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An Integral Representation of a Family of Slit Mappings

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ABSTRACT

We consider a normalized family *F* of analytic functions *f*, whose common domain is the complement of a closed ray in the complex plane. If $f(z)$ is real when *z* is real and the range of *f* does not intersect the nonpositive real axis, then *f* can be reproduced by integrating the biquadratic kernel $\frac{t(t-1)z^2-z+1}{(1-tz)^2}$ against a probability measure $\mu(t)$. It is shown that while this integral representation does not characterize the family *F*, it applies to a large class of functions, including a collection of functions which multiply the Hardy space H^p into itself.

Keywords: Herglotz Formula; Integral Representations; Subordination; Slit Mappings; Hardy Spaces; Multipliers; Hadamard Product

1. Introduction

Let $\Delta=\{z\in\mathbb{C}:|z|<1\}$, and let $\partial\Delta=\{z\in C:|z|=1\}$. Suppose *f* is analytic in Δ with the real part of *f* nonnegative. Then there is a nondecreasing function μ defined on $[0,2\pi]$ such that $f(z)=\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\mathrm{d}\mu(t)+ib$, where *b* is a real constant. This representation of such functions by integrating a bilinear kernel against a measure is due to G. Herglotz ([1], pp. 21-24) and ([2], pp. 27-30). In this paper, we examine a family of functions defined on the complex plane with a closed ray removed, which may be represented by integrating a biquadratic kernel against a probability measure (A measure μ is called a probability measure on $[0,1]$ provided μ is nonnegative with $\int_0^1\mathrm{d}\mu(t)=1$). In what follows, given functions *f* and *g* analytic in Δ, we say that *f* is subordinate to *g* (written $f\prec g$) provided $f(z)=g(\omega(z))$ for some ω analytic in Δ with $|\omega(z)|\le|z|$.

2. The Main Results

Theorem 1. *Let* $\Omega=C-[1,\infty)$, $\Phi=C-(-\infty,0]$, *and let F be the family of functions f having the following properties*:

1) *f* is analytic in Ω;
2) $f(0)=1$;
3) $f(z)\in R$ whenever $-\infty<z<1$;
4) $f(\Omega)\subseteq\Phi$.

Then

$$F\subseteq\left\{f:f(z)=\int_0^1\frac{t(t-1)z^2-z+1}{(1-tz)^2}\mathrm{d}\mu(t)\right\},$$

where μ is a probability measure.

Proof. Let $\varphi(w)=-\left(\frac{1-w}{1+w}\right)^2+1$. Then φ is an analytic, bijective mapping of Δ in the *w*-plane onto Ω in the *z*-plane with $\varphi(0)=0$. Let $f\in F$. Then $f(\Omega)\subseteq\Phi$ by 4). Let $g=f\circ\varphi$, and let $G(w)=\left(\frac{1+w}{1-w}\right)^2$. Then *G* is an analytic, bijective mapping of Δ onto Φ with $g\prec G$. Define $s(G)$ to be the collection of all functions *h* analytic in Δ with $h\prec G$. By a result due to D. A. Brannan, J. G. Clunie, and W. E. Kirwan [3],

$$\overline{co}\,s(G)=\left\{h\text{ analytic in }\Delta: h(z)=\int_{\partial\Delta}\left(\frac{1+\bar\zeta z}{1-\bar\zeta z}\right)^2\mathrm{d}\nu(\zeta)\right\},$$

where ν is a probability measure and $\overline{co}\,s(G)$ denotes the closed convex hull of $s(G)$. Let $F(z)=-z+1$. Then $F:\Omega\to\Phi$ is an analytic bijection with $F(0)=1$. Since $g\in s(G)$,

$$g(w)=\int_{\partial\Delta}\left(\frac{1+\bar\zeta w}{1-\bar\zeta w}\right)^2\mathrm{d}\nu(\zeta)$$

for $w\in\Delta$ and ν a probability measure. Since φ is injective with $\varphi(\Delta)=\Omega$, we have
$g(w)=f(\varphi(w))=f(z)$.

