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α-Times Integrated C-Semigroups

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ABSTRACT

The α-times integrated C semigroups, $\alpha > 0$, are introduced and analyzed. The Laplace inverse transformation for α-times integrated C semigroups is obtained, some known results are generalized.

Keywords: α-Times Integrated C Semigroups; Laplace Inverse Transformation; Pseudo-Resolvent Identity

1. Introduction

Integrated semigroups are more general than strongly continuous semigroups (*i.e.*, C_0 semigroups), cosine operator functions and exponentially bounded distribution semigroups. Integrated exponentially bounded semigroups were investigated in [1-15]. In this paper, we will introduce and analyze α-times integrated C semigroups, $\alpha \in R^+$. In Theorem 2.6 we give a necessary and sufficient condition for an $R_C(\lambda)$ to be the pseudo-resolvent of an α-times integrated C semigroups $S(t)$. At the same time we discuss the Laplace inverse transformation for α-times integrated C semigroups. The results obtained are generalizations of the corresponding results for integrated semigroups.

Throughout this paper, X is a Banach space, $B(X)$ is the space of bounded linear operators from X into X, $D(A)$, $R(A)$, $K(A)$ denote the domain, range, core of operator A respectively, $C \in B(X)$.

2. Definitions and Properties of α-Times Integrated C Semigroups

For $\alpha \ge 0$, $[\alpha]$, (α) denote the integral part and decimal part of α respectively. $\Gamma(\cdot)$ is well known Gamma function, and $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\mathrm{d}x$, $s\Gamma(s) = \Gamma(s+1)$.

For $\beta \ge -1$, we definite the function $j_\beta : (0,\infty) \to R$, and $j_\beta(t) = \dfrac{t^\beta}{\Gamma(\beta+1)}$ j_{-1} denotes 0-point Dirac measure δ_0.

For continuous function $f(\cdot)$, $\beta \ge -1$, the definition of convolution product is as following

$$(j_\beta * f)(t) = \begin{cases} \int_0^t \dfrac{(t-s)^\beta}{\Gamma(\beta+1)} f(s)\mathrm{d}s, & \beta > -1 \\ f(t), & \beta = -1 \end{cases}.$$

At first we introduce the fractional differential and integral of function.

For arbitrary $\alpha > 0$, α -order differential of function u denotes

$$(D_\alpha u)(t_0) = \omega^{(n-1)}(t_0).$$

For arbitrary $\alpha > 0$, α -times cumulative integral of function u denotes

$$(I_\alpha u) = (j_{\alpha-1} * u)(t).$$

Definition 2.1. Let $\alpha \in R^+$, a strongly continuous family $\{S(t)\}_{t\ge0} \in B(X)$ is called α-times integrated C-semigroups, if

(V_1) $S(t)C = CS(t)$, and $S(0) = 0$;

(V_2)

$$S(t)S(s)x = \frac{1}{\Gamma(\alpha)}\left[\int_t^{s+t}(t+s-r)^{\alpha-1}S(r)Cx\mathrm{d}r - \int_0^s (t+s-r)^{\alpha-1}S(r)Cx\mathrm{d}r\right], \forall t,s \ge 0 \tag{2.1}$$

If $\alpha = n\ (n \in N)$, then $\{S(t)\}_{t\ge0}$ is called n-times integrated C semigroups.

If $\alpha = n\ (n \in N)$, and $C = I$, then $\{S(t)\}_{t\ge0}$ is called n-times integrated semigroups.

If $\alpha > 0$, $S(t)x = 0$ $(t \ge 0)$ implies $x = 0$, then α-times integrated C semigroups $\{S(t)\}_{t\ge0}$ is non-degenerated.

If there exists $M > 0$, $\omega \in R$, such that $\|S(t)\| \le Me^{\omega t}$, $t \ge 0$, then $\{S(t)\}_{t\ge0}$ is called exponentially bounded.

