Scientific Research

Difference Sets of Null Density Subsets of ℕ

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ABSTRACT

Let $B \subset \mathbb{N}$, $D^1(B) := B - B = \{a - b : a > b, a, b \in B\}$ and for any $k \geq 2$, $D^k(B) = D^{k-1}(B) - D^{k-1}(B)$. If $\bar{d}(B) = \limsup_{n \to \infty} \frac{|B \cap \{1, \cdots, n\}|}{n}$ is positive, then B is considered as a large set with $\sum_{b \in B} \frac{1}{b} = \infty$. Its difference set $D^1(B)$ has both high density and rich structure. The set $A \subset \mathbb{N}$ with $\sum_{a \in A} \frac{1}{a} = \infty$ is also relatively large and it is a long standing conjecture that like sets with positive upper density they have arithmetic progression of arbitrary length. Here we show their difference set may not be substantial; for any $k \in \mathbb{N}$ there exists $A_k \subset \mathbb{N}$ such that $\sum_{a \in A_k} \frac{1}{a} = \infty$ and $d^*(D^{k-1}(A)) = 0$.

Keywords: Difference Set; Density; Δ^*-Set

1. Introduction

A subset *B* of ℕ has null density if

$$d(B) = \lim_{n \to \infty} \frac{|B \cap \{1, \cdots, n\}|}{n} = 0.$$

Perhaps, the most prominent general result in this case is a theorem conjectured by Erdős and proved by Rusza [1]. It states that if $d(B) = 0$ then

$$\lim_{n \to \infty} \frac{|D^1(B) \cap \{1, \cdots, n\}|}{|B \cap \{1, \cdots, n\}|} = \infty \qquad (1)$$

where $D^1(B) := B - B = \{a - b : a > b, a, b \in B\}$.

Our concern here is to consider the null density subsets in the family $\mathcal{F} = \left\{ A \subset \mathbb{N} : \sum_{a \in A} \frac{1}{a} = \infty \right\}$. These sets can be considered "large" sets among null density sets. To have a comparison with sets of higher densities, note that if $A \in \mathcal{F}$ and $n \in \mathbb{N}$, $A - n = \{a - n : a \in A\} \in \mathcal{F}$ and $\mathcal{F}$ has Ramsey property, that is, if $A \in \mathcal{F}$ and *A* is partitioned into finitely many sets then at least one of the elements of the partition lies in $\mathcal{F}$. These two properties, being invariant under translation and having Ramsey property, hold for

$$\mathcal{D} = \left\{ B \subset \mathbb{N} : \bar{d}(B) = \limsup_{n \to \infty} \frac{|A \cap \{1, \cdots, n\}|}{n} > 0 \right\}$$

as well. However, if $B \in \mathcal{D}$ then $D^1(B)$ is a Δ^*-set [2]. This means $D^1(B)$ is both large and structured set. For instance, it is syndetic: there exists a *p* such that $[n, n + p] \cap D^1(B) \neq \varnothing$ for any *n* in ℕ. By definition, a Δ^*-set intersects the difference set of any infinite natural sequence and amongst many structures it is *IP* which means there is a sequence of natural numbers $\{b_n\}_{n \in \mathbb{N}}$ such that all of its finite sums are in *B* [3].

From largeness point of view, $\mathcal{F}$ may be considered next to $\mathcal{D}$ for $\mathcal{D} \subset \mathcal{F}$ [4]. Note that there is also a family

$$\mathcal{D}^* = \left\{ A \subset \mathbb{N} : d^*(A) = \limsup_{m-n \to \infty} \frac{|A \cap \{m, \cdots, n\}|}{m - n} > 0 \right\}$$

which $d^*(A)$ is called the upper Banach density of *A*. We have that $\mathcal{D} \subset \mathcal{D}^*$ and $\mathcal{D}^*$ is invariant under translation and satisfies Ramsey property [5]. Also, if $B \subset \mathcal{D}^*$ then $D^1(B)$ is Δ^* [5] and *B* has arithmetic progression of arbitrary length [6]. However, *B* may not be $\mathcal{F}$-large. For instance, $B = \{2^n, 2^n + 1, \cdots, 2^n + n : n \in \mathbb{N}\} \in \mathcal{D}^*$ but

$$\sum_{b \in B} \frac{1}{b} < \infty.$$

Erdös conjectured that elements of $\mathcal{F}$ have arithmetic progression of arbitrary length [7]. An important subset of natural numbers, which is the set of prime numbers lies in $\mathcal{F}$ and the conjecture was proved positively by

Green-Tao [8].

