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A Family of Non-Self Maps Satisfying ϕ_i-Contractive Condition and Having Unique Common Fixed Point in Metrically Convex Spaces*

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ABSTRACT

Class of 5-dimensional functions Φ was introduced and a convergent sequence determined by non-self mappings satisfying certain ϕ_i-contractive condition was constructed, and then that the limit of the sequence is the unique common fixed point of the mappings was proved. Finally, several more general forms were given. Our main results generalize and unify many same type fixed point theorems in references.

Keywords: Metrically Convex Space; 5-Dimensional Functions Φ; ϕ_i-Contractive Condition; Common Fixed Point; Complete

1. Introduction

There have appeared many fixed point theorems for single-valued self-map of closed subset of Banach space. However, in many applications, the mapping under considerations is a not self-mapping of closed sets. 1976, Assad [1] gave sufficient condition for such single-valued mapping to obtain a fixed point by proving a fixed point theorem for Kannan mappings on a Banach space and putting certain boundary conditions on the mapping. Similar results for multi-valued mappings were respecttively given by Assad [2] and Assad and Kirk [3]. Later, some authors generalized the same type results on complete metrically convex metric spaces, see [4-8]. Those above results were discussed under some contractive conditions or certain boundary condition. Recently, the author discussed unique common fixed point theorems for a family of contractive or quasi-contractive type mappings on metrically convex spaces or 2-metric spaces, see [9-13], these results improve many known common fixed point theorems. In order to generalize and unify further these results, in this note, we shall discuss and obtain some unique common fixed point theorems for a family of more general non-self maps satisfying ϕ-contractive condition on closed subset of a complete metrically convex metric space.

We need the following definition and lemma in the sequel.

Definition 1.1. ([4,5]) A metric space (X,d) is said to be metrically convex if, for any $x,y\in X$, with $x\neq y$, there exists $z\in X$, $z\neq x$, $z\neq y$ such that $d(x,z)+d(z,y)=d(x,y)$.

Lemma 1.2. ([3,4]) If K is a nonempty closed subset of a complete metrically convex metric space (X,d), then for any $x\in K$ and $y\notin K$, there exists $z\in\partial K$ such that $d(x,z)+d(z,y)=d(x,y)$.

Let Φ denotes a family of mappings such that each $\phi\in\Phi$, $\phi:(\mathbb{R}^+)^5\to\mathbb{R}^+$ is continuous and increasing in each coordinate variable, and also $\lambda(t)=\phi(t,t,2t,t,t)<t$ for every $t>0$, where $t\in\mathbb{R}^+=[0,\infty)$.

Obviously, $\phi(0,0,0,0,0)=0$.

There exist many functions ϕ which belongs to Φ:

Example 1.3. Let $\phi:(\mathbb{R}^+)^5\to\mathbb{R}^+$ be defined by

$$\phi(t_1,t_2,t_3,t_4,t_5)=\frac{1}{7}(t_1+t_2+t_3+t_4+t_5).$$

Then obviously, $\phi\in\Phi$

Example 1.4. Let $\phi:(\mathbb{R}^+)^5\to\mathbb{R}^+$ be defined by

$$\phi(t_1,t_2,t_3,t_4,t_5)=\frac{1}{7}\ln^{(1+t_1)(1+t_2)(1+t_3)(1+t_4)(1+t_5)}.$$

Then obviously, ϕ is continuous and increasing in each coordinate variable, and

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$$\lambda(t)=\phi(t,t,2t,t,t)=\frac{1}{7}\ln^{(1+t)(1+t)(1+2t)(1+t)(1+t)}$$
$$<\frac{6}{7}\ln^{(1+t)}<\frac{6}{7}t<t.$$

Hence $\phi\in\Phi$.