Definition 2.2. Let $\alpha \ge 0$, a strongly continuous family $\{S(t)\}_{t\ge0} \in B(X)$ is called α-times exponentially bounded integrated C semigroups generated by A, if $S(0) = 0$, and there exists $M > 0$, $\omega > 0$, such that $(\omega,\infty) \subset \rho(A)$, $\|S(t)\| \le Me^{\omega t}$, $t \ge 0$, and for arbitrary

$\lambda > \omega$, $x \in X$, we have

$$R_C(\lambda, A)x = (\lambda - A)^{-1} Cx = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t) x \mathrm{d}t. \quad (2.2)$$

Proposition 2.3. Let *A* be the generator of an *α*-times integrated *C* semigroups $\{S(t)\}_{t\geq 0}$, $\alpha \geq 0$. Then

1) For all $x \in D(A)$ and $t \geq 0$,

$$S(t)x \in D(A), AS(t)x = S(t)Ax \quad (2.3)$$

$$S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)} Cx + \int_0^t S(s) Ax \mathrm{d}s \quad (2.4)$$

2) $\int_0^t S(s) x \mathrm{d}s \in D(A)$, for all $x \in X$, and $t \geq 0$ and

$$A\int_0^t S(s) x \mathrm{d}s = S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)} Cx \quad (2.5)$$

Proof. Letting $R_C(\lambda)x = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t) x \mathrm{d}t$, $\mathrm{Re}\lambda > \omega$ Fix $u \in \rho(A)$, then

$$\int_0^\infty e^{-\lambda t} S(t) R_C(u, A) x \mathrm{d}t = \lambda^{-\alpha} R_C(\lambda, A) R_C(u, A) x$$
$$= \int_0^\infty e^{-\lambda t} R_C(u, A) S(t) x \mathrm{d}t,$$

for all $\mathrm{Re}\lambda > \omega$, and $x \in X$. By the uniqueness theorem it follows that

$$R_C(u, A) S(t) = S(t) R_C(u, A), u \in \rho(A), t \geq 0 \quad (2.6)$$

This implies (2.3). Let $x \in D(A)$, then for all $\mathrm{Re}\lambda > \omega$,

$$\begin{aligned} Cx &= \int_0^\infty \lambda^{\alpha+1} e^{-\lambda t} \frac{t^\alpha}{\Gamma(\alpha+1)} Cx \mathrm{d}t \\ &= \lambda R_C(\lambda, A)x - R_C(\lambda, A)Ax \\ &= \int_0^\infty \lambda^{\alpha+1} e^{-\lambda t} S(t) x \mathrm{d}t - \int_0^\infty \lambda^\alpha e^{-\lambda t} S(t) Ax \mathrm{d}t \\ &= \int_0^\infty \lambda^{\alpha+1} e^{-\lambda t} S(t) x \mathrm{d}t - \int_0^\infty \lambda^\alpha e^{-\lambda t} d\int_0^t S(s) Ax \mathrm{d}s \\ &= \int_0^\infty \lambda^{\alpha+1} e^{-\lambda t} S(t) x \mathrm{d}t - \int_0^\infty \lambda^{\alpha+1} e^{-\lambda t} \int_0^t S(s) Ax \mathrm{d}s \mathrm{d}t. \end{aligned}$$

Then (2.4) follows from the uniqueness theorem.

In order to prove (2.5), let $x \in X$, and $t \geq 0$, $\mathrm{Re}\lambda > \omega$, then by (2.3), (2.4), (2.6) we have

$$\begin{aligned} C\int_0^t S(s) x \mathrm{d}s &= \lambda R_C(\lambda, A) \int_0^t S(s) x \mathrm{d}s \\ &\quad - R_C(\lambda, A) \int_0^t S(s) Ax \mathrm{d}s \\ &= \lambda R_C(\lambda, A) \int_0^t S(s) x \mathrm{d}s \\ &\quad - R_C(\lambda, A) \left[S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)} Cx \right] \end{aligned} \quad (2.7)$$

Noting that $\lambda R_C(\lambda, A)x - Cx = R_C(\lambda, A)Ax$ Hence, $\int_0^t S(s) x \mathrm{d}s \in D(A)$, and by (2.7), (2.5) follows.

