(f(\mathbf{Y}))$ *such that* (1) *for any* $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, *any* ε > 0, $\cap_{f\in H} f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \in \mathbf{Q}$ *and* (2) *for any* $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, *any* ε > 0, $\cap_{f\in H} f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \neq \phi$. (*See* Cor. 2.2 *in the* Ref. [2, p. 1164].)

Therefore, for a given open ultrafilter **Q**, **Q** contains a unique open filter base $\mathbf{V}_r = \{\cap_{f\in H} f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) | \cap_{f\in H} f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \neq \phi$ for any **H** in $[\boldsymbol{D}]^{<\omega}$, ε > 0}. $\mathbf{V}_r$ is called an *open* C_D^* *-filter base*. An open filter $\mathcal{E}_r$ on **Y** containing an open C_D^* -filter base $\mathbf{V}_r$ is called an *open* C_D^* *-filter*. An open filter $\mathbf{Å}_r$ on **Y** generated by an open C_D^* -filter base $\mathbf{V}_r$ is called a *basic open* C_D^* *-filter*. For each f ∈ ***D***, if $r_f = f(x)$ for an x in **Y**, then $\mathbf{V}_r$ and $\mathbf{Å}_r$ are called the *open* C_D^* *-filter base* and the *basic open* C_D^* *-filter at x*, denoted by $\mathbf{V}_x$ and $\mathbf{Å}_x$, respectively.

Definition 2.3 *Let* ***L*** *be a family of continuous functions on* **Y**. *A net* $\{x_i\}$ *in* **Y** *is called a* ***L*-net**, *iff* $\{f(x_i)\}$ *converges for each f* ∈ ***L***.

Theorem 2.4 *Let* ***L*** *be a set of continuous functions on* **Y**. *Then* **Y** *is compact iff* (1) *f*(**Y**) *is contained in a compact set* $\mathbf{C}_f$ *for each f in* ***L***, *and* (2) *every* ***L****-net has a cluster point in* **Y**.

Proof. Let $\{x_i\}$ be an ultranet in **Y**. For each f in ***L***, $\{f(x_i)\}$ is an ultranet in $\mathbf{C}_f$, hence converges in $\mathbf{C}_f$; *i.e.*, $\{x_i\}$ is a ***L***-net. (2) implies that $\{x_i\}$ has a cluster point x in **Y**. Since $\{x_i\}$ is an ultranet, $\{x_i\}$ converges to x. Thus, **Y** is compact. The converse is obvious. □

Corollary 2.5 *Let* ***D*** ⊆ **C***(**Y**). *If every* ***D****-net converges in* **Y**, *then* **Y** *is compact.*

Definition 2.6 *If* $\mathcal{F}$ *is a filter on* **Y**, *let* $\Lambda_{\mathcal{F}} = \{(x, \mathbf{F}) | x \in \mathbf{F} \in \mathcal{F}\}$. *Then* $\Lambda_{\mathcal{F}}$ *is directed by the relation* $(x_1, \mathbf{F}_1) \le (x_2, \mathbf{F}_2)$ *iff* $\mathbf{F}_2 \subset \mathbf{F}_1$, *so the map* **P**: $\Lambda_{\mathcal{F}} \to \mathbf{Y}$ *defined by* **P**(x, **F**) = x *is a net in* **Y**. *It is called the net based on* $\mathcal{F}$. (*See* Def.12.16 *in the* Ref. [1, p. 81].)

Corollary 2.7 *If* $\mathcal{F}$ *is a filter on* **Y**, {**P**(x, **F**)} *is the net based on* $\mathcal{F}$, *then* $\mathcal{F} = \{\mathbf{S} \subseteq \mathbf{Y} | \mathbf{P}(x, \mathbf{F})\}$ *is eventually in* **S**}. (*See* **L**2) *in the* Ref. [3, p. 83].)

Lemma 2.8 *Let* ***D*** ⊆ **C***(**Y**). 1) *For each open* C_D^* *-filter* $\mathcal{E}$, *let* $\mathbf{V}_r$, *as the* $\mathbf{V}_r$ *defined in* Section 1, *be an open* C_D^* *-filter base such that* $\mathbf{V}_r \subset \mathcal{E}$. *Then the net* $\{x_F\}$ *based on* $\mathcal{E}$ *is a* ***D****-net such that* $\lim\{f(x_F)\} = r_f$ *for each* f ∈ ***D***. 2) *For each* ***D****-net* $\{x_i\}$ *in* **Y**, $\{x_i\}$ *induces a unique open* C_D^* *-filer base* $\mathbf{V}\{x_i\}$ *on* **Y**.

Proof. 1) By Cor. 2.7, $\{x_F\}$ is eventually in $f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \in \mathbf{V}_r \subset \mathcal{E}$ for each f ∈ ***D*** and any ε > 0. Thus $\lim\{f(x_F)\} = r_f$ for each f ∈ ***D***; *i.e.*, $\{x_F\}$ is a ***D***-net. 2) Let $\{x_i\}$ be a ***D***-net. For each f ∈ ***D***, let $t_f = \lim\{f(x_i)\}$. Then $\cap_{f\in H} f^{-1}((t_f - \varepsilon, t_f + \varepsilon)) \neq \phi$ for any $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, any ε > 0. Let $\mathbf{V}\{x_i\} = \{\cap_{f\in H} f^{-1}((t_f - \varepsilon, t_f + \varepsilon)) | \cap_{f\in H} f^{-1}((t_f - \varepsilon, t_f + \varepsilon)) \neq \phi$ for any $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, ε > 0}, then $\mathbf{V}\{x_i\}$ is an open C_D^* -filter base on **Y**. Since t_f is unique for each f ∈ ***D***, thus $\mathbf{V}\{x_i\}$ is uniquely induced by $\{x_i\}$. □

Theorem 2.9 *Let* ***D*** ⊆ **C***(**Y**). *Then*, 1) *and* 2) *in the following are equivalent*: 1) *Every* ***D****-net converges in* **Y**. 2) *Every open* C_D^* *-filter* $\mathcal{E}$ *converges in* **Y.**

Proof. 1) ⇒ 2) is obvious by Lemma 2.8 1) above and Thm. 12.17 (a) in the Ref. [1, p. 81]. For 2) ⇒ 1): Let $\{x_i\}$ be a ***D***-net in **Y**, let $\mathcal{F}$ = {**O**|**O** is open and $\{x_i\}$ is eventually in **O**}. Clearly, $\mathcal{F}$ is an open filter. For each f in ***D***, let $t_f = \lim\{f(x_i)\}$, then $\{x_i\}$ is eventually in $f^{-1}((t_f - \varepsilon, t_f + \varepsilon))$ for any ε > 0; *i.e.*, for each f in ***D***, any ε > 0, $f^{-1}((t_f - \varepsilon, t_f + \varepsilon)) \in \mathcal{F}$, so $\mathcal{F}$ is an open C_D^* -filter. 2) implies that $\mathcal{F}$ converges to a point x. Thus, for any open nhood $\mathbf{U}_x$ of x, $\mathbf{U}_x \in \mathcal{F}$; *i.e.*, $\{x_i\}$ is eventually in $\mathbf{U}_x$. So $\{x_i\}$ converges to x. □