Example 1.5. Let $\phi:(\mathbb{R}^+)^5\to\mathbb{R}^+$ be defined by

$$\phi(t_1,t_2,t_3,t_4,t_5)=\frac{1}{7}(\arctan t_1+\arctan t_2+\arctan t_3$$
$$+\arctan t_4+\arctan t_5).$$

Then obviously, ϕ is continuous and increasing in each coordinate variable, and

$$\lambda(t)=\phi(t,t,2t,t,t)$$
$$=\frac{1}{7}(\arctan t+\arctan t+\arctan 2t+\arctan t+\arctan t)$$
$$=\frac{1}{7}(4\arctan t+\arctan 2t)<\frac{1}{7}(4t+2t)<t.$$

Hence $\phi\in\Phi$.

2. Unique Common Fixed Point

Here, we will discuss unique existence problems of common fixed points for a family of non-self mappings satisfying certain ϕ_i-contractive condition and certain boundary condition in complete metrically convex spaces.

Theorem 2.1. Let *K* be a nonempty closed subset of a complete metrically convex metric space (X,d), and $\{T_i:K\to X\}_{i\in\mathbb{N}}$ a family of non-self maps such that for each $x,y\in K$ and $i,j\in\mathbb{N}$ with $i<j$,

$$d(T_ix,T_jy)\le q\phi_i\big(d(x,T_ix),d(y,T_jy),d(x,T_jy),d(y,T_ix),d(x,y)\big),\quad(1)$$

where $0<q<\frac{1}{2}$ and $\phi_i\in\Phi$ for each $i\in\mathbb{N}$. Further, if $T_i(\partial K)\subset K$ for each $i\in\mathbb{N}$, then $\{T_i\}_{i\in\mathbb{N}}$ has an unique common fixed point in *K*.

Proof. Take an $x_0\in K$. We will construct two sequences $\{x_n\}$ and $\{x'_n\}$ in the following manner:

Define $x'_1=T_1x_0$. If $x'_1\in K$, put $x_1=x'_1$; if $x'_1\notin K$, then by Lemma 1.2 there exists $x_1\in\partial K$ such that $d(x_0,x_1)+d(x_1,x'_1)=d(x_0,x'_1)$. Define $x'_2=T_2x_1$. If $x'_2\in K$, put $x_2=x'_2$; if $x'_2\notin K$, then by Lemma 1.2 there exists $x_2\in\partial K$ such that $d(x_1,x_2)+d(x_2,x'_2)=d(x_1,x'_2)$. Continuing in this way, we obtain $\{x_n\}$ and $\{x'_n\}$:

i) $x'_n=T_nx_{n-1}$;

ii) if $x'_n\in K$, then $x_n=x'_n$;

iii) if $x'_n\notin K$, then there exists $x_n\in\partial K$ such that $d(x_{n-1},x_n)+d(x_n,x'_n)=d(x_{n-1},x'_n)$ by Lemma 1.2.

Let $P=\{x_i\in\{x_n\}:x_i=x'_i\}$ and $Q=\{x_i\in\{x_n\}:x_i\ne x'_i\}$, then since $T_i(\partial K)\subset K$ for all $i\in\mathbb{N}$, it is easy to show that $x_n\in Q$ implies $x_{n-1},x_{n+1}\in P$ (2)

Now, we wish to estimate $d(x_n,x_{n+1})$. We can divide the proof into three cases in view of (2).

Case I. $x_n,x_{n+1}\in P$, we have

$$\begin{aligned}
&d(x_n,x_{n+1})=d(x_{n'},x'_{n+1})=d(T_nx_{n-1},T_{n+1}x_n)\\
&\le q\phi_n\big(d(x_{n-1},T_nx_{n-1}),d(x_n,T_{n+1}x_n),\\
&\quad d(x_{n-1},T_{n+1}x_n),d(x_n,T_nx_{n-1}),d(x_{n-1},x_n)\big)\\
&=q\phi_n\big(d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_{n-1},x_{n+1}),0,d(x_{n-1},x_n)\big\}\\
&\le q\phi_n\big(d(x_{n-1},x_n),d(x_n,x_{n+1}),[d(x_{n-1},x_n)+d(x_n,x_{n+1})],\\
&\quad 0,d(x_{n-1},x_n)\big).
\end{aligned}\quad(3)$$