In order to present a substantial contrast of sets in $\mathcal{F}$ with those in $\mathcal{D}$, we will show that for any $k \in \mathbb{N}$ there is $A \in \mathcal{F}$ such that $D^k(A) = D(D^{k-1}(A))$; the kth difference set of A even does not lie in $\mathcal{D}^*$.

Throughout this note, unless otherwise stated, by an interval we mean an interval of integers. So for instance, $[a,b) = \{x \in \mathbb{Z} : a \leq x < b\}$.

2. Reference Set

Here we introduce and investigate the properties of a subset of non-negative integers suitably defined for our later use.

Let $Y = \{y_0 = 0, y_1, y_2, \cdots\}$ be a set of non-negative integers with the property that $y_n \in \mathbb{N}$, $y_{n+1} > 2y_n$ for any $n \geq 1$. For any $n \in \mathbb{N}$ let l_n be the largest integer strictly less than $\frac{y_{n+1}}{2y_n}$. Set $c_{k_1,\cdots,k_n} := 2k_1 y_1 + 2k_2 y_2 + \cdots + 2k_n y_n$ for all those $0 \leq k_i \leq l_i$, $i = 1, \cdots, n$ such that $2k_1 y_1 + 2k_2 y_2 + \cdots + 2k_n y_n \in [y_0, y_{n+1})$. Also set

$$d_{k_1,\cdots,k_n} := \begin{cases} c_{k_1,\cdots,k_n} + y_1, & \text{if } c_{k_1,\cdots,k_n} + y_1 \leq y_{n+1} \\ y_{n+1}, & \text{otherwise.} \end{cases}$$

Definition 1.1. *Let Y be as above. The set of non-negative integers*

$$R_Y = \bigcup_{n=1}^{\infty} \left(\bigcup_{k_1=0}^{l_1} \bigcup_{k_2=0}^{l_2} \cdots \bigcup_{k_n=0}^{l_n} \left([c_{k_1,\cdots,k_n}, d_{k_1,\cdots,k_n}) \cap [y_0, y_n) \right) \right)$$

is called the reference set *associated to Y.*

The reference set can also be seen as follows. Set $\Omega_n := [y_0, y_n) \cap R_Y$, then

$$\Omega_1 = [0, y_1),$$
$$\Omega_2 = (\Omega_1 \cup \cdots \cup (2k_1 y_1 + \Omega_1) \cup$$
$$\cdots \cup (2l_1 y_1 + \Omega_1)) \cap [0, y_2),$$
$$\Omega_3 = (\Omega_2 \cup \cdots \cup (2k_2 y_2 + \Omega_2) \cup$$
$$\cdots \cup (2l_2 y_2 + \Omega_2)) \cap [0, y_3),$$
$$\vdots$$
$$\Omega_{n+1} = (\Omega_n \cup \cdots \cup (2k_n y_n + \Omega_n) \cup$$
$$\cdots \cup (2l_n y_n + \Omega_n)) \cap [0, y_n)$$
$$\vdots$$

where $2k_i y_i + \Omega_i = \{2k_i y_i + x : x \in \Omega_i\}$.

Let us now examine when $R_Y \setminus \{0\}$ for a given Y belongs to $\mathcal{F}\left(= \left\{ A \subset \mathbb{N} : \sum_{a \in A} \frac{1}{a} = \infty \right\} \right)$.

Lemma 1.2. *Suppose* $Y = \{y_0 = 0, y_1, y_2, \cdots\} \subset \tilde{\mathbb{N}}$ *with* $y_{n+1} > 2y_n$ *and let* R_Y *be its associated reference set. Suppose for sufficiently large n,*

$$\frac{y_{n+2}}{y_{n+1}} > \left(\frac{y_{n+1}}{y_n} \right)^{2+\varepsilon}, \tag{2}$$

for some $\varepsilon > 0$. *Then* $R_Y \setminus \{0\} \in \mathcal{F}$.

Proof. First note that (2) implies

$$\lim_{n \to \infty} \frac{y_{n+1}}{y_n} = \infty; \tag{3}$$

and also if we set

$$\alpha_n := \frac{\ln(y_{n+2}) - \ln(y_{n+1})}{2(\ln(y_{n+1}) - \ln(y_n))} \tag{4}$$

then $\liminf_{n \to \infty} \alpha_n > 1$.