Corollary 2.10 *If every open* C_D^* *-filter* $\mathcal{E}$ *on* **Y** *converges in* **Y**, *then* **Y** *is compact.*

3. An Open C_D^*-Filter Process of Compactification

For each x ∈ **Y**, let $\mathbf{N}_x = \{\{x\}\} \cup \{\mathbf{O} | \mathbf{O}$ is open, x ∈ **O**}. $\mathbf{N}_x$ is a $\wp$-filter (See 12E. in the Ref. [1, p. 83] for its definition and convergence.) with $\wp = \mathbf{N}_x$. For each x ∈ **Y**, $\mathbf{N}_x$ is called a $\wp_x$*-filter*. Let $\mathbf{Y}_E = \{\mathbf{N}_x | \mathbf{N}_x$ is a $\wp_x$-filter, x ∈ **Y**}, $\mathbf{Y}_S = \{\mathcal{E} | \mathcal{E}$ is an open C_D^* -filter that does not converge in **Y**}, $\mathbf{Y}_T$ = {**Å**|**Å** is a basic open C_D^* -filter that does not converge in **Y**}, $\mathbf{Y}_S^W = \mathbf{Y}_E \cup \mathbf{Y}_S$ and $\mathbf{Y}_T^W = \mathbf{Y}_E \cup \mathbf{Y}_T$.

Lemma 3.11 *For each* $\mathcal{F} \in \mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$), *there is a unique* $r_f \in Cl(f(\mathbf{Y}))$ *for each* f ∈ ***D*** *such that* $f^{-1}(r_f - \varepsilon, r_f + \varepsilon) \in \mathbf{V}_r \subset \mathcal{F}$ *for all* ε > 0.

Proof. If $\mathcal{F} = \mathbf{N}_x$ for an x ∈ **Y**, then for each f ∈ ***D***, $f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \in \mathbf{V}_x \subset \mathbf{N}_x$ for all ε > 0, where $r_f = f(x)$. If $\mathcal{F} = \mathcal{E}$ (or **Å**), then there is an open C_D^* -filter base $\mathbf{V}_r$, as the $\mathbf{V}_r$ defined in Sec. 1, such that for each f ∈ ***D***, $f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \in \mathbf{V}_r \subset \mathcal{E}$ (or **Å**) for all ε > 0. The uniqueness of r_f for each f ∈ ***D*** follows from Cor. 2.2. □

Definition 3.12 *For each open set* **U** ≠ ϕ *in* **Y**, *define* $\mathbf{U}^* = \{\mathcal{F} \in \mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$)$| \mathbf{U} \in \mathcal{F}\}$.

Lemma 3.13 1) *For any open set* **U** *in* **Y**, **U** ≠ ϕ ⇔ **U*** ≠ ϕ; 2) **U** = **Y** ⇔ $\mathbf{U}^* = \mathbf{Y}_S^W$ (*or* $\mathbf{Y}_T^W$); *and* 3) *for any* $\mathcal{F}$ *in* $\mathbf{Y}_S^W$ (*or* $\mathbf{Y}_T^W$), *any open set* **U** ≠ ϕ *in* **Y**, $\mathcal{F} \in \mathbf{U}^* \Leftrightarrow \mathbf{U} \in \mathcal{F}$.

Proof. 1) If **U** ≠ ϕ, pick an x ∈ **U**, then $\mathbf{U} \in \mathbf{N}_x \Rightarrow \mathbf{N}_x \in \mathbf{U}^*$; *i.e.*, **U*** ≠ ϕ. If **U*** ≠ ϕ, pick a $\mathcal{F} \in \mathbf{U}^*$, then $\mathbf{U} \in \mathcal{F} \Rightarrow \mathbf{U} \neq \phi$. 2) and 3) are obvious from Def. 3.12. □

Lemma 3.14 *For any two nonempty open sets* **S** *and* **T** *in* **Y**, 1) **S** ⊆ **T** *iff* **S*** ⊆ **T***, *and* 2) (**S** ∩ **T**)* = **S*** ∩ **T***, *if* **S** ∩ **T** ≠ ϕ.

Proof. 1)**:** (⇒): $\mathcal{F} \in \mathbf{S}^* \Rightarrow \mathbf{S} \in \mathcal{F} \Rightarrow \mathbf{T} \in \mathcal{F} \Rightarrow \mathcal{F} \in \mathbf{T}^*$. (⇐)**:** **S** ⊈ **T** ⇒ there is a y ∈ (**S** − **T**) ⇒ $\mathbf{N}_y \in (\mathbf{S}^* - \mathbf{T}^*) \Rightarrow$ **S*** ⊈ **T***. 2): By 1) above, (**S** ∩ **T**)* ⊆ **S*** ∩ **T***. If $\mathcal{F} \in \mathbf{S}^* \cap \mathbf{T}^*$, then $\mathbf{S} \in \mathcal{F}$, $\mathbf{T} \in \mathcal{F}$ and $\mathbf{S} \cap \mathbf{T} \in \mathcal{F}$; *i.e.*, $\mathcal{F} \in (\mathbf{S} \cap \mathbf{T})^*$. □

Proposition 3.15 $\mathcal{B}$ = {**U***|**U** ≠ ϕ *is an open set in* **Y**} *is a base for* $\mathbf{Y}_S^W$ (*or* $\mathbf{Y}_T^W$).