If $d(x_{n-1},x_n)<d(x_n,x_{n+1})$, then (3) becomes

$$\begin{aligned}
d(x_n,x_{n+1})&\le q\phi_n\big(d(x_n,x_{n+1}),d(x_n,x_{n+1})\big\},\\
&\quad 2d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_n,x_{n+1})\big)\\
&<qd(x_n,x_{n+1}).
\end{aligned}$$

Which is a contradiction since $q<\frac{1}{2}$, hence we have that $d(x_n,x_{n+1})\le d(x_{n-1},x_n)$. In this case, (3) further becomes the following

$$\begin{aligned}
d(x_n,x_{n+1})&\le q\phi_n\big(d(x_{n-1},x_n),d(x_{n-1},x_n)\big\},\\
&\quad 2d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_{n-1},x_n)\big)\quad(4)\\
&\le qd(x_{n-1},x_n).
\end{aligned}$$

Case II. $x_n\in P$, $x_{n+1}\in Q$, then by iii) and (2), we have

$$\begin{aligned}
&d(x_n,x_{n+1})\le d(x_n,x_{n+1})+d(x_{n+1},x'_{n+1})\\
&=d(x_n,x'_{n+1})=d(x'_n,x'_{n+1})=d(T_nx_{n-1},T_{n+1}x_n)\\
&\le q\phi_n\big(d(x_{n-1},T_nx_{n-1}),d(x_n,T_{n+1}x_n),\\
&\quad d(x_{n-1},T_{n+1}x_n),d(x_n,T_nx_{n-1}),d(x_{n-1},x_n)\big)\\
&=q\phi_n\big(d(x_{n-1},x_n),d(x_n,x'_{n+1}),d(x_{n-1},x'_{n+1}),0,d(x_{n-1},x_n)\big)\\
&\le q\phi_n\big(d(x_{n-1},x_n),d(x_n,x'_{n+1}),[d(x_{n-1},x_n)+d(x_n,x'_{n+1})],\\
&\quad 0,d(x_{n-1},x_n)\big).
\end{aligned}\quad(5)$$

If $d(x_{n-1},x_n)<d(x_n,x'_{n+1})$, then we obtain from (5) that

$$\begin{aligned}
d(x_n,x'_{n+1})&\le q\phi_n(d(x_n,x'_{n+1}),d(x_n,x'_{n+1}),\\
&\quad 2d(x_n,x'_{n+1}),d(x_n,x'_{n+1}),d(x_n,x'_{n+1})\big)\\
&<qd(x_n,x'_{n+1}).
\end{aligned}$$

Which is a contradiction since $q < \frac{1}{2}$, hence $d(x_{n-1}, x_n) \ge d(x_n, x'_{n+1})$. In this case, we obtain from (5) that

$$\begin{aligned} d(x_n, x_{n+1}) &\le d(x_n, x'_{n+1}) \\ &\le q\phi_n\big(d(x_{n-1}, x_n), d(x_{n-1}, x_n), \\ &\quad 2d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n)\big) \\ &\le qd(x_{n-1}, x_n). \end{aligned} \tag{6}$$

Case III. $x_n \in Q$, $x_{n+1} \in P$. By (2) and iii), we know $x_{n-1} \in P$, and we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) &\le d(x_n, x'_n) + d(x'_n, x_{n+1}) \\ &\le d(x_{n-1}, x_n) + d(x_n, x'_n) + d(x'_n, x_{n+1}) \\ &= d(x_{n-1}, x'_n) + d(x'_n, x_{n+1}) \\ &= d(x_{n-1}, x'_n) + d(x'_n, x'_{n+1}) \\ &= d(x_{n-1}, x'_n) + d(T_n x_{n-1}, T_{n+1} x_n) \\ &\le d(x_{n-1}, x'_n) + q\phi_n\big(d(x_{n-1}, T_n x_{n-1}), d(x_n, T_{n+1} x_n), \\ &\quad d(x_{n-1}, T_{n+1} x_n), d(x_n, T_n x_{n-1}), d(x_{n-1}, x_n)\big) \\ &= d(x_{n-1}, x'_n) + q\phi_n\big(d(x_{n-1}, x'_n), d(x_n, x_{n+1}), \\ &\quad d(x_{n-1}, x_{n+1}), d(x_n, x'_n), d(x_{n-1}, x_n)\big). \end{aligned} \tag{7}$$