Corollary 2.4. Let $\alpha \in R^+$. Then $S(t)x \in \overline{D(A)}$ for all $x \in X$ and $t \geq 0$. Then $S(\cdot)x$ is right differentiable in $t \geq 0$ if $S(t)x \in D(A)$. In that case

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t)x = AS(t)x + \frac{t^{\alpha-1}}{\Gamma(\alpha)} Cx, t \geq 0, x \in X.$$

Proposition 2.5. Let $A: D(A) \to X$ be closed linear operator, when λ, $u \in \rho(A)$, we have

1) The pseudoresolvent identity

$$R_C(\lambda, A)C - R_C(\mu, A)C = (\mu - \lambda) R_C(\lambda, A) R_C(\mu, A) \quad (2.8)$$

2) $$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} R_C(\lambda, A) C^n = (-1)^n n! \left[R_C(\lambda, A) \right]^{n+1}$$

$$n = 1, 2, \cdots \quad (2.9)$$

Proof. 1)

$$\begin{aligned} R_C(\lambda, A)C &= (\mu - A)^{-1} (\mu - A) R_C(\lambda, A) C \\ &= (\mu - A)^{-1} C(\mu - \lambda + \lambda - A) R_C(\lambda, A) \\ &= (\mu - A)^{-1} CC + (\mu - \lambda) R_C(\mu, A) R_C(\lambda, A) \\ &= R_C(\mu, A) C + (\mu - \lambda) R_C(\mu, A) R_C(\lambda, A) \end{aligned}$$

It follows that

$$R_C(\lambda, A)C - R_C(\mu, A)C = (\mu - \lambda) R_C(\lambda, A) R_C(\mu, A)$$

2) We apply the mathematical induction when $n = 1$, by (2.8)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} R_C(\lambda, A) C = -\left[R_C(\lambda, A) \right]^2$$

we suppose $n = k$, (2.9) is complete. *i.e.*,

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} R_C(\lambda, A) C^k = (-1)^k k! \left[R_C(\lambda, A) \right]^{k+1}$$

then

$$\begin{aligned} \frac{\mathrm{d}^{k+1}}{\mathrm{d}\lambda^{k+1}} R_C(\lambda, A) C^{k+1} &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} R_C(\lambda, A) C^k \right) C \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda} (-1)^k k! \left[R_C(\lambda, A) \right]^{k+1} C \\ &= (-1)^{k+1} (k+1)! \left[R_C(\lambda, A) \right]^{k+2} \end{aligned}$$

i.e., it follows $n = k + 1$. The proof is complete.

Theorem 2.6. Let $S(t)$ be a stongly continuous operator function, and $\|S(t)\| \leq Me^{\omega t}$, $t \geq 0$, letting $R_C(\lambda) = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t) x \mathrm{d}t$, $\mathrm{Re}\lambda > \omega$. Then $\{R_C(\lambda)\}_{\mathrm{Re}\lambda > \omega}$ satisfies the pseudoresolvent

$$R_C(\lambda)C - R_C(\mu)C = (\mu - \lambda) R_C(\lambda) R_C(\mu) \quad (2.10)$$

if and only if $S(t)$ satisfies (V_2).

Proof. One can easily prove the necessary condition.

Let us prove that it is sufficient.

Letting $\mathrm{Re}\lambda$, $\mathrm{Re}u > \omega$, and $\lambda \neq u$. Then the resolvent equation implies

$$\frac{R_C(\lambda)R_C(\mu)}{\lambda^\alpha u^\alpha}=\frac{R_C(\lambda)C-R_C(\mu)C}{(u-\lambda)\lambda^\alpha u^\alpha}=\frac{\lambda^{-\alpha}R_C(\lambda)C-u^{-\alpha}R_C(\mu)C}{(u-\lambda)u^\alpha}+\frac{R_C(\mu)C\left(u^{-\alpha}-\lambda^{-\alpha}\right)}{(u-\lambda)u^\alpha} \quad (2.11)$$

$$\frac{R_C(\lambda)R_C(\mu)}{\lambda^\alpha u^\alpha}=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-ut}S(t)S(s)\mathrm{d}t \quad (2.12)$$