Hence

$$f(z)=\int_{\partial\Delta}\left(\frac{1+\bar{\zeta}\varphi^{-1}(z)}{1-\bar{\zeta}\varphi^{-1}(z)}\right)^2 \mathrm{d}\nu(\zeta)$$

$$=\int_{\partial\Delta}\left(\frac{1+\bar{\zeta}\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}}{1-\bar{\zeta}\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}}\right)^2 \mathrm{d}\nu(\zeta)$$

$$=\int_{\partial\Delta}\left(\frac{(1+\bar{\zeta})+(1-\bar{\zeta})\sqrt{1-z}}{(1-\bar{\zeta})+(1+\bar{\zeta})\sqrt{1-z}}\right)^2 \mathrm{d}\nu(\zeta).$$

By 3) $f(z)=\overline{f(\bar{z})}$ whenever $z\in(-\infty,1)$. Since Ω is symmetric about the real axis, by the identity theorem $f(z)=\overline{f(\bar{z})}$ throughout Ω. Let $X=\{\zeta\in\partial\Delta:\operatorname{Im}\zeta\le 0\}$. For any measurable subset A of X define $\nu^*(A)=1/2\left(\nu(A)+\nu(\bar{A})\right)$. We have

$$f(z)=\frac{1}{2}\left[f(z)+\overline{f(\bar{z})}\right]$$

$$=\frac{1}{2}\int_{\partial\Delta}\left\{\left(\frac{(1+\bar{\zeta})+(1-\bar{\zeta})\sqrt{1-z}}{(1-\bar{\zeta})+(1+\bar{\zeta})\sqrt{1-z}}\right)^2 +\left(\frac{(1+\zeta)+(1-\zeta)\sqrt{1-z}}{(1-\zeta)+(1+\zeta)\sqrt{1-z}}\right)^2\right\}\mathrm{d}\nu(\zeta)$$

$$=\int_X \frac{\left([\operatorname{Re}\zeta]^2-1\right)z^2-4z+4}{(\operatorname{Re}\zeta+1)^2-4(\operatorname{Re}\zeta+1)z+4}\mathrm{d}\nu^*(\zeta)$$

$$=\int_{-\pi}^{0}\frac{1/4\left(\cos^2\theta-1\right)z^2-z+1}{\left(1-\frac{1+\cos\theta}{2}z\right)^2}\mathrm{d}\sigma(\theta)$$

$$=\int_0^1\frac{t(t-1)z^2-z+1}{(1-tz)^2}\mathrm{d}\mu(t).$$

where $\sigma(\theta)=\nu^*\left(e^{i\theta}\right)$ and $\mu(t)=\sigma\left(\cos^{-1}(2t-1)\right)$. This integral representation does not characterize F, as the following theorem shows.

Theorem 2. *Suppose* $f:C-[1,\infty)\to C$ *is defined via*

$$f(z)=\int_0^1\frac{t(t-1)z^2-z+1}{(1-tz)^2}\mathrm{d}\mu(t)$$

where μ is a probability measure.

1) If μ has support $\{0,1\}$, then $f\notin F$.

2) If μ is a point mass, $f\in F$ if and only if μ has support $\{0\}$ or $\{1\}$.

Proof. Let f be as defined in the theorem. Suppose μ has support $\{0,1\}$, and the weight at 0 is a, where $a\in(0,1)$. Since μ is a probability measure, the corresponding weight at 1 is $1-a$. We have $f(z)=\frac{az^2-2az+1}{1-z}$. Since $0<a<1$, the value $z=1+\sqrt{1-1/a}$ lies in the domain of f, and is mapped to the origin in the w-plane. Therefore $f\notin F$, proving 1).

Observe that point mass at 0 gives $f(z)=-z+1$ and point mass at 1 gives $f(z)=\frac{1}{1-z}$, each of which is an analytic bijection from Ω onto Φ, and clearly in F. Suppose μ has support $\{t\}$, where $0<t<1$. Then

$$f(z)=\frac{t(t-1)z^2-z+1}{(1-tz)^2}.$$

Let

$$\zeta(t)=\frac{1+\sqrt{1-4t(t-1)}}{2t(t-1)}.$$

Then $\zeta'(t)=0$ precisely when $t=1/2$. It follows that ζ lies in the domain of f for each $t\in(0,1)$, and $f(\zeta)=0$. Therefore $f\notin F$.

3. An Application

In [4], T. H. MacGregor and M. P. Sterner investigate multipliers of Hardy spaces of analytic functions using asymptotic expansions and power functions of the form $(1-z)^{-b}$, where b is a complex constant. A subclass of F which multiplies H^p into H^p is given in the following theorem. Suppose $f(z)=\sum_{n=0}^{\infty}a_n z^n$ and $g(z)=\sum_{n=0}^{\infty}b_n z^n$ are analytic in Δ. Then the Hadamard product of f and g is defined by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_n b_n z^n$$

for $z\in\Delta$. We say that f *multiplies* H^p into H^p provided $f*g\in H^p$ whenever $g\in H^p$.