Since $d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x_{n-1}, x'_n)$, $d(x_{n-1}, x_n) \le d(x_{n-1}, x'_n)$ and $d(x_n, x'_n) \le d(x_{n-1}, x'_n)$, hence (7) can be restated the following

$$\begin{aligned} & d(x_n, x_{n+1}) \\ &\le d(x_{n-1}, x'_n) + q\phi_n\big(d(x_{n-1}, x'_n)), d(x_n, x_{n+1}), \\ &\quad d(x_{n-1}, x_{n+1}), d(x_{n-1}, x'_n), d(x_{n-1}, x_{n'})\big) \\ &\le d(x_{n-1}, x'_n) + q\phi_n\big(d(x_{n-1}, x'_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) \\ &\quad + d(x_n, x_{n+1}), d(x_{n-1}, x'_n), d(x_{n-1}, x_{n'})\big) \\ &\le d(x_{n-1}, x'_n) + q\phi_n\big(d(x_{n-1}, x'_n), d(x_n, x_{n+1}), d(x_{n-1}, x'_n) \\ &\quad + d(x_n, x_{n+1}), d(x_{n-1}, x'_n), d(x_{n-1}, x'_n)\big) \end{aligned} \tag{8}$$

If $d(x_{n-1}, x_{n'}) \le d(x_n, x_{n+1})$, then (8) becomes

$$\begin{aligned} d(x_n, x_{n+1}) &\le d(x_{n-1}, x'_n) + q\phi_n\big(d(x_n, x_{n+1}), d(x_n, x_{n+1}), \\ &\quad 2d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1})\big) \\ &\le d(x_{n-1}, x'_n) + qd(x_n, x_{n+1}), \end{aligned}$$

hence

$$d(x_n, x_{n+1}) \le \frac{1}{1-q} d(x_{n-1}, x'_n),$$

and therefore, by (6) in Case II, we obtain

$$d(x_n, x_{n+1}) \le \frac{1}{1-q} d(x_{n-1}, x'_n) \le \frac{q}{1-q} d(x_{n-2}, x_{n-1}). \tag{9}$$

If $d(x_n, x_{n+1}) \le d(x_{n-1}, x_{n'})$, then (8) becomes

$$\begin{aligned} & d(x_n, x_{n+1}) \\ &\le d(x_{n-1}, x'_n) + q\phi_n\big(d(x_{n-1}, x'_n), d(x_{n-1}, x'_n), \\ &\quad 2d(x_{n-1}, x'_n), d(x_{n-1}, x'_n), d(x_{n-1}, x_{n'})\big) \\ &\le d(x_{n-1}, x'_n) + qd(x_{n-1}, x_{n'}) \\ &= (1+q) d(x_{n-1}, x'_n). \end{aligned}$$

By (6) in Case II again, we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) &\le (1+q) d(x_{n-1}, x'_n) \\ &\le (1+q) qd(x_{n-2}, x_{n-1}). \end{aligned} \tag{10}$$

Thus in two situations, we obtain from (9) and (10) that

$$d(x_n, x_{n+1}) \le \max\left\{\frac{q}{1-q}, (1+q)q\right\} d(x_{n-2}, x_{n-1}).$$

Hence in all three cases (see, (4), (6), (9) and (10)), we find that

$$\begin{aligned} & d(x_n, x_{n+1}) \\ &\le \max\left\{q, \frac{q}{1-q}, (1+q)q\right\} \max\left\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\right\}, \\ & \forall n \in \mathbb{N}, n > 2. \end{aligned}$$

Let $M = \max\left\{q, \frac{q}{1-q}, (1+q)q\right\}$, then $0 < M < 1$ since $0 < q < \frac{1}{2}$, and we have that

$$d(x_n, x_{n+1}) \le M \max\left\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\right\}$$
for all $n \in \mathbb{N}$ with $n > 2$.