$$\begin{aligned}
&\frac{\lambda^{-\alpha}R_C(\lambda)C-u^{-\alpha}R_C(\mu)C}{(u-\lambda)u^\alpha}\\
&=\left(-\int_0^\infty \frac{1}{\lambda-u}e^{-\lambda t}S(t)C\mathrm{d}t+\int_0^\infty e^{-(\lambda-u)t}u^{-\alpha}R_C(u)C\mathrm{d}t\right)\frac{1}{u^\alpha}\\
&=\left(-\int_0^\infty e^{-(\lambda-u)t}\int_0^t e^{-us}S(s)C\mathrm{d}s\mathrm{d}t\right.\\
&\quad\left.+\int_0^\infty e^{-(\lambda-u)t}\int_0^\infty e^{-us}S(s)C\mathrm{d}s\mathrm{d}t\right)\frac{1}{u^\alpha}\\
&=\left(\int_0^\infty e^{-(\lambda-u)t}\int_t^\infty e^{-us}S(s)C\mathrm{d}s\mathrm{d}t\right)\frac{1}{u^\alpha}\\
&=\left(\int_0^\infty e^{-\lambda t}\int_t^\infty e^{-u(s-t)}S(s)C\mathrm{d}s\mathrm{d}t\right)\frac{1}{u^\alpha}\\
&=\left(\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-us}S(s+t)C\mathrm{d}s\mathrm{d}t\right)\frac{1}{u^\alpha}
\end{aligned}$$

Noting that $\int_0^\infty e^{-uv}v^{\alpha-1}\mathrm{d}v=\frac{\Gamma(\alpha)}{u^\alpha}$

Then

$$\begin{aligned}
&\frac{\lambda^{-\alpha}R_C(\lambda)C-u^{-\alpha}R_C(\mu)C}{(u-\lambda)u^\alpha}\\
&=\int_0^\infty e^{-\lambda t}\int_0^\infty S(s+t)\int_0^\infty \frac{e^{-u(s+v)}}{\Gamma(\alpha)}v^{\alpha-1}C\mathrm{d}v\mathrm{d}s\mathrm{d}t\\
&=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-ur}\int_0^r \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)}S(s+t)C\mathrm{d}v\mathrm{d}s\mathrm{d}t\\
&=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-ur}\int_t^{t+s} \frac{(t+s-r)^{\alpha-1}}{\Gamma(\alpha)}S(r)C\mathrm{d}v\mathrm{d}s\mathrm{d}t
\end{aligned} \quad (2.13)$$

Moreover,

$$\begin{aligned}
&\frac{R_C(\mu)u^{-\alpha}}{(u-\lambda)u^\alpha}C\\
&=\int_0^\infty e^{(\lambda-u)t}\int_0^\infty e^{-us}S(s)C\int_0^\infty \frac{e^{-uv}}{\Gamma(\alpha)}v^{\alpha-1}\mathrm{d}v\mathrm{d}s\mathrm{d}t\\
&=\int_0^\infty e^{\lambda t}\int_0^\infty S(s)C\int_{t+s}^\infty \frac{e^{-ur}}{\Gamma(\alpha)}(r-t-s)^{\alpha-1}\mathrm{d}r\mathrm{d}s\mathrm{d}t \quad (2.14)\\
&=\int_0^\infty e^{\lambda t}\int_t^\infty e^{-ur}\int_0^{r-t} \frac{(r-t-s)^{\alpha-1}}{\Gamma(\alpha)}S(s)C\mathrm{d}r\mathrm{d}s\mathrm{d}t\\
&=\int_0^\infty e^{-ut}\int_{-t}^0 e^{-\lambda r}\int_0^{t+r} \frac{(t+r-s)^{\alpha-1}}{\Gamma(\alpha)}S(s)C\mathrm{d}r\mathrm{d}s\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
&\frac{R_C(\mu)\lambda^{-\alpha}}{(u-\lambda)u^\alpha}C\\
&=\int_0^\infty e^{(\lambda-u)t}\int_0^\infty e^{-us}S(s)C\int_0^\infty \frac{e^{-\lambda v}}{\Gamma(\alpha)}v^{\alpha-1}\mathrm{d}v\mathrm{d}s\mathrm{d}t\\
&=\int_0^\infty e^{-ut}\int_{-t}^0 e^{-\lambda r}\int_0^{t+r} \frac{(t+r-s)^{\alpha-1}}{\Gamma(\alpha)}S(s)C\mathrm{d}r\mathrm{d}s\mathrm{d}t\\
&\quad+\int_0^\infty e^{-ut}\int_0^\infty e^{-\lambda r}\int_0^t \frac{(t+r-s)^{\alpha-1}}{\Gamma(\alpha)}S(s)C\mathrm{d}r\mathrm{d}s\mathrm{d}t
\end{aligned} \quad (2.15)$$

Using (2.14) and (2.15), we obtain

$$\begin{aligned}
&\frac{R_C(\mu)C\left(u^{-\alpha}-\lambda^{-\alpha}\right)}{(u-\lambda)u^\alpha}\\
&=\int_0^\infty e^{-ut}\int_0^\infty e^{-\lambda r}\int_0^t \frac{(t+r-s)^{\alpha-1}}{\Gamma(\alpha)}S(s)C\mathrm{d}r\mathrm{d}s\mathrm{d}t
\end{aligned} \quad (2.16)$$

Assertion (V_2) follows from (2.13) and (2.16) and the uniqueness of the Laplace transformation.