Proof. For (a) in Thm. 5.3 in the Ref. [1, p. 38]: For each $\mathcal{F} \in \mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$), pick a $\mathbf{O} \in \mathcal{F}$. Then **O** ≠ ϕ, $\mathcal{F} \in \mathbf{O}^*$ and $\mathbf{O}^* \in \mathcal{B}$. Thus $\mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$) = ∪{**U***|**U*** ∈ $\mathcal{B}$}. For (b): If $\mathcal{F} \in \mathbf{S}^* \cap \mathbf{T}^*$ for **S***, **T*** ∈ $\mathcal{B}$, then **S**, **T** ∈ $\mathcal{F}$, ϕ

$\neq \mathbf{S} \cap \mathbf{T} \in \mathcal{F}$, $(\mathbf{S} \cap \mathbf{T})^* \in \mathcal{B}$ and $\mathcal{F} \in (\mathbf{S} \cap \mathbf{T})^* \subseteq \mathbf{S}^* \cap \mathbf{T}^* \in \mathcal{B}$. □

Equip $\mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$) with the topology induced by $\mathcal{B}$. For each f in $\boldsymbol{D}$, define $f^*: \mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$) $\to \mathbb{R}$ by $f^*(\mathcal{F}) = r_f$, if $f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \in \mathbf{V}_r \subset \mathcal{F}$ for all $\varepsilon > 0$. By Lemma 3.11, for each $f \in \boldsymbol{D}$, f* is well-defined and $f^*(\mathbf{Y}_S^W)$ (or $f^*(\mathbf{Y}_T^W)$) $\subseteq \mathrm{Cl}(f(\mathbf{Y}))$, thus f* is a bounded real-valued function on $\mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$) such that $f^*(\mathbf{N}_x) = f(x)$ for all $x \in \mathbf{Y}$.

Proposition 3.16 *For each* f *in* $\boldsymbol{D}$, *let* $t \in \mathrm{Cl}(f(\mathbf{Y}))$. *For any* δ, ε *with* $0 < \delta < \varepsilon$, 1) $[f^{-1}((t - \delta, t + \delta))]^* \subseteq f^{*-1}((t - \varepsilon, t + \varepsilon))$, 2) $f^{*-1}((t - \varepsilon, t + \varepsilon)) \subseteq [f^{-1}((t\text{-}\varepsilon, t\text{+}\varepsilon))]^*$.

Proof. 1): If $\mathcal{F} \in [f^{-1}((t - \delta, t + \delta))]^*$, then $f^{-1}((t - \delta, t + \delta)) \in \mathcal{F}$. If $f^*(\mathcal{F}) = e$, then $f^{-1}((e - \gamma, e + \gamma)) \in \mathcal{F}$ for all $\gamma > 0$. Since $f^{-1}((t - \delta, t + \delta) \cap (e - \gamma, e + \gamma)) = f^{-1}((t - \delta, t + \delta)) \cap f^{-1}((e - \gamma, e + \gamma)) \in \mathcal{F}$ for all $\gamma > 0$, so $(t - \delta, t + \delta) \cap (e - \gamma, e + \gamma) \neq \phi$ for all $\gamma > 0$. Thus $f^*(\mathcal{F}) = e \in [t - \delta, t+\delta] \subset (t - \varepsilon, t + \varepsilon)$; *i.e.*, $\mathcal{F} \in f^{*-1}((t - \varepsilon, t + \varepsilon))$. 2): If $\mathcal{F} \in f^{*-1}((t - \varepsilon, t + \varepsilon))$, then $f^*(\mathcal{F}) = s \in (t - \varepsilon, t + \varepsilon)$ and $f^{-1}((s - \gamma, s + \gamma)) \in \mathcal{F}$ for all $\gamma > 0$. Pick $\rho > 0$ such that $(s - \rho, s + \rho) \subset (t - \varepsilon, t + \varepsilon)$. Then $\mathbf{S} = f^{-1}((s - \rho, s + \rho)) \subset f^{-1}((t - \varepsilon, t + \varepsilon))$ and $\mathbf{S} \in \mathcal{F}$. Thus $f^{-1}((t - \varepsilon, t + \varepsilon)) \in \mathcal{F}$; *i.e.*, $\mathcal{F} \in [f^{-1}((t - \varepsilon, t + \varepsilon))]^*$. □

Proposition 3.17 *For each* $f \in \boldsymbol{D}$, f* *is a bounded real-valued continuous function on* $\mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$).

Proof. For any $\mathcal{F} \in \mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$), let $f^*(\mathcal{F}) = t$. We show that for any $\varepsilon > 0$, there is a $\mathbf{U}^* \in \mathcal{B}$ such that $\mathcal{F} \in \mathbf{U}^* \subset f^{*-1}((t - \varepsilon, t + \varepsilon))$. Let $\mathbf{U} = f^{-1}((t - \varepsilon/2, t + \varepsilon/2))$. Since $f^{-1}((t - \gamma, t + \gamma)) \in \mathcal{F}$ for all $\gamma > 0$. Thus, $\mathbf{U} = f^{-1}((t - \varepsilon/2, t + \varepsilon/2)) \in \mathcal{F}$; *i.e.*, $\mathcal{F} \in \mathbf{U}^*$. By Prop. 3.16 1), $\mathcal{F} \in \mathbf{U}^* \subset f^{*-1}((t - \varepsilon, t + \varepsilon))$. Thus f* is continuous on $\mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$). □

Lemma 3.18 *Let* k: $\mathbf{Y} \to \mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$) *be defined by* $k(x) = \mathbf{N}_x$. *Then*, 1) k *is well-defined*, *one-one and* $k^{-1}(\mathbf{U}^*) = \mathbf{U}$ *for all nonempty open set* $\mathbf{U}$ *in* $\mathbf{Y}$ *and all* $\mathbf{U}^* \in \mathcal{B}$; *i.e.*, k *is continuous*; 2) $f^* \circ k = f$ *for all* $f \in \boldsymbol{D}$; 3) $k(\mathbf{Y})$ *is dense in* $\mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$).

Proof. 1) For any x, y in $\mathbf{Y}$, $x = y \Leftrightarrow \mathbf{N}_x = \mathbf{N}_y$, thus $x \neq y \Leftrightarrow \mathbf{N}_x \neq \mathbf{N}_y$, so k is well-defined and one-one. For any $\mathbf{U}^* \in \mathcal{B}$, by Def. 3.12 and Lemma 3.13 1), $\mathbf{U}^* \neq \phi$, $\mathbf{U}$ is open, $\mathbf{U} \neq \phi$. So (a): $x \in k^{-1}(\mathbf{U}^*) \Leftrightarrow$ (b): $\mathbf{N}_x = k(x) \in \mathbf{U}^*$. By Lemma 3.13 3), (b) $\Leftrightarrow$ (c): $\mathbf{U} \in \mathbf{N}_x$. By the setting of $\mathbf{N}_x$, (c) $\Leftrightarrow$ (d): $x \in \mathbf{U}$. Thus $k^{-1}(\mathbf{U}^*) = \mathbf{U}$ for all $\mathbf{U}^* \in \mathcal{B}$, $\mathbf{U} \neq \phi$ and $\mathbf{U}$ is open in $\mathbf{Y}$; *i.e.*, k is continuous. 2) is obvious from $(f^* \circ k)(x) = f^*(\mathbf{N}_x) = f(x)$ for all x in $\mathbf{Y}$ and all f in $\boldsymbol{D}$. 3) For any $\mathbf{U}^* \in \mathcal{B}$, pick a $\mathcal{F} \in \mathbf{U}^*$, then $\mathbf{U} \in \mathcal{F}$ and $\mathbf{U} \neq \phi$. Pick an $x \in \mathbf{U}$, by 1) above, $x \in \mathbf{U} \Leftrightarrow k(x) \in \mathbf{U}^*$; *i.e.*, $k(x) \in \mathbf{U}^* \cap k(\mathbf{Y}) \neq \phi$. Hence $k(\mathbf{Y})$ is dense in $\mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$). □