And therefore,

$$d(x_n, x_{n+1}) \le (M)^{\frac{n}{2}-1} \max\left\{d(x_0, x_1), d(x_1, x_2)\right\}$$
for all $n \in \mathbb{N}$ with $n \ge 2$.

Let $\delta = (M)^{-1} \max\left\{d(x_0, x_1), d(x_1, x_2)\right\}$, then $d(x_n, x_{n+1}) \le (M)^{\frac{n}{2}} \delta$ for all $n \in \mathbb{N}$ with $n \ge 2$. Hence for $m > n > N$,

$$d(x_m, x_n) \le \sum_{i=N}^{+\infty} d(x_i, x_{i+1}) \le \sum_{i=N}^{+\infty} (M)^{\frac{i}{2}} \delta \to 0$$

as $N \to +\infty$. Which means that $\{x_n\}$ is a Cauchy sequence. Let p be a limit of $\{x_n\}$, then $p \in K$ since K is closed and $x_n \in K$ for all $n \in \mathbb{N}$. From (2), we are easy to know that there exists an infinite subsequence

$\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k+1} \in P$. Hence $T_{n_k} x_{n_k} = x'_{n_k+1} = x_{n_k+1}$. For any fixed $n \in \mathbb{N}$, we can take $k \in \mathbb{N}$ such that $n < n_k$, then

$$\begin{aligned} d(T_n p, p) &\le d(T_n p, T_{n_k} x_{n_k}) + d(T_{n_k} x_{n_k}, p) \\ &= d(T_n p, T_{n_k} x_{n_k}) + d(x_{n_k+1}, p) \\ &\le q\phi_n\big(d(p, T_n p), d(x_{n_k}, T_{n_k} x_{n_k}), d(p, T_{n_k} x_{n_k}), \\ &\quad d(T_n p, x_{n_k}), d(p, x_{n_k})\big) + d(x_{n_k+1}, p) \\ &= q\phi_n\big(d(p, T_n p), d(x_{n_k}, x_{n_k+1}), d(p, x_{n_k+1}), \\ &\quad d(T_n p, x_{n_k}), d(p, x_{n_k})\big) + d(x_{n_k+1}, p). \end{aligned}$$

Let $k \to +\infty$, then by the continuity of ϕ, the above becomes

$$\begin{aligned} d(T_n p, p) &\le q\phi_n\big(d(T_n p, p), d(p, p), d(p, p), \\ &\quad d(T_n p, p), d(p, p)\big) + d(p, p) \\ &\le q\phi_n\big(d(T_n p, p), d(T_n p, p), \\ &\quad 2d(T_n p, p), d(T_n p, p), d(T_n p, p)\big) \\ &\le qd(T_n p, p). \end{aligned}$$

Since $0 < q < 1$, $d(T_n p, p) = 0$, *i.e.*, $T_n p = p$. This completes that *p* is the common fixed point of $p\{T_n\}_{n\in\mathbb{N}}$.

If *u* and *v* are common fixed points of $\{T_n\}_{n\in\mathbb{N}}$, then

$$\begin{aligned} &d(u,v) = d(T_1 u, T_2 v) \\ &\le q\phi_1\big(d(u, T_1 u), d(v, T_2 v), d(u, T_2 v), d(v, T_1 u), d(u,v)\big) \\ &= q\phi_1\big(0, 0, d(u,v), d(u,v), d(u,v)\big) \\ &\le q\phi_1\big(d(u,v), d(u,v), 2d(u,v), d(u,v), d(u,v)\big) \\ &\le qd(u,v) \end{aligned}$$

Hence $d(u,v) = 0$ since $0 < q < 1$; this is, $u = v$. This completes that $\{T_n\}_{n\in\mathbb{N}}$ has an unique common fixed point *p*.