Now let for $n \geq 1$, $I_n := \text{card}([y_{n-1}, y_n] \cap R_Y)$. Then

$$\frac{y_n - 2y_{n-1}}{2y_{n-1}} \left(\text{card}(R_Y \cap [0, y_{n-1})) \right) \leq I_n$$
$$\leq \frac{y_n - y_{n-1}}{2y_{n-1}} \left(\text{card}(R_Y \cap [0, y_{n-1})) \right). \tag{5}$$

But for $0 \leq k_n < l_n$,

$$\text{card}(R_Y \cap [0, y_n)) = \text{card}(R_Y \cap [2k_n y_n, (2k_n + 1) y_n))$$
$$= (I_1 + I_2 + \cdots + I_n).$$

So by estimating any $\frac{1}{r}$, $r \in [2k_n y_n, (2k_n + 1) y_n)$ with $\frac{1}{(2k_n + 1) y_n}$,

$$\sum_{r \in R_Y \cap (0, y_{m+1})} \frac{1}{r} > \sum_{n=1}^{m} e_n \tag{6}$$

where

$$e_n := (I_1 + \cdots + I_n)(1/y_n + 1/3y_n + \cdots + 1/(y_{n+1} - 3y_n)).$$

Note that by (3) for sufficiently large n we have $y_{n+1} - 3y_n > 0$. Now

$$\frac{e_{n+1}}{e_n} = \frac{I_1 + \cdots + I_{n+1}}{I_1 + \cdots + I_n} \times \frac{\dfrac{1}{y_{n+1}} + \dfrac{1}{3y_{n+1}} + \cdots + \dfrac{1}{y_{n+2} - 3y_{n+1}}}{\dfrac{1}{y_n} + \dfrac{1}{3y_n} + \cdots + \dfrac{1}{y_{n+1} - 3y_n}}$$

$$= \left(1 + \frac{I_{n+1}}{I_1 + \cdots + I_n} \right) \times \frac{\dfrac{1}{y_{n+1}} + \dfrac{1}{3y_{n+1}} + \cdots + \dfrac{1}{y_{n+2} - 3y_{n+1}}}{\dfrac{1}{y_n} + \dfrac{1}{3y_n} + \cdots + \dfrac{1}{y_{n+1} - 3y_n}}.$$

Also by (5),

$$I_{n+1} \geq \left(\frac{y_{n+1} - 2y_n}{2y_n} \right) (I_1 + \cdots + I_n).$$

Therefore,

$$\frac{e_{n+1}}{e_n} \geq \left(1 + \frac{y_{n+1} - 2y_n}{2y_n}\right) \times \frac{y_n}{y_{n+1}} \times \frac{\int_1^{\frac{y_{n+2} - 3y_{n+1}}{y_{n+1}}} \frac{1}{2x-1} dx}{1 + \int_1^{\frac{y_{n+1} - 3y_n}{y_n}} \frac{1}{2x-1} dx}$$

$$= \frac{1}{2} \times \frac{\int_1^{\frac{y_{n+2} - 3y_{n+1}}{y_{n+1}}} \frac{1}{2x-1} dx}{1 + \int_1^{\frac{y_{n+1} - 3y_n}{y_n}} \frac{1}{2x-1} dx}.$$

By considering (3) this implies

$$\liminf_{n \to \infty} \frac{e_{n+1}}{e_n} \geq \liminf_{n \to \infty} \left[\frac{1}{2} \times \frac{\ln\left(\frac{y_{n+2}}{y_{n+1}}\right)}{\ln\left(\frac{y_{n+1}}{y_n}\right)}\right] \qquad (7)$$

$$= \liminf_{n \to \infty} \alpha_n > 1.$$

Now the proof is complete by the ratio test.

Let $b \geq 2$, then an example for Y satisfying the above lemma is

$$Y = \left\{0, 2b^{(2b)}, \cdots, (2b)^{(2b)^n}, \cdots\right\}.$$

For this example $\liminf_{n \to \infty} \alpha_n = \lim_{n \to \infty} \alpha_n = b$.

Remark 1.3. *Similar arguments as the proof of the above lemma shows that if either* (3) *does not hold or if* (3) *holds but there is some* $\varepsilon > 0$ *such that for infinitely many n,*

$$\frac{y_{n+2}}{y_{n+1}} < \left(\frac{y_{n+1}}{y_n}\right)^{2-\varepsilon}, \qquad (8)$$

then $\sum_{r \in R_Y} \frac{1}{r} < \infty$. We give a sketch of proof for the latter.