Let $\boldsymbol{D}^* = \{f^* | f \in \boldsymbol{D}\}$. Then, $\boldsymbol{D}^* \subseteq \mathbf{C}^*(\mathbf{Y}_S^W)$ (or $\mathbf{C}^*(\mathbf{Y}_T^W)$). For each open $\boldsymbol{C}^*_{\boldsymbol{D}^*}$-filter $\mathcal{E}_t^*$ on $\mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$), let $\mathbf{V}_t^* = \{\cap_{f^* \in H^*} f^{*-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) | \cap_{f^* \in H^*} f^{*-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) \neq \phi$ for any $\mathbf{H}^* \in [\boldsymbol{D}^*]^{<\omega}$, $\varepsilon > 0\}$ be the open $\boldsymbol{C}^*_{\boldsymbol{D}^*}$-filter base on $\mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$) such that $\mathbf{V}_t^* \subseteq \mathcal{E}_t^*$. Since $f^* \circ k = f$, k is one-one and $k(\mathbf{Y})$ is dense in $\mathbf{Y}_S^W$ (or $\mathbf{Y}_T^W$), so $k(\cap_{f \in H} f^{-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon))) = [\cap_{f^* \in H^*} f^{*-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon))] \cap k(\mathbf{Y}) \neq \phi$ for any $\mathbf{H}^* \in [\boldsymbol{D}^*]^{<\omega}$, $\mathbf{H} = \{f \in \boldsymbol{D} | f^* \in \mathbf{H}^*\}$ and any $\varepsilon > 0$. Thus $\mathbf{V}_t = \{\cap_{f \in H} f^{-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) | \cap_{f \in H} f^{-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) \neq \phi$ for any $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, $\varepsilon > 0\}$ is a well-defined open $\boldsymbol{C}_{\boldsymbol{D}}^*$-filter base on $\mathbf{Y}$. Let $\mathcal{L}_S = \{\mathbf{U} \subseteq \mathbf{Y} \mid \mathbf{U}$ is open, $\mathbf{U} \neq \phi$ and $\mathbf{U}^* \in \mathcal{E}_t^*\}$ and $\mathcal{L}_T = \mathring{\mathbf{A}}_t$, the basic open $\boldsymbol{C}_{\boldsymbol{D}}^*$-filter generated by $\mathbf{V}_t$. Since $\mathcal{E}_t^*$ is a filter, clearly, by Lemma 3.14, $\mathcal{L}_S$ is an open filter on $\mathbf{Y}$.

Lemma 3.19 $\mathcal{L}_S$ *is an open* $\boldsymbol{C}_{\boldsymbol{D}}^*$ *-filter on* $\mathbf{Y}$.

Proof. For any $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, $\varepsilon > 0$, let $\mathbf{H}^* = \{f^* | f \in \mathbf{H}\}$, $\mathbf{O} = \cap_{f \in H} f^{-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon))$ and $\mathbf{P} = \cap_{f^* \in H^*} f^{*-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon))$. Then $\phi \neq \mathbf{P} \in \mathbf{V}_t^* \subset \mathcal{E}_t^*$. By Lemmas 3.13, 14 and Prop. 3.16 2), $\mathbf{P} \subseteq \mathbf{O}^*$, $\phi \neq \mathbf{O}^* \in \mathcal{E}_t^*$, $\mathbf{O} \neq \phi$ and $\mathbf{O} \in \mathcal{L}_S$. This implies that $\mathbf{V}_t \subseteq \mathcal{L}_S$. □

Theorem 3.20 ($\mathbf{Y}_S^W$, k) *is a compactification of* $\mathbf{Y}$.

Proof. ***Case* 1:** If $\mathcal{L}_S$ converges to a point p in $\mathbf{Y}$. Let $\mathbf{U}$ be any open set in $\mathbf{Y}$ such that $k(p) \in \mathbf{U}^* \in \mathcal{B}$. By Lemma 3.18 1), $p \in \mathbf{U} = k^{-1}(\mathbf{U}^*)$, thus $\mathbf{U} \in \mathcal{L}_S$; *i.e.*, $\mathbf{U}^* \in \mathcal{E}_t^*$. This implies that $\mathcal{E}_t^*$ converges to k(p) in $\mathbf{Y}_S^W$. ***Case* 2:** If $\mathcal{L}_S$ does not converge in $\mathbf{Y}$, then $\mathcal{L}_S \in \mathbf{Y}_S^W$. For any $\mathbf{U}^*$ in $\mathcal{B}$ such that $\mathcal{L}_S \in \mathbf{U}^*$, $\mathbf{U} \in \mathcal{L}_S$ and therefore $\mathbf{U}^* \in \mathcal{E}_t^*$. This shows that $\mathcal{E}_t^*$ converges to $\mathcal{L}_S$ in $\mathbf{Y}_S^W$. By Cor. 2.10, $\mathbf{Y}_S^W$ is compact and by Lemma 3.18 3), ($\mathbf{Y}_S^W$, k) is a compactification of $\mathbf{Y}$. □

Lemma 3.21 *For each open set* $\mathbf{U} \in \mathcal{L}_T = \mathring{\mathbf{A}}_t$, $\mathbf{U}^* \in \mathcal{E}_t^*$.

Proof. If $\mathbf{U} \in \mathring{\mathbf{A}}_t$, then there exist a $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, an $\varepsilon > 0$ such that $\mathbf{E} = \cap_{f \in H} f^{-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) \in \mathbf{V}_t$ and $\mathbf{E} \subseteq \mathbf{U}$. Lemma 3.14 and Prop. 3.16 2) imply that $\mathbf{F} = \cap_{f^* \in H^*} f^{*-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) \subseteq \mathbf{E}^* \subseteq \mathbf{U}^*$ and $\mathbf{F} \in \mathcal{E}_t^*$. Thus, $\mathbf{U}^* \in \mathcal{E}_t^*$. □

Theorem 3.22 ($\mathbf{Y}_T^W$, k) *is a compactification of* $\mathbf{Y}$.

