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On Commutativity of Semiprime Right Goldie C_k-Rings

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ABSTRACT

This short exposition is about some commutativity conditions on a semiprime right Goldie C_k-ring. In particular, it is observed here that a semiprime right Goldie C_k-ring with symmetric quotient is commutative. The statement holds if the symmetric ring is replaced by reduced, 2-primal, left duo, right duo, abelian, NI, NCI, IFP, or Armendariz ring.

Keywords: Semiprime Right Goldie C_k-Rings; Reduced; Symmetric; Von Neumann Regular Rings

In this short note we expose some commutativity conditions on a semiprime right Goldie C_k-ring. All rings here are associative with an identity. A ring A is said to be a C_k-ring, as introduced by Chuang and Lin in 1989 in [1], if for very pair of elements $x, y \in A$, there exist integers $m = m(x, y)$ and $n = n(x, y)$ such that

$$\left[x^m, y^n\right]_k = 0,$$

where $[x, y]_k$ is the kth-commutator defined by Klein, Nada, and Bell in [2] in 1980, as

$$[x,y]_k = \left[[x,y]_{k-1}, y\right] \text{ where } [x,y]_1 = [x,y].$$

A ring is called a *symmetric ring* (in the sense of Lambek [3]), if whenever $rab = 0$, then $rba = 0$, $\forall r,a,b \in A$, *semiprime* (respt. *reduced*) if A has no non-zero nilpotent ideal (respt. element) and *von Neumann regular* if for each $a \in A$, there exists $r \in A$ such that $ara = a$. A ring is right Goldie in case it has finite right uniform dimension and satisfies acc on right annihilators.

In [1; Theorem 1] Chaung-Lin proved that:

Lemma 1: *Every reduced C_k-ring is commutative.*

We use it to prove the following.

Theorem: *A semiprime right Goldie C_k-ring with symmetric right quotient is commutative.*

Proof: Lambek in [3; Section 1G] proved that every reduced ring is symmetric. We prove that the converse holds for von Neumann regular rings. In deed, one may deduce easily that A is symmetric if and only if

$$a_1 a_2 \cdots a_k = 0,$$

then

$$a_{p(1)} a_{p(2)} \cdots a_{p(k)} = 0,$$

where $a_i, a_{p(i)} \in A$ and p is a one-to-one correspondence on the set $\{1, 2, \cdots, k\}$. Let $a \in N(A)$ be a non-zero element of some index n. Since A is von Neumann regular, for some $x \in A$,

$$a = axa = (ax)^{n-1} a.$$

But A is symmetric and $a^n = 0$, which implies that $a = a^n x^{n-1} = 0$. Hence A is reduced.

The famous Goldie's Theorem states that a ring A is semiprime right Goldie if and only if A has a right quotient ring B which is semisimple Artinian [4; Theorem 2.3.6]. But a semisimple Artinian ring is von Neumann regular [5; Theorem 1.7]. Since B is symmetric and now von Neumann regular, therefore B is reduced. This means that A is reduced. Since A is a C_k-ring, by the Lemma 1, we get that A is commutative. ■

The statement of the Theorem remains unchanged if we replace the condition of the ring being symmetric by 2-primal, abelian, left or right duo, NI, NCI, IFP, quasi-IFP, near-IFP, Armendariz, weak-Armendariz, and some other relations that are listed in Lemma 2.

Let us denote by $N(A)$ the set of all nilpotent elements of A. For a reduced ring $N(A) = 0$, and a ring is *NI* if $N(A)$ is an ideal [6], *NCI* if $N(A)$ contains a nonzero ideal [7], and 2-*primal* if N(A) is the intersection of prime ideals [8]. A ring A is said to have "Insertion of factor property (in short, *IFP*) [9] in case for any pair of elements a, b of A, if $ab = 0$, then $arb = 0$ for all $r \in A$. Such rings are also termed as *semicommutative* in literature, we simply call them *IFP rings. Near-IFP* (respt. *quasi-IFP*) *rings* are introduced recently in [10] (respt. in [11]), and are characterized as AaA contains a non-zero nilpotent ideal of A for any $0 \neq a \in A$ in [10; Proposition 1.2] (respt. AaA is a nilpotent ideal of A for any $0 \neq a \in A$ in [11; Lemma 1.3]).

By definitions, every reduced ring is an *IFP* ring, an *IFP* ring is a quasi-*IFP* ring, and a quasi-*IFP* ring is a near-*IFP* ring. The converse need not be true in general (see the Example below) but for a semiprime ring it holds.

A ring A is called Armendariz in [12] if whenever polynomials in $A[x]$, $f(x)=a_0+a_1x+\cdots+a_mx^m$ and $g(x)=b_0+b_1x+\cdots+b_nx^n$ satisfy $f(x)g(x)=0$, then for each *i*, *j*, $a_ib_j=0$ and weak Armendariz in [13] if whenever $(a_0+a_1x)(b_0+b_1x)=0$, then for each *i*, *j*, $a_ib_j=0$.

For several interactions and various characterizations with examples and counter examples of these rings which we have discussed above the interested reader may refer to the articles [4,11,12,14,15].

Example: It is clear from the examples and counter examples in above citations that the rings listed above are different from each other, but we found no example for near-*IFP* and quasi-*IFP* rings to be different in literature. By definitions, quasi-*IFP* is near-*IFP*, we prove that the opposite may not be true.