The following is the very particular form of Theorem 2.1:

Corollary 2.2. Let *K* be a nonempty closed subset of a complete metrically convex metric space (X, d), and $\{T_i : K \to X\}_{i\in\mathbb{N}}$ a family of non-self maps such that for each $x, y \in K$ and $i, j \in \mathbb{N}$ with $i < j$,

$$\begin{aligned} d(T_i x, T_j y) \le \frac{1}{15}\big(&\arctan d(x, T_i x) + \arctan d(y, T_j y) \\ &+ \arctan d(x, T_j y) + \arctan d(y, T_i x) + \arctan d(x, y)\big). \end{aligned}$$

Further, if $T_i(\partial K) \subset K$ for each $i \in \mathbb{N}$, then $\{T_i\}_{i\in\mathbb{N}}$ has an unique common fixed point in *K*.

Proof. Let $q = \frac{7}{15}$ and $\phi_i = \phi$ be that in Example 3 for each $i \in \mathbb{N}$, then *q* and ϕ_i satisfy all conditions of Theorem 1, hence $\{T_i\}_{i\in\mathbb{N}}$ has an unique common fixed point in *K* by Theorem 2.1.

From Theorem 2.1, we can obtain more general forms than Theorem 2.1.

Theorem 2.3. Let *K* be a nonempty closed subset of a complete metrically convex metric space (X, d), and $\{T_i : K \to X\}_{i\in\mathbb{N}}$ a family of maps and $\{m_i\}_{i\in\mathbb{N}}$ a family of positive integers such that for each $x, y \in K$ and $i, j \in \mathbb{N}$ with $i < j$,

$$\begin{aligned} d(T_i^{m_i} x, T_j^{m_j} y) \le q\phi_i\big(&d(x, T_i^{m_i} x), d(y, T_j^{m_j} y), \\ &d(x, T_j^{m_j} y), d(y, T_i^{m_i} x), d(x, y)\big) \end{aligned}$$

where $0 < q < \frac{1}{2}$ and $\phi_i \in \Phi$ for each $i \in \mathbb{N}$. Further, if $T_i^{m_i}(\partial K) \subset K$ for all $i \in \mathbb{N}$, then $\{T_i\}_{i\in\mathbb{N}}$ has unique common fixed point in *K*.

Proof. Let $S_i = T_i^{m_i}$ for each $i \in \mathbb{N}$, then $\{S_i\}_{i\in\mathbb{N}}$ satisfy all conditions of Theorem 2.1, hence $\{S_i\}_{i\in\mathbb{N}}$ has an unique common fixed point $p \in K$.

Next, we prove that *p* is also an unique common fixed point of $\{T_i\}_{i\in\mathbb{N}}$. In fact, for any fixed $i \in \mathbb{N}$,

$$S_i(T_i(p)) = T_i^{m_i+1}(p) = T_i(S_i(p)) = T_i(p),$$

hence $T_i(p)$ is a fixed point of S_i for each $i \in \mathbb{N}$. For any $j \in \mathbb{N}$ with $j > i$,

$$\begin{aligned} &d(T_i(p), S_j(T_i(p))) = d(S_i(T_i(p)), S_j(T_i(p))) \\ &\le q\phi_i\big(d(T_i(p), S_i(T_i(p))), d(T_i(p), S_j(T_i(p))), \\ &\quad d(T_i(p), S_j(T_i(p))), d(T_i(p), S_i(T_i(p))), \\ &\quad d(T_i(p), T_i(p))\big) \\ &\le q\phi_i\big(d(T_i(p)), S_j(T_i(p))), d(T_i(p), S_j(T_i(p))), \\ &\quad 2d(T_i(p), S_j(T_i(p))), d(T_i(p), S_j(T_i(p))), \\ &\quad (d(T_i(p)), S_j(T_i(p)))\big) \\ &\le q\big(d(T_i(p)), S_j(T_i(p))\big) \end{aligned}$$