Proof. *Case* 1: If $\mathcal{L}_T = \mathring{\mathbf{A}}_t$ converges to a point p in $\mathbf{Y}$, let $\mathbf{U}$ be any open set in $\mathbf{Y}$ such that $k(p) \in \mathbf{U}^*$, Lemma 3.18 1) implies that $p \in \mathbf{U}$, thus $\mathbf{U} \in \mathcal{L}_T = \mathring{\mathbf{A}}_t$. So by Lemma 3.21, $\mathbf{U}^* \in \mathcal{E}_t^*$. This implies that $\mathcal{E}_t^*$ converges to k(p) in $\mathbf{Y}_T^W$. *Case* 2: If $\mathcal{L}_T = \mathring{\mathbf{A}}_t$ does not converge in $\mathbf{Y}$, then $\mathcal{L}_T = \mathring{\mathbf{A}}_t \in \mathbf{Y}_T^W$. For any $\mathbf{U}^* \in \mathcal{B}$ such that $\mathring{\mathbf{A}}_t \in \mathbf{U}^*$, $\mathbf{U} \in \mathring{\mathbf{A}}_t$ and by Lemmas 3.21, $\mathbf{U}^* \in \mathcal{E}_t^*$. Thus $\mathcal{E}_t^*$ converges to $\mathcal{L}_T = \mathring{\mathbf{A}}_t$ in $\mathbf{Y}_T^W$. Cor.2.10 implies that $\mathbf{Y}_T^W$ is compact and by Lemma 3.18 3), ($\mathbf{Y}_T^W$, k) is a compactification of $\mathbf{Y}$. □

4. An Arbitrary Hausdorff Compactification of a Tychonoff Space

For an arbitrary Hausdorff compactification ($\mathbf{Z}$, h) of a Tychonoff space $\mathbf{X}$, let $\boldsymbol{D} = \{f | f = {}^\circ f \circ h, {}^\circ f \in {}^\circ\boldsymbol{D} = \mathbf{C}(\mathbf{Z})\}$. Then $\boldsymbol{D} \subseteq \mathbf{C}^*(\mathbf{X})$, $\boldsymbol{D}$ separates points of $\mathbf{X}$ and the topology on $\mathbf{X}$ is the weak topology induced by $\boldsymbol{D}$. For each $x \in \mathbf{X}$, let $\mathbf{V}_x$, as the $\mathbf{V}_x$ defined in Section 2, be the open $\boldsymbol{C}_{\boldsymbol{D}}^*$-filter base at x induced by $\boldsymbol{D}$. Obviously, we can easily get Lemma 4.21 as follows**:**

Lemma 4.21 $\mathbf{G}_{\boldsymbol{D}} = \cup\{\mathbf{V}_x | x \in \mathbf{X}\}$ *is a base for the topology on* $\mathbf{X}$ *and for each* $x \in \mathbf{X}$, $\mathbf{V}_x$ *is an open nhood base at x.*

Let $\mathbf{X}^{\mathbf{W}} = \{\text{Å}|\text{Å}$ is a basic open $C^*_{\boldsymbol{D}}$-filter on $\mathbf{X}\}$. For each $\text{Å}_r \in \mathbf{X}^{\mathbf{W}}$, let $\mathbf{V}_r$, as the $\mathbf{V}_r$ defined in Sec. 1, be the open $C^*_{\boldsymbol{D}}$-filter base that generates Å_r. If Å_r converges to an $x \in \mathbf{X}$, then for each $f \in \boldsymbol{D}$, $x \in \text{Cl}(f^{-1}((r_f - \varepsilon/2, r_f + \varepsilon/2)) \subseteq f^{-1}([r_f - \varepsilon/2, r_f + \varepsilon/2]) \subset f^{-1}((r_f - \varepsilon, r_f + \varepsilon))$ for all $\varepsilon > 0$; *i.e.*, $r_f = f(x)$ for all $f \in \boldsymbol{D}$, so $\mathbf{V}_r = \mathbf{V}_x$ and $\text{Å}_r = \text{Å}_x$. Thus $\mathbf{X}^{\mathbf{W}} = \mathbf{X}_E \cup \mathbf{X}_F$ and $\mathbf{X}_E \cap \mathbf{X}_F = \phi$, where $\mathbf{X}_E = \{\text{Å}_x|x \in \mathbf{X}\}$ and $\mathbf{X}_F = \{\text{Å}|\text{Å}$ is a basic open $C^*_{\boldsymbol{D}}$-filter that does not converge in $\mathbf{X}\}$. Similar to what we have done in Section 3, we can get the similar definitions and results for $\mathbf{X}^{\mathbf{W}}$ in the following:

4.22-1. For each open set $\mathbf{U} \neq \phi$ in $\mathbf{X}$, define $\mathbf{U}^* = \{\text{Å} \in \mathbf{X}^{\mathbf{W}}|\mathbf{U} \in \text{Å}\}$.

4.22-2. 1) for any open set $\mathbf{U}$ in $\mathbf{X}$, $\mathbf{U} \neq \phi \Leftrightarrow \mathbf{U}^* \neq \phi$; 2) $\mathbf{U} = \mathbf{X} \Leftrightarrow \mathbf{U}^* = \mathbf{X}^{\mathbf{W}}$; and (c) for any Å in $\mathbf{X}^{\mathbf{W}}$, any open set $\mathbf{U} \neq \phi$, $\text{Å} \in \mathbf{U}^* \Leftrightarrow \mathbf{U} \in \text{Å}$.

4.22-3. For any two nonempty open sets $\mathbf{S}$ and $\mathbf{T}$ in $\mathbf{X}$, 1) $\mathbf{S} \subseteq \mathbf{T}$ iff $\mathbf{S}^* \subseteq \mathbf{T}^*$, and 2) $(\mathbf{S} \cap \mathbf{T})^* = \mathbf{S}^* \cap \mathbf{T}^*$, if $\mathbf{S} \cap \mathbf{T} \neq \phi$.

4.22-4. $\mathcal{B} = \{\mathbf{U}^*|\mathbf{U} \neq \phi, \mathbf{U}$ is an open set in $\mathbf{X}\}$ is a base for a topology on $\mathbf{X}$.

4.22-5. For each $f \in \boldsymbol{D}$, $f^*: \mathbf{X}^{\mathbf{W}} \to \mathbb{R}$ is defined by $f^*(\text{Å}_r) = r_f$, if $f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \in \mathbf{V}_r \subset \text{Å}_r$ for all $\varepsilon > 0$. Then $f^*(\text{Å}_x) = f(x)$ for all $x \in \mathbf{X}$.