Let *R* be a ring and $I\neq 0$ a nilpotent ideal of *R* such that every element of $R-I$ is a unit. For example, a local ring is of this type. By Proposition 1.10 [10] $R_n=Mat_n(R)$ is near-IFP. Let $0\neq(r_{ij})\in N(R_n)$. It is clear that

$$R_n(r_{ij})R_n=R_n$$

is nilpotent if $r_{ij}\in I$, otherwise not nilpotent in general. The ideal $R_nMat_n(I)R_n$ of R_n is proper and nilpotent. Hence we conclude that R_n is not quasi-IFP. ■

Now we give a list of rings which coincide on the condition of von Neumann regularity. Here P stands for some property, for instance, property for being reduced, etc.

Lemma 2: *Let A be a von Neumann regular ring. Then the following are equivalent.*

(P1) *A is reduced*;
(P2) *A is left* (*or right*) *duo*;
(P3) *A is abelian*;
(P4) *A is* 2-*primal*;
(P5) *A is symmetric*;
(P6) *A is NI*;
(P7) *A is NCI*;
(P8) *A is IFP*;
(P9) *A is quasi-IFP*;
(P10) *A is near-IFP*;
(P11) *A is a subdirect product of division ring*;
(P12) *A is Armendariz*;
(P13) *A is weak Armendariz*;
(P14) *If* $a,a',a''\in A$, *such that* $aa''=0=a'^2$ *with* $n\geq 1$, *then* $aa'a''=0$.
(P15) *If* $a,a',a''\in A$, *such that* $aa''=0=a'^2$, *then* $aa'a''=0$.

Proof: The equivalence (P1) $\Leftrightarrow$ (P5) is proved in the Theorem above. Equivalences of (P1)-(P4) and (P6) and (P7) hold in [11; Proposition 1.4]. It is clear by definitions that every reduced ring is IFP, every IFP ring is quasi-IFP and every quasi-IFP ring is near IFP. Thus

$$\text{(P1)} \Rightarrow \text{(P8)} \Rightarrow \text{(P9)} \Rightarrow \text{(P10)}.$$

Because a von Neumann regular ring is semiprime, by [10; Proposition 1.4], for a semiprime ring a near-IFP ring is reduced, giving the equivalence (P1) $\Leftarrow$ (P10). Equivalences of (P1)-(P3), (P6), (P8) and (P10) also hold in [10; Proposition 1.6]. The equivalence of a von Neumann regular ring to be reduced, Armendariz, and weak Armendariz is proved in [14; Lemma 2.4]. Finally, the equivalences

$$\text{(P1)} \Leftrightarrow \text{(P2)} \Leftrightarrow \text{(P3)} \Leftrightarrow \text{(P8)} \Leftrightarrow \text{(P11)} \Leftrightarrow \text{(P12)} \Leftrightarrow \text{(P13)} \Leftrightarrow \text{(P14)} \Leftrightarrow \text{(P15)}$$

are established in [14; Lemma 2.4]. ■

Lemma 3 *Let A be a semiprime ring of bounded index of nilpotency. Then the following conditions are equivalent*:

(P1) *A is reduced*;
(P4) *A is* 2-*primal*;
(P6) *A is NI*;
(P7) *A is NCI*;
(P8) *A is IFP*;
(P9) *A is quasi-IFP.*

Proof: (P1) $\Leftrightarrow$ (P4) $\Leftrightarrow$ (P6) $\Leftrightarrow$ (P7) hold by [7; Proposition 1.3], (P1) $\Leftrightarrow$ (P6) $\Leftrightarrow$ (P8) $\Leftrightarrow$ (P10) hold by [10; Proposition 1.5] while (P1) $\Leftrightarrow$ (P4) $\Leftrightarrow$ (P6) $\Leftrightarrow$ (P8) $\Leftrightarrow$ (P9) hold in [11; Proposition 1.6]. ■

The consequences of the Theorem and above lemmas are the following.

Corollary 1: *A* C_k*-von Neumann regular ring is commutative if any one of the properties* (*P*1)-(*P*15) *of Lemma* 2 *is satisfied.*

Corollary 2: *A* C_k*-semiprime ring of bounded index of nilpotency is commutative if any one of the properties* (*P*1), (*P*4), (*P*6)-(*P*9) *of Lemma* 3 *is satisfied.*

Corollary 3: *Let A be a semiprime right Goldie ring and B its classical ring of quotients. Then the ring B satisfies all equivalent conditions from* (*P*1) *to* (*P*15) *of Lemma* 2. *Moreover*, *for the ring A*, *the conditions* (*P*1), (*P*4), (*P*8), (*P*9), (*P*10), (*P*12) *and* (*P*13) *of Lemma* 2 *are mutually equivalent and are also equivalent to above each of fifteen conditions for the ring B.*

Proof: Equivalence of (P1) to (P15) is followed from [14; Theorem 2.6] and Lemma 1.2.

If *A* is near-IFP and semiprime, and if *a* is nilpotent, then every ideal of *AaA* is zero. Hence, in particular, *a* is zero, and *A* is reduced. So, (P1), (P8)-(P10) are equivalent for the ring *A*. Equivalence of (P1) and (P4) for the ring *A* is obvious and for the same ring (P12) and (P13) are followed from [14; Theorem 2.6]. ■

Corollary 4: *A semiprime right Goldie* C_k*-ring is commutative if its classical ring of quotient satisfies any* *one of the properties* (P1)-(P15) *as listed in Lemma* 2.

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