3. Laplace Inverse Transformation for *α*-Times Integrated *C*-Semigroups

Lemma 3.1. [16] Let $\omega\geq 0$, $F(\lambda):(\omega,\infty)\to X$, $F(\lambda)$ is Laplace-type expression: $F(\lambda)=\lambda\int_0^{+\infty}e^{-\lambda t}\alpha(t)\mathrm{d}t$, $\alpha(0)=0$, and $\left\|\alpha(t+h)-\alpha(t)\right\|\leq Mhe^{\omega(t+h)}$, $t,h\geq 0$, then

$$\alpha(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}F(\lambda)\frac{\mathrm{d}\lambda}{\lambda},(\gamma>\omega)$$

Theorem 3.2. Let $\alpha\geq 0$, then the following conditions are equivalent:

1) A generates an α-times exponentially bounded integrated semigroups $\{S(t)\}_{t\geq 0}$;

2) There exists $\omega>0$, such that $(\omega,\infty)\subset\rho(A)$, and for all $u>\omega$, A generates an $(u-A)^{-\alpha}C$ exponentially bounded semigroups $\{T(t)\}_{t\geq 0}$, and

$$S(t)=(u-A)^{-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{[\alpha]+1}\left(j_{-(\alpha)}*T\right)(t).$$

Proof. 1) If A generates an α-times exponentially bounded integrated semigroups $\{S(t)\}_{t\geq 0}$, then

$$A\int_0^t W(r)x\mathrm{d}r=W(t)x-\frac{t^\alpha}{\Gamma(\alpha+1)}Cx,$$

By ([17], Proposition 3.7(a)), $\{W(t)\}_{t\geq 0}$ is an $(u-A)^{-\alpha}C$ semigroup generated by $\tilde{A}$ is the extention of A, By ([17], Proposition 3.11), $\tilde{A}=A$.

2) Combing [18] with [17, Theorem 3.4], we can prove $A\int_0^t S(r)x\mathrm{d}r=S(t)x-\frac{t^\alpha}{\Gamma(\alpha+1)}Cx$, and the space of op-

erator is exchangeable, by Proposition 2.3, This ends the proof .

Theorem 3.3. Let A be closed linear operator on X, $\rho(A)\neq\Phi$, $\lambda\in\rho(A)$, an α-times exponentially bounded integrated C semigroups $\{S(t)\}_{t\ge 0}$ with infinitesimal generator A, and $\|S(t)\|\le Me^{\omega t}$, $\omega\ge 0$, $\gamma>\omega$, then for $\forall x\in D(A)$,

$$\begin{aligned}\int_0^t S(s)x\mathrm{d}s &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{R_C(\lambda,A)x}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda}\\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(I_\alpha e^{\lambda t}R_C(\lambda,A)x\right)\frac{\mathrm{d}\lambda}{\lambda}\end{aligned} \tag{3.1}$$

Proof. Let $\alpha(t)=\int_0^t S(s)\mathrm{d}s$,

$$F(\lambda)=\frac{(\lambda-A)^{-1}Cx}{\lambda^\alpha}, \forall x\in D(A)$$

by Lemma 3.1

$$\begin{aligned}F(\lambda)&=\frac{(\lambda-A)^{-1}Cx}{\lambda^\alpha}=\int_0^{+\infty}e^{-\lambda t}S(t)x\mathrm{d}t\\ &=\int_0^{+\infty}e^{-\lambda t}x\mathrm{d}\int_0^t S(s)\mathrm{d}s\\ &=\lambda\int_0^{+\infty}e^{-\lambda t}\left(\int_0^t S(s)x\mathrm{d}s\right)\mathrm{d}t\ (\lambda>\omega)\end{aligned}$$

So $F(\lambda)$ satisfies Lemma 3.1,

$$\int_0^t S(s)x\mathrm{d}s=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)x}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda},(\gamma>\omega)$$

On the other hand, by Theorem 3.2 A generates $(u-A)^{-\alpha}C$ exponentially bounded semigroups $\{T(t)\}_{t\ge 0}$.