4.22-6. For each f in $\boldsymbol{D}$, let $t \in \text{Cl}(f(\mathbf{X}))$. For any δ, ε with $0 < \delta < \varepsilon$, 1) $[f^{-1}((t - \delta, t + \delta))]^* \subseteq f^{*-1}((t - \varepsilon, t + \varepsilon))$, 2) $f^{*-1}((t - \varepsilon, t + \varepsilon)) \subseteq [f^{-1}((t - \varepsilon, t + \varepsilon))]^*$.

4.22-7. For each f in $\boldsymbol{D}$, f* is a bounded real-valued continuous function on $\mathbf{X}^{\mathbf{W}}$.

4.22-8. Define $k: \mathbf{X} \to \mathbf{X}^{\mathbf{W}}$ by $k(x) = \text{Å}_x$, then 1) k is well-defined, one-one, and $\mathbf{U} = k^{-1}(\mathbf{U}^*)$ for all open set $\mathbf{U} \neq \phi$ in $\mathbf{X}$ and all $\mathbf{U}^* \in \mathcal{B}$; *i.e.*, k is continuous, 2) $f^* \text{ o } k = f$ for all f in $\boldsymbol{D}$ and 3) $k(\mathbf{X})$ is dense in $\mathbf{X}^{\mathbf{W}}$.

4.22-9. Let $\boldsymbol{D}^* = \{f^*|f \in \boldsymbol{D}\}$. Then $\boldsymbol{D}^* \subseteq \mathbf{C}^*(\mathbf{X}^{\mathbf{W}})$.

Lemma 4.23 $\boldsymbol{D}^*$ *separates points of* $\mathbf{X}^{\mathbf{W}}$.

Proof. For $\text{Å}_s, \text{Å}_t \in \mathbf{X}^{\mathbf{W}}$, let $\mathbf{V}_s = \{\cap_{f\in\mathbf{H}} f^{-1}((s_f - \varepsilon, s_f + \varepsilon))|\cap_{f\in\mathbf{H}} f^{-1}((s_f - \varepsilon, s_f + \varepsilon)) \neq \phi$ for any $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}, \varepsilon > 0\}$ be the open $C^*_{\boldsymbol{D}}$-filter base that generates Å_s and similarly for $\mathbf{V}_t$. Since $\text{Å}_s = \text{Å}_t$, $\mathbf{V}_s = \mathbf{V}_t$ and that $s_f = t_f$ for all f in $\boldsymbol{D}$ are equivalent, thus $\text{Å}_s \neq \text{Å}_t$, $\mathbf{V}_s \neq \mathbf{V}_t$ and that there is a g in $\boldsymbol{D}$ such that $s_g \neq t_g$ are equivalent. So, if $\text{Å}_s \neq \text{Å}_t$, then $g^*(\text{Å}_s) = s_g \neq t_g = g^*(\text{Å}_t)$ for some $g^* \in \boldsymbol{D}^*$. □

Lemma 4.24 *The topology on* $\mathbf{X}^{\mathbf{W}}$ *is the weak topology induced by* $\boldsymbol{D}^*$.

Proof. For each $\text{Å}_r \in \mathbf{X}^{\mathbf{W}}$, let $\mathbf{V}_r$, as the $\mathbf{V}_r$ defined in Sec. 1, be the open $C^*_{\boldsymbol{D}}$-filter base that generates Å_r and let $\mathbf{U}^* \in \mathcal{B}$ such that $\text{Å}_r \in \mathbf{U}^*$, then $\mathbf{U} \in \text{Å}_r$. So there exist a $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, an $\varepsilon > 0$ such that $\cap_{f\in\mathbf{H}} f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \subset \mathbf{U}$, where $\cap_{f\in\mathbf{H}} f^{-1}((r_f - \delta, r_f + \delta)) \in \mathbf{V}_r \subset \text{Å}_r$ for all $\delta > 0$. By 4.22-2 (c), 4.22-3 and 4.22-6 2), $\text{Å}_r \in [\cap_{f\in\mathbf{H}} f^{-1}((r_f - \varepsilon/2, r_f + \varepsilon/2))]^* \subset \cap_{f^*\in\mathbf{H}^*} f^{*-1}((r_f - \varepsilon, r_f + \varepsilon)) \subset [\cap_{f\in\mathbf{H}} f^{-1}((r_f - \varepsilon, r_f + \varepsilon))]^* \subset \mathbf{U}^*$; *i.e.*, $\text{Å}_r \in \cap_{f^*\in\mathbf{H}^*} f^{*-1}((r_f - \varepsilon, r_f + \varepsilon)) \subset \mathbf{U}^*$. □

For any open $C^*_{\boldsymbol{D}^*}$-filter $\mathcal{E}^*_t$ on $\mathbf{X}^{\mathbf{W}}$, let $\mathbf{V}^*_t = \{\cap_{f^*\in\mathbf{H}^*} f^{*-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon))|\cap_{f^*\in\mathbf{H}^*} f^{*-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) \neq \phi$ for any $\mathbf{H}^* \in [\boldsymbol{D}^*]^{<\omega}, \varepsilon > 0\}$ be the open $C^*_{\boldsymbol{D}^*}$-filter base that is contained in $\mathcal{E}^*_t$. Since $f^* \text{ o } k = f$ for all $f \in \boldsymbol{D}$, k is one-one and $k(\mathbf{X})$ is dense in $\mathbf{X}^{\mathbf{W}}$, so $k(\cap_{f\in\mathbf{H}} f^{-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon))) = \cap_{f^*\in\mathbf{H}^*} f^{*-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) \cap k(\mathbf{X}) \neq \phi$ for any $\mathbf{H}^* \in [\boldsymbol{D}^*]^{<\omega}$, $\mathbf{H} = \{f \in \boldsymbol{D}|f^* \in \mathbf{H}^*\}$) and any $\varepsilon > 0$. Thus $\mathbf{V}_t = \{\cap_{f\in\mathbf{H}} f^{-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon))|\cap_{f\in\mathbf{H}} f^{-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) \neq \phi$ for any $\mathbf{H} \in [\mathbf{D}]^{<\omega}, \varepsilon > 0\}$ is a well-defined open $C^*_{\boldsymbol{D}}$-filter base on $\mathbf{X}$. Let Å_t be the basic open $C^*_{\boldsymbol{D}}$-filter on $\mathbf{X}$ generated by $\mathbf{V}_t$.

Lemma 4.25 *For any open set* $\mathbf{U} \in \text{Å}_t$, $\mathbf{U}^* \in \mathcal{E}^*_t$.