Theorem 3. *Let* μ *be a finite complex-valued Borel measure defined on* $[0,1]$ *and let*

$$f(z)=\int_0^1\frac{1}{1-tz}\mathrm{d}\mu(t)\,(z\in\Delta).$$

Then f is a multiplier of H^p into H^p for every $p>0$. Moreover, there is a constant C_p depending only on p such that $\|f*g\|_{H^p}\le\|\mu\|C_p\|g\|_{H^p}$ for all $g\in H^p$.

Proof. Let f be as described in the hypotheses of the theorem, and suppose $g\in H^p$ for some $p>0$. Then for $z\in\Delta$ and $r\in[0,1)$ we have

$$(f*g)(rz)=\frac{1}{2\pi}\int_0^{2\pi}f\left(ze^{-i\theta}\right)g\left(re^{i\theta}\right)\mathrm{d}\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\frac{1}{1-tze^{-i\theta}}\mathrm{d}\mu(t)\,g\left(re^{i\theta}\right)\mathrm{d}\theta$$

$$=\int_0^1\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{g\left(re^{i\theta}\right)}{1-tze^{-i\theta}}\mathrm{d}\theta\right\}\mathrm{d}\mu(t).$$

By Cauchy's formula,

$$g(z)=\frac{1}{2\pi i}\oint_{|\xi|=r}\frac{g(\xi)}{\xi-z}\mathrm{d}\xi\left(|z|<r,0<r<1\right)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{g\left(re^{i\theta}\right)}{1-\frac{z}{r}e^{-i\theta}}\mathrm{d}\theta.$$

Hence

$$(f*g)(rz)=\int_0^1 g(rtz)\mathrm{d}\mu(t).$$

Therefore for $0\le\rho<1$ and $0\le\varphi<2\pi$ we have

$$(f*g)\left(\rho e^{i\varphi}\right)=\int_0^1 g\left(\rho te^{i\varphi}\right)\mathrm{d}\mu(t).$$

Let $G(\varphi)=\sup_{0\le x<1}\left|g\left(xe^{i\varphi}\right)\right|$ for $0\le\varphi<2\pi$. Then G is the Hardy-Littlewood maximal function for g, and so lies in $L^p[0,2\pi]$ ([5], p. 12). Moreover, there is a constant C_p depending only on p such that $\|G\|_{L^p}\le C_p\|g\|_{H^p}$ (In fact, for $p\ge1$, $C_p=1$). Since $0\le\rho<1$ and $0\le t\le1$, we obtain

$$\left|(f*g)\left(\rho e^{i\varphi}\right)\right|\le\int_0^1\sup_{0\le x<1}\left|g\left(xe^{i\varphi}\right)\right|\left|\mathrm{d}\mu(t)\right|=G(\varphi)\|\mu\|.$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi}\left|(f*g)\left(\rho e^{i\varphi}\right)\right|^p\mathrm{d}\varphi\le\frac{1}{2\pi}\int_0^{2\pi}\left|G(\varphi)\|\mu\|\right|^p\mathrm{d}\varphi$$
$$\le\|\mu\|^p C_p^p\|g\|_{H^p}^p.$$

Therefore

$$\sup_{0\le\rho<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|(f*g)\left(\rho e^{i\varphi}\right)\right|^p\mathrm{d}\varphi\right\}^{1/p}\le\|\mu\|C_p\|g\|_{H^p}.$$

If we restrict the measure μ to be a probability measure, then the formula implies the analyticity of f on $C-[1,\infty)$, the value of f is unity at the origin, and $f(z)$ is real when z is real $(-\infty<z<1)$. Finally, observe that the range of f is contained in $C-(-\infty,0]$. To see this last statement, fix $z\in C-[1,\infty)$. Then $\{tz:0\le t\le1\}$ is the line segment from 0 to z. Hence $\left\{\frac{1}{1-tz}:0\le t\le1\right\}$ is the arc of the circle determined by 1, $\frac{1}{1-z}$, and 0, having endpoints 1 and $\frac{1}{1-z}$ and not including the origin. Since μ is a probability measure, $\int_0^1\frac{1}{1-tz}\mathrm{d}\mu(t)$ lies in the circular segment which is the closed convex hull of that arc, and this circular segment does not intersect $(-\infty,0]$. Hence each such multiplier function f lies in F.

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