Suppose α_n is defined as (4). Then by (8), we have $\limsup_{n \to \infty} \alpha_n < 1$. Let I_i be as above and set $e_n := (I_1 + \cdots + I_n)\left(\frac{1}{2y_n} + \frac{1}{4y_n} + \cdots + \frac{1}{2l_n y_n}\right)$. Using the left inequality in (5), we have

$$\sum_{r \in R_Y \cap (0, y_m)} \frac{1}{r} \leq I_1 + \sum_{n=1}^{m} e_n.$$

Now

$$\frac{e_{n+1}}{e_n} \leq \frac{y_{n+1} + y_n}{2y_{n+1}} \times \frac{1 + \int_2^{l_{n+1}} \frac{1}{x} dx}{\int_2^{l_n} \frac{1}{x} dx}.$$

From (3), we have $\int_2^{l_n} \frac{1}{x} \to \infty$. This and the fact that $\frac{y_n}{y_{n+1}} \to 0$ implies

$$\limsup_{n \to \infty} \frac{e_{n+1}}{e_n} \leq \limsup_{n \to \infty} \alpha_n < 1$$

and we are done.

3. Main Result

We will show that comparing to a set of positive upper Banach density, the difference set, $D(A) = A - A$ of a set A with $\sum_{a \in A} \frac{1}{a} = \infty$ and $d^*(A) = 0$ can be very sparse. The next theorem gives a class of examples with this property.

Theorem 2.1. *Let* $k \in \mathbb{N}$, *then there exists* $A_k \subset \mathbb{N}$ *with* $\sum_{a \in A_k} \frac{1}{a} = \infty$ *and* $D^k(A_k) \subset A_0$ *such that* A_0 *is not* Δ^*. *In particular,* $d^*(D^{k-1}(A_k)) = 0$.

Proof. Let $b \geq 2$, $y_0 = 0$, $y_1 = 2^{k+1}b$, $y_n = y_1^{y_1^{n-1}}$ and $Y = \{y_0, y_1, y_2, \cdots\}$. Consider the following symmetric subsets of integers:

$$F_0^j = \{0\},$$

$$F_{i-1}^j = \left\{\frac{-y_i}{2^{j+2} y_{i-1}} + 1, \cdots, -1, 0, 1, \cdots, \frac{y_i}{2^{j+2} y_{i-1}} - 1\right\} \qquad (9)$$

for $0 \leq j \leq k$ and $i \geq 2$. Set

$$A_j := \bigcup_{n=0}^{\infty} \left(\bigcup_{a_0 \in F_0^j} \bigcup_{a_1 \in F_1^j} \cdots \bigcup_{a_n \in F_n^j} I_a^j\right)$$

where $\boldsymbol{a} := a_0, a_1, \cdots a_n$ and $I_{a_0, a_1, \cdots, a_n}^j$ is the interval

$$\left[2a_0 y_0 + 2a_1 y_1 + \cdots + 2a_n y_n - \frac{y_1}{2^{j+1}}, \right.$$
$$\left. 2a_0 y_0 + 2a_1 y_1 + \cdots + 2a_n y_n + \frac{y_1}{2^{j+1}}\right)$$

in $\mathbb{N}$. Hence $A_{j+1} \subset A_j$ and A_j is a union of subintervals of non-zero integers with the first subinterval of length $\frac{y_1}{2^{j+1}}$ and all others of length $\frac{y_1}{2^j}$. To have a picture of A_j's consider A_0 which is a prototype for others. Then A_0 may be considered as $\bigcup_{n=1}^{\infty} A_0^n$ in which

$$A_0^1 = A_0 \cap [0, y_1) = \left[0, \frac{y_1}{2}\right) = R_Y \cap \left[y_0, \frac{y_1}{2}\right) - \frac{y_0}{2},$$

$$A_0^2 = A_0 \cap [y_1, y_2) = R_Y \cap \left[y_1, \frac{y_2}{2}\right) - \frac{y_0}{2} - \frac{y_1}{2},$$

$$\vdots$$

$$A_0^{n+1} = A_0 \cap [y_n, y_{n+1}) = R_Y \cap \left[y_n, \frac{y_{n+1}}{2}\right) - \sum_{i=1}^{n} \frac{y_i}{2}$$

$$\vdots$$

That is, A_0 has as half elements as R_Y in $[0, y_{n+1})$ which are shifted to the left appropriately. So

$$2 \sum_{a \in A_0 \cap (0, y_{n+1})} \frac{1}{a} > \sum_{r \in R_Y \cap (0, y_{n+1})} \frac{1}{r}.$$

By Lemma 1.2, one obtains that $\sum_{r \in R_Y \setminus \{0\}} \frac{1}{r}$ and hence $\sum_{a \in A_0 \setminus \{0\}} \frac{1}{a} = \infty$. (Later we will prove that $\sum_{a \in A_k} \frac{1}{a} = \infty$. So divergence of $\sum_{a \in A_0 \setminus \{0\}} \frac{1}{a}$ will be a consequence of that as well).