Proof. For any $\mathbf{U} \in \text{Å}_t$, there exist a $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, an $\varepsilon > 0$ such that $\cap_{f\in\mathbf{H}} f^{-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) = \mathbf{S} \in \mathbf{V}_t$ and $\mathbf{S} \subseteq \mathbf{U}$. By 4.22-3 and 4.22-6, $\mathbf{T} = \cap_{f^*\in\mathbf{H}^*} f^{*-1}((t_{f^*} - \varepsilon, t_{f^*} + \varepsilon)) \subseteq \mathbf{S}^* \subseteq \mathbf{U}^*$ and $\mathbf{T} \in \mathbf{V}^*_t$. Thus $\mathbf{U}^* \in \mathcal{E}^*_t$.

Theorem 4.26 $(\mathbf{X}^{\mathbf{W}}, k)$ *is a Hausdorff compactification of* $\mathbf{X}$.

Proof. We show that the open $C^*_{\boldsymbol{D}^*}$-filter $\mathcal{E}^*_t$ converges to Å_t in $\mathbf{X}^{\mathbf{W}}$. For any open set $\mathbf{U}$ in $\mathbf{X}$ such that $\text{Å}_t \in \mathbf{U}^*$, by 4.22-2 (c), $\mathbf{U} \in \text{Å}_t$, by Lemma 4.25, $\mathbf{U}^* \in \mathcal{E}^*_t$. This implies that $\mathcal{E}^*_t$ converges to Å_t in $\mathbf{X}^{\mathbf{W}}$. By Cor. 2.10, $\mathbf{X}^{\mathbf{W}}$ is compact. Thus, by 4.22-8 3) and Lemma 4.23, $(\mathbf{X}^{\mathbf{W}}, k)$ is a Hausdorff compactification of $\mathbf{X}$. □

5. The Homeomorphism between $(\mathbf{X}^{\mathbf{W}}, \mathbf{k})$ and $(\mathbf{Z}, \mathbf{h})$

For each basic open $C^*_{\boldsymbol{D}}$-filter $\text{Å}_r \in \mathbf{X}^{\mathbf{W}}$, let $\mathbf{V}_r$, as the $\mathbf{V}_r$ defined in Sec. 1, be the open $C^*_{\boldsymbol{D}}$-filter base that generates Å_r. Since $h^{-1}: h(\mathbf{X}) \to \mathbf{X}$ is one-one, $f = {}^\circ f \text{ o } h$ and $h(\mathbf{X})$ is dense in $\mathbf{Z}$, so $h^{-1}(\cap_{{}^\circ f\in{}^\circ\mathbf{H}} {}^\circ f^{-1}((r_f - \varepsilon, r_f + \varepsilon))) = \cap_{f\in\mathbf{H}} f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \neq \phi$ for any ${}^\circ\mathbf{H} \in [{}^\circ\boldsymbol{D}]^{<\omega}$, $\mathbf{H} = \{f|{}^\circ f \in {}^\circ\mathbf{H}\}$ and any $\varepsilon > 0$. Thus, ${}^\circ\mathbf{V}_r = \{\cap_{{}^\circ f\in{}^\circ\mathbf{H}} {}^\circ f^{-1}((r_f - \varepsilon, r_f + \varepsilon))|\cap_{{}^\circ f\in{}^\circ\mathbf{H}} {}^\circ f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \neq \phi$ for any ${}^\circ\mathbf{H} \in [{}^\circ\boldsymbol{D}]^{<\omega}, \varepsilon > 0\}$ is a well-defined open $C^*_{{}^\circ\boldsymbol{D}}$-filter base on $\mathbf{Z}$. Let ${}^\circ\text{Å}_r$ be the basic open $C^*_{{}^\circ\boldsymbol{D}}$-filter on $\mathbf{Z}$ generated by ${}^\circ\mathbf{V}_r$. Since $\mathbf{Z}$ is compact, ${}^\circ\text{Å}_r$ clusters at a $z_r \in \mathbf{Z}$. For each ${}^\circ f \in {}^\circ\boldsymbol{D}$, $z_r \in \text{Cl}({}^\circ f^{-1}((r_f - \varepsilon/2, r_f + \varepsilon/2))) \subseteq {}^\circ f^{-1}([r_f - \varepsilon/2, r_f + \varepsilon/2]) \subset {}^\circ f^{-1}((r_f - \varepsilon, r_f + \varepsilon)) \in {}^\circ\mathbf{V}_r$ for all $\varepsilon > 0$; *i.e.*, ${}^\circ f(z_r) = r_f$ for all ${}^\circ f \in {}^\circ\boldsymbol{D}$. So ${}^\circ\mathbf{V}_r = {}^\circ\mathbf{V}z_r$ and ${}^\circ\text{Å}_r = {}^\circ\text{Å}z_r$. The z_r is called *the w-point in* Z *induced by* Å_r such that ${}^\circ f(z_r) = r_f = f^*(\text{Å}_r)$ for all ${}^\circ f \in {}^\circ\boldsymbol{D}$ and $f^* \in \boldsymbol{D}^*$. ${}^\circ\mathbf{V}z_r$ and ${}^\circ\text{Å}z_r$ are called the *open* $C^*_{{}^\circ\boldsymbol{D}}$*-filter base* and the *basic open* $C^*_{{}^\circ\boldsymbol{D}}$*-filter at* z_r *in* Z *induced by* $\mathbf{V}_r$ *or* Å_r, If $z_s \neq z_r$ in $\mathbf{Z}$, there is a ${}^\circ f \in {}^\circ\boldsymbol{D}$ such that ${}^\circ f(z_s) \neq {}^\circ f(z_r) = r_f = f^*(\text{Å}_r)$, so z_r is the unique w-point in $\mathbf{Z}$ induced by Å_r. If $\text{Å}_t \neq \text{Å}_r$, let z_t be the w-point in $\mathbf{Z}$ induced by Å_t. By Lemma 4.23, there is a $g^* \in \boldsymbol{D}^*$ such that ${}^\circ g(z_t) = g^*(\text{Å}_t) \neq g^*(\text{Å}_r) = {}^\circ g(z_r)$; *i.e.*, $z_t \neq z_r$. So, if $\mathcal{H}: \mathbf{X}^{\mathbf{W}} \to \mathbf{Z}$ is defined by $\mathcal{H}(\text{Å}_r) = z_r$, where z_r is the w-point in $\mathbf{Z}$ induced by Å_r, then $\mathcal{H}$ is well-defined and one-one. For any $z \in \mathbf{Z}$, let ${}^\circ\text{Å}_z$ be the basic open $C^*_{{}^\circ\boldsymbol{D}}$-filter at $z \in \mathbf{Z}$ generated by ${}^\circ\mathbf{V}_z = \{\cap_{{}^\circ f\in{}^\circ\mathbf{H}} {}^\circ f^{-1}(({}^\circ f(z) - \varepsilon, {}^\circ f(z) + \varepsilon))|{}^\circ\mathbf{H} \in [{}^\circ\boldsymbol{D}]^{<\omega}, \varepsilon > 0\}$. Since h is one-one, $f = {}^\circ f \text{ o } h$ and $h(\mathbf{X})$ is dense in $\mathbf{Z}$, so $h(\cap_{f\in\mathbf{H}} f^{-1}(({}^\circ f(z) - \varepsilon, {}^\circ f(z) + \varepsilon))) = \cap_{{}^\circ f\in{}^\circ\mathbf{H}} {}^\circ f^{-1}$