Since $0 < q < 1$, $d(T_i(p), S_j(T_i(p))) = 0$, which implies that $T_i(p) = S_j(T_i(p))$. Similarly, for any $j \in \mathbb{N}$ with $j < i$, we can obtain that $d(T_i(p), S_j(T_i(p))) \le qd(T_i(p), S_j(T_i(p)))$, hence $T_i(p) = S_j(T_i(p))$. Therefore $T_i(p)$ is a common fixed point of $\{S_j\}_{j\in\mathbb{N}}$ for all $i \in \mathbb{N}$. By the uniqueness of common fixed point of $\{S_j\}_{j\in\mathbb{N}}$, we have $T_i(p)=p$ for all $i\in\mathbb{N}$, this means that *p* is a common fixed point of $\{T_i\}_{i\in\mathbb{N}}$. If *u* and *v* are common fixed points of $\{T_i\}_{i\in\mathbb{N}}$, then *u* and *v* are also common fixed points of $\{S_i\}_{i\in\mathbb{N}}$, hence again by uniqueness of common fixed points of $\{S_i\}_{i\in\mathbb{N}}$, we have that $u=v=p$, This completes that *p* is the unique common fixed point of $\{T_i\}_{i\in\mathbb{N}}$.

Theorem 2.4. Let *K* be a nonempty closed subset of a complete metrically convex metric space (X,d), and $\{T_{i,j}:K\to X\}_{i,j\in\mathbb{N}}$ a family of maps and $\{m_{i,j}\}_{i,j\in\mathbb{N}}$ a family of positive integers such that for each $x,y\in K$ and $i_1,i_2,j\in\mathbb{N}$ with $i_1<i_2$,

$$d\left(T_{i_1,j}^{m_{i_1,j}}x,T_{i_2,j}^{m_{i_2,j}}y\right)\le q\phi_{i_1,j}\left(d\left(x,T_{i_1,j}^{m_{i_1,j}}x\right),d\left(y,T_{i_2,j}^{m_{i_2,j}}y\right),\right.$$
$$\left.d\left(x,T_{i_2,j}^{m_{i_2,j}}y\right),d\left(y,T_{i_1,j}^{m_{i_1,j}}x\right),d(x,y)\right),$$

where $0<q<\frac{1}{2}$ and $\phi_{i_1,j}\in\Phi$ for each $(i_1,j)\in\mathbb{N}\times\mathbb{N}$. Further, if 1) $T_{i,j}^{m_{i,j}}(\partial K)\subset K$ for all $i,j\in\mathbb{N}$; 2) $T_{i_1,\mu}T_{i_2,\nu}=T_{i_2,\nu}T_{i_1,\mu}$ for all i_1,i_2,μ,ν with $\mu\ne\nu$, then $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ has an unique common fixed point in *K*.

Proof For any fixed $j\in\mathbb{N}$, $\{T_{i,j}\}_{i\in\mathbb{N}}$ has an unique common fixed point $p_j\in K$ by 1) and Theorem 2.3. Now, we prove that $p_\mu=p_\nu$ for all $\mu,\nu\in\mathbb{N}$. In fact, for each $i_1,i_2,\mu,\nu\in\mathbb{N}$ with $\mu\ne\nu$, since $T_{i_1,\mu}(p_\mu)=p_\mu$ and $T_{i_2,\nu}(p_\nu)=p_\nu$, hence $T_{i_1,\mu}(T_{i_2,\nu}(p_\nu))=T_{i_1,\mu}(p_\nu)$, and therefore $T_{i_2,\nu}(T_{i_1,\mu}(p_\nu))=T_{i_1,\mu}(T_{i_2,\nu}(p_\nu))=T_{i_1,\mu}(p_\nu)$ by 2). This means that $T_{i_1,\mu}(p_\nu)$ is a common fixed point of $\{T_{i_2,\nu}\}_{i_2\in\mathbb{N}}$ for all $i_1\in\mathbb{N}$, But $\{T_{i_2,\nu}\}_{i_2\in\mathbb{N}}$ has an unique common fixed p_ν, hence $T_{i_1,\mu}(p_\nu)=p_\nu$ for all $i_1\in\mathbb{N}$, which implies that p_ν is a common fixed point of $\{T_{i_1,\mu}\}_{i_1\in\mathbb{N}}$, hence $p_\nu=p_\mu$ since p_μ is the unique common fixed point of $\{T_{i_1,\mu}\}_{i_1\in\mathbb{N}}$. Let $p^*=p_j$, then p^* is the unique fixed point of $\{T_{i,j}\}_{i,j\in\mathbb{N}}$. This completes our proof.

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