Now we claim that A_0 is not Δ^*. First note that $\{y_{n+1} - y_i : 1 \leq i \leq n\} \subseteq [y_{n+1} - y_n, y_{n+1}]$. Then the claim is established by noting that by the above definition for A_0

$$A_0 \cap [y_{n+1} - y_n, y_{n+1}] = \varnothing$$

which implies A_0 does not intersect $D^1(Y \setminus \{0\}) = \{y_i - y_j : i > j > 1\}$ and so it is not Δ^*.

The sets F_{i-1}^k's and A_j's are defined in such a way that $D^1(A_j) \subseteq A_{j-1}$, for $1 \leq j \leq k$. To see this, suppose x and y are two elements of A_j and $x > y$. Then there exists $n \in \mathbb{N}$ such that

$$2a_0 y_0 + 2a_1 y_1 + \cdots + 2a_n y_n - \frac{y_1}{2^{j+1}}$$
$$\leq x < 2a_0 y_0 + 2a_1 y_1 + \cdots + 2a_n y_n + \frac{y_1}{2^{j+1}}$$

and also $m \leq n$ with

$$2b_0 y_0 + 2b_1 y_1 + \cdots + 2b_m y_m - \frac{y_1}{2^{j+1}}$$
$$\leq y < 2b_0 y_0 + 2b_1 y_1 + \cdots + 2b_m y_m + \frac{y_1}{2^{j+1}}$$

where $a_i, b_i \in F_{i-1}^j$. So a_i and b_i are in

$$\left[\frac{-y_i}{2^{j+2} y_{i-1}} + 1, \frac{y_i}{2^{j+2} y_{i-1}} - 1 \right] \text{ or equivalently,}$$

$$\frac{-y_i}{2^{j+1} y_{i-1}} + 2 \leq a_i - b_i \leq \frac{y_i}{2^{j+1} y_{i-1}} - 2.$$

Therefore, $x - y \in A_{j-1}$.

To complete the proof, it remains to show that for each A_j, $\sum_{a \in A_j \setminus \{0\}} \frac{1}{a} = \infty$. We already have proved this fact for A_0 and we will prove that for A_k and since $A_k \subset A_j$, $0 \leq j \leq k-1$ we are done. So consider F_i^k in (9) and let

$$F_0'^k := \{0\} = F_0^k,$$

$$F_i'^k := F_i^k + \frac{y_{i+1}}{2^{k+2} y_i} - 1 = \left\{ 0, 1, \cdots, \frac{y_{i+1}}{2^{k+1} y_{i-1}} - 2 \right\}, \ \forall i \geq 1.$$

Also set

$$A_k' := \bigcup_{n=0}^{\infty} \bigcup_{b_0 \in F_0'^j} \cdots \bigcup_{b_n \in F_n'^k} \left(2b_0 y_0 + \cdots + 2b_n y_n, \right.$$
$$\left. 2b_0 y_0 + \cdots + 2b_n y_n + \frac{y_1}{2^k} \right).$$

Then $\sum_{a \in A_k} \frac{1}{a} \geq \sum_{b \in A_k'} \frac{1}{b}$. Now using the same arguments as in Lemma 1.2, we prove that $\sum_{b \in A_k'} \frac{1}{b}$ is divergent.

If we let

$$I_1' := \operatorname{card}([0, y_1) \cap A_k'), \cdots,$$
$$I_n' := \operatorname{card}([y_{n-1}, y_n) \cap A_k'), \cdots,$$

then

$$I_1' = \frac{1}{2^k} y_1, \cdots,$$
$$I_n' \geq \frac{1}{2^k} \left(\frac{y_n - 2y_{n-1}}{2 y_{n-1}} - 2 \right) (I_1' + \cdots + I_{n-1}'), \cdots.$$