$((^\circ f(z) - \varepsilon, ^\circ f(z) + \varepsilon)) \cap h(\mathbf{X}) \neq \phi$ for any $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}$, $^\circ\mathbf{H} = \{^\circ f | f \in \mathbf{H}\}$, $\varepsilon > 0$. Thus $\mathbf{V}_z = \{\cap_{f\in H} f^{-1}((^\circ f(z) - \varepsilon, ^\circ f(z) + \varepsilon)) | \cap_{f\in H} f^{-1}((^\circ f(z) - \varepsilon, ^\circ f(z) + \varepsilon)) \neq \phi$ for any $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}, \varepsilon > 0\}$ is a well-defined open C_D^*-filter base on $\mathbf{X}$. Let $\mathring{\mathbf{A}}_z$ be the basic open C_D^*-filter on $\mathbf{X}$ generated by $\mathbf{V}_z$. If z_o is the w-point in $\mathbf{Z}$ induced by $\mathring{\mathbf{A}}_z$. Then $^\circ f(z_o) = {}^\circ f(z) = f^*(\mathring{A}_z)$ for all $^\circ f \in {}^\circ\boldsymbol{D}$ and $f^* \in \boldsymbol{D}^*$. This implies that $z = z_o$ in $\mathbf{Z}$. So, for any $z \in \mathbf{Z}$, there is a unique $\mathring{\mathbf{A}}_z$ in $\mathbf{X}^\mathbf{W}$ such that $\mathcal{H}(\mathring{\mathbf{A}}_z) = z$. Hence, $\mathcal{H}$ is well-defined, one-one and onto.

Theorem 5.27 $(\mathbf{X}^\mathbf{W}, k)$ *is homeomorphic to* $(\mathbf{Z}, h)$ *under the mapping* $\mathcal{H}$ *such that* $\mathcal{H}(k(x)) = h(x)$.

Proof. Since the topologies on $\mathbf{Z}$ and $\mathbf{X}^\mathbf{W}$ are the weak topologies induced by $^\circ\boldsymbol{D}$ and $\boldsymbol{D}^*$, respectively, to show the continuity of $\mathcal{H}$, it is enough to show that for any $^\circ f \in {}^\circ\boldsymbol{D}$ (or $f^* \in \boldsymbol{D}^*$), any $\varepsilon > 0$, $\mathcal{H}^{-1}(^\circ f^{-1}((t_f - \varepsilon, t_f + \varepsilon))) = f^{*-1}((t_f - \varepsilon, t_f + \varepsilon))$. For each $\mathring{\mathbf{A}}_s$ in $\mathbf{X}^\mathbf{W}$, let $\mathbf{V}_s = \{\cap_{f\in H} f^{-1}((s_f - \varepsilon, s_f + \varepsilon)) | \cap_{f\in H} f^{-1}((s_f - \varepsilon, s_f + \varepsilon)) \neq \phi$ for any $\mathbf{H} \in [\boldsymbol{D}]^{<\omega}, \varepsilon > 0\}$ be the open C_D^*-filter base on $\mathbf{X}$ that generates $\mathring{\mathbf{A}}_s$. Let z_s be the w-point in $\mathbf{Z}$ induced by $\mathring{\mathbf{A}}_s$, then $^\circ f(z_s) = s_f = f^*(\mathring{\mathbf{A}}_s)$. Thus (a): $[\mathring{\mathbf{A}}_s \in f^{*-1}((t_f - \varepsilon, t_f + \varepsilon))]$ iff (b): $[^\circ f(z_s) = f^*(\mathring{\mathbf{A}}_s) = s_f \in (t_f - \varepsilon, t_f + \varepsilon)]$. Since $\mathcal{H}(\mathring{\mathbf{A}}_s) = z_s$, so (b) iff (c): $[\mathcal{H}(\mathring{\mathbf{A}}_s) = z_s \in {}^\circ f^{-1}((t_f - \varepsilon, t_f + \varepsilon))]$ and (c) iff (d): $[\mathring{\mathbf{A}}_s \in \mathcal{H}^{-1}(^\circ f^{-1}((t_f - \varepsilon, t_f + \varepsilon)))]$; *i.e.*, $f^{*-1}((t_f - \varepsilon, t_f + \varepsilon)) = \mathcal{H}^{-1}(^\circ f^{-1}((t_f - \varepsilon, t_f + \varepsilon)))$. So, $\mathcal{H}$ is continuous. Since $\mathcal{H}$ is one-one, onto and $\mathbf{Z}$, $\mathbf{X}^\mathbf{W}$ are compact Hausdorff, by Theorem 17.14 in the Ref. [1, p. 123], $\mathcal{H}$ is a homeomorphism. For that $\mathcal{H}(k(x)) = h(x)$ is obvious from the definitions of k and h. □

Corollary 5.28 *Let* $(\beta\mathbf{X}, h)$ *be the Stone-Čech compactification of a Tychonoff space* $\mathbf{X}$, $\boldsymbol{D} = \{f | f = {}^\circ f \circ h, {}^\circ f \in \mathbf{C}(\beta\mathbf{X})\}$ *and* $\mathcal{H}_\beta: \mathbf{X}^\mathbf{W} \to \beta\mathbf{X}$ *is defined similarly to* $\mathcal{H}$ *as above. Then* $(\beta\mathbf{X}, h)$ *is homeomorphic to* $(\mathbf{X}^\mathbf{W}, k)$ *such that* $\mathcal{H}_\beta(k(x)) = h(x)$.

Corollary 5.29 *Let* $(\gamma\mathbf{X}, h)$ *be the Wallman compactification of a normal* $\mathbf{T}_1$*-space* $\mathbf{X}$, $\boldsymbol{D} = \{f | f = {}^\circ f \circ h, {}^\circ f \in \mathbf{C}(\gamma\mathbf{X})\}$ and $\mathcal{H}_\gamma: \mathbf{X}^\mathbf{W} \to \gamma\mathbf{X}$ is defined similarly to $\mathcal{H}$ as above. Then $(\gamma\mathbf{X}, h)$ *is homeomorphic to* $(\mathbf{X}^\mathbf{W}, k)$ *such that* $\mathcal{H}_\gamma(k(x)) = h(x)$.

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