So for $\forall x\in D(A)$, we have

$$\begin{aligned}&\int_0^t T(s)x\mathrm{d}s\\ &=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}(\lambda-A)^{-1}(u-A)^{-\alpha}Cx\frac{\mathrm{d}\lambda}{\lambda}\\ &=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}(\lambda-A)^{-1}R(u,A)^{\alpha}Cx\frac{\mathrm{d}\lambda}{\lambda}\\ &=\frac{R(u,A)^\alpha}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}(\lambda-A)^{-1}Cx\frac{\mathrm{d}\lambda}{\lambda}\end{aligned}$$

It follows that $S(t)=(u-A)^\alpha(I_\alpha T)(t)$.

Whence

$$\begin{aligned}&\int_0^t S(s)x\mathrm{d}s=(u-A)^\alpha\int_0^t(I_\alpha T)(s)x\mathrm{d}s\\ &=(u-A)^\alpha\left(I_\alpha\int_0^t T(s)x\mathrm{d}s\right)\\ &=(u-A)^\alpha\frac{R(u,A)^\alpha}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(I_\alpha e^{\lambda t}(\lambda-A)^{-1}Cx\right)\frac{\mathrm{d}\lambda}{\lambda}\\ &=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(I_\alpha e^{\lambda t}R_C(\lambda,A)x\right)\frac{\mathrm{d}\lambda}{\lambda}.\end{aligned}$$

And the integral on the right converges uniformly on any bounded intervals.

Corollary 3.4. The conditions are same as Theorem 3.3, then for $\forall x\in X$,

$$\begin{aligned}S(t)x&=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)x}{\lambda^\alpha}\mathrm{d}\lambda\\ &=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(I_\alpha e^{\lambda t}R_C(\lambda,A)x\right)\mathrm{d}\lambda\end{aligned} \tag{3.2}$$

Proof. by Theorem 3.3

$$\int_0^t S(s)x\mathrm{d}s=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)x}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda} \tag{3.3}$$

Then $A\int_0^t S(s)x\mathrm{d}s=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)Ax}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda}$.

By (2.5) and noting that

$$\frac{t^\alpha}{\Gamma(\alpha+1)}=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\lambda^{-\alpha-1}\mathrm{d}\lambda$$

Therefore

$$\begin{aligned}S(t)x&=\frac{t^\alpha}{\Gamma(\alpha+1)}Cx+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)Ax}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda}\\ &=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\lambda^{-\alpha-1}Cx\mathrm{d}\lambda\\ &\quad+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)Ax}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda}\\ &=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)x}{\lambda^\alpha}(\lambda-A+A)\frac{\mathrm{d}\lambda}{\lambda}\\ &=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)x}{\lambda^\alpha}\mathrm{d}\lambda\end{aligned}$$

Combining Theorem 3.3 we can prove the next part.

Corollary 3.5. The conditions are same as Theorem 3.3, then for $\forall x\in X$,

$$\begin{aligned}\int_0^t S(s)x\mathrm{d}s&=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)x}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda^2}\\ &=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(I_\alpha e^{\lambda t}R_C(\lambda,A)x\right)\frac{\mathrm{d}\lambda}{\lambda^2}\end{aligned} \tag{3.4}$$

Proof. by Theorem 3.3

$$\int_0^t S(s)\mathrm{d}s=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)x}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda} \tag{3.5}$$

integrating (3.5) from 0 to t, *i.e.*,

$$\begin{aligned}\int_0^t(t-s)S(s)x\mathrm{d}s&=\frac{1}{2\pi i}\int_0^t\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\frac{R_C(\lambda,A)x}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda}\mathrm{d}s\\ &=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(e^{\lambda t}-1\right)\frac{R_C(\lambda,A)x}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda^2}\end{aligned}$$

Noting that $\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{R_C(\lambda,A)x}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda}=0$

Consequently,

$$\int_0^t (t-s)S(s)x\mathrm{d}s = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{R_C(\lambda,A)x}{\lambda^\alpha}\frac{\mathrm{d}\lambda}{\lambda^2}.$$

The next part is easy to prove.

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