Let l_n be as in the definition of R_Y; then

$$\sum_{b \in A_k' \setminus \{0\}} \frac{1}{b} \geq \sum_{n \in \mathbb{N}} e_n'$$

where

$$e_n' := (I_1' + \cdots + I_n') \left(\frac{1}{y_n} + \frac{1}{3 y_n} + \cdots + \frac{1}{2^{-k} (2l_n - 3) y_n} \right)$$
$$= (I_1' + \cdots + I_n') \left(\frac{1}{y_n} + \frac{1}{3 y_n} + \cdots + \frac{1}{2^{-k} (y_{n+1} - 5 y_n)} \right).$$

So

$$\liminf_{n \to \infty} \frac{e_{n+1}'}{e_n'} \geq \lim_{n \to \infty} \frac{I_1' + \cdots + I_{n+1}'}{I_1' + \cdots + I_n'}$$

$$\times \frac{\dfrac{1}{y_{n+1}} + \cdots + \dfrac{1}{2^{-k}(y_{n+2} - 5y_{n+1})}}{\dfrac{1}{y_n} + \cdots + \dfrac{1}{2^{-k}(y_{n+1} - 5y_n)}}$$

$$\geq \liminf_{n \to \infty} \left(\frac{1}{2^{k+1}} \times \frac{\displaystyle \int_2^{\frac{y_{n+2} - 5y_{n+1}}{2^k y_{n+1}}} \frac{1}{2x - 1} dx}{1 + \displaystyle \int_1^{\frac{y_{n+1} - 5y_n}{2^k y_n}} \frac{1}{2x - 1} dx} \right)$$

$$= \frac{1}{2^{k+1}} \times \liminf_{n \to \infty} \frac{y_1^{n+2} - y_1^{n+1}}{y_1^{n+1} - y_1^n} = \frac{y_1}{2^{k+1}} > 1.$$

Let A_k be the set defined in the proof of the above theorem and let $S \subset \mathbb{N}$ be a syndetic set with the larg-

est gap smaller than b. Then the conclusion of the above theorem applies for $A_k \cap S$. This conclusion also holds if we let $Y = \{y_0 = 0, y_1, y_2, \cdots\}$ with $y_1 = e^{k+1} b_1$, $y_n = b_n y_1^{y_1^{p(n)}}$, $2 \leq b_1 \leq \cdots \leq b_n \leq b_{n+1} \leq \cdots$ where e is an even integer and $p(n)$ is an increasing integer valued function with $p(1) \geq 1$.

Also recall that the theorem of Rusza [1] for sets of null density states that the difference set is considerably larger than the set itself, that is, if $d(B) = 0$ then (1) holds. However, in examples such as those in the above theorem, one tries to have a difference set which is as small as possible. Our approach was to have the arithmetic progression of long possible lengths. Therefore, such examples not only do not contradict the Erdös conjecture, but strongly are in the favor of it.

REFERENCES

[1] I. Z. Rusza, "On Difference-Sequences," *Acta Arithmetica*, Vol. 25, 1974, pp. 151-157.

[2] V. Bergelson and N. Hindman, "Additive and Multiplicative Ramsey Theorems in $\mathbb{N}$ -Some Elementary Results," *Combinatorics*, *Probability and Computing*, Vol. 2, 1993, pp. 221-241. doi:10.1017/S0963548300000638

[3] V. Bergelson, "Partition Regular Structures Contained in Large Sets Are Abundant," *Journal of Combinatorial Theory*, *Series A*, Vol. 93, No. 1, 2001, pp. 18-36. doi:10.1006/jcta.2000.3061

[4] T. C. Brown and A. R. Freedman, "Arithmetic Progressions in Lacunary Sets," *Rocky Mountain Journal of Mathematics*, Vol. 17, No. 3, 1987, pp. 587-596. doi:10.1216/RMJ-1987-17-3-587

[5] V. Bergelson, N. Hindman and R. McCutchen, "Notions of Size and Combinatorial Properties of Quotient Sets in Semigroups," *Topology Proceedings*, Vol. 23, 1998, pp. 23-60.

[6] E. Szemerdi, "On Sets of Integers Containing No k Elements in Arithmetic Progression," *Acta Arithmetica*, Vol. 27, 1975, pp. 199-245.

[7] P. Erdös, "Problems and Results in Combinatorial Number Theory," *Astrisque*, 1975, pp. 295-310.

[8] B. Green and T. Tao, "The Primes Contain Arbitrarily Long Arithmetic Progressions," *Annals of Mathematics*, Vol. 167, No. 2, 2004, pp. 481-547. doi:10.4007/annals.2008.167.481