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Tensor Product of Krammer's Representations of the Pure Braid Group, P_3

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ABSTRACT

We consider the complex specializations of Krammer's representation of the pure braid group on three strings, namely, $K(q,t): P_3 \to GL(3,\mathbb{C})$, where q and t are non-zero complex numbers. We then specialize the indeterminate t by one and replace $K(q,1)$ by $K(q)$ for simplicity. Then we present our main theorem that gives us sufficient conditions that guarantee the irreducibility of the tensor product of two irreducible complex specializations of Krammer's representations $K(q_1)\otimes K(q_2): P_3 \to GL(\mathbb{C}^3 \otimes \mathbb{C}^3)$.

Keywords: Braid Group; Pure Braid Group; Magnus Representation; Krammer's Representation

1. Introduction

Let B_n be the braid group on n strings. It has many kinds of linear representations. The earliest was the Artin representation, which is an embedding $B_n \to Aut(F_n)$. Applying the free differential calculus to elements of $Aut(F_n)$ sometimes gives rise to linear representations of B_n and its normal subgroup, the pure braid group denoted by P_n. The Lawrence-Krammer representation named after Ruth Lawrence and Daan Krammer arises this way. Krammer's representation is a representation of the braid group B_n in $GL\left(m,\mathbb{Z}\left[t^{\pm 1},q^{\pm 1}\right]\right) = Aut(V_0)$, where $m = n(n-1)/2$ and V_0 is the free module of rank m over $\mathbb{Z}\left[t^{\pm 1},q^{\pm 1}\right]$. It is denoted by $K(q,t)$. For simplicity, we write K instead of $K(q,t)$. In previous work, we considered Krammer's representations of B_3 and P_3 and we specialized the indeterminates to non zero complex numbers. We then found necessary and sufficient conditions that guarantee the irreducibility of such representations. For more details, see [1,2]. Note that in a previous work of Abdulrahim and Al-Tahan [2], a necessary and sufficient condition for the irreducibility of Krammer's representation of degree three was found. However, in our current work, we are dealing with a representation of higher degree (degree nine) and which also has two indeterminates. This made our work seem more difficult. For this reason, we had to be satisfied in this current work with only a sufficient condition for irreducibility, so we fell short of finding a necessary and sufficient condition for irreducibility. To make computations easier, we had to specialize the indeterminate t by one in order to have a one parameter complex specialization.

In Section 2, we introduce the pure braid group and Krammer's representation. In Section 3, we present our main theorem, Theorem 1, which gives sufficient conditions that guarantee the irreducibility of the tensor product of two irreducible complex specializations of Krammer's representations of P_3. In this way, we will have succeeded in constructing a representation of the pure braid group, P_3, of degree nine and which is also irreducible.

2. Definitions

Definition 1. [3] *The braid group on n strings, B_n, is the abstract group with presentation*

$$B_n = \left\{ \begin{array}{l} \sigma_1,\cdots,\sigma_{n-1} / \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \text{for } i = 1,2,\cdots,n-2, \\ \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j|>1 \end{array} \right\}.$$

The generators $\sigma_1,\cdots,\sigma_{n-1}$ are called the standard generators of B_n.

Definition 2. *The Pure braid group on n strands, denoted by P_n, is the kernel of the group homomorphism $B_n \to S_n$. It consists of those braids which connect the ith item of the left set to the ith item of the right set for all i. It is generated by the generators A_{ij}, $1 \le i < j \le n$*

where $A_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$.

Definition 3. [4] *The image of each Artin generator under Krammer's representation with respect to* $\{x_{ij}\}_{1\le i<j\le n}$, *the free basis of* V_0, *is written as follows*:

$$K(\sigma_k)(x_{i,j}) = \begin{cases} tq^2 x_{k,k+1}, & i=k,\ j=k+1;\\ (1-q)x_{i,k}+qx_{i,k+1}, & j=k, i<k;\\ x_{i,k}+tq^{k-i+1}(q-1)x_{k,k+1}, & j=k+1, i<k;\\ tq(q-1)x_{k,k+1}+qx_{k+1,j}, & i=k, k+1<j;\\ x_{k,j}+(1-q)x_{k+1,j}, & i=k+1, k+1<j;\\ x_{i,j}, & i<j<k \text{ or } k+1<i<j;\\ x_{i,j}+tq^{k-i}(q-1)^2 x_{k,k+1}, & i<k<k+1<j. \end{cases}$$

Now, we determine Krammer's representation

$$K(q,t): P_3 \to GL\left(3,\mathbb{Z}\left[t^{\pm1},q^{\pm1}\right]\right)$$

using the Magnus representation of subgroups of the automorphisms group of free group with $n(n-1)/2$ generators, where $\mathbb{Z}\left[t^{\pm1},q^{\pm1}\right]$ is the ring of Laurent polynomials on two variables. The images of the generators under Krammer's representation are as follows

$$K(A_{1,2}) = \begin{pmatrix} t^2q^4 & 0 & 0\\ t^2q^3(q-1) & q & q(1-q)\\ tq(q-1) & 1-q & 1-q+q^2 \end{pmatrix},$$

$$K(A_{2,3}) = \begin{pmatrix} 1-q+q^2 & q(1-q) & tq^3(q-1)\\ 1-q & q & t^2q^4(q-1)\\ 0 & 0 & t^2q^4 \end{pmatrix}$$

and

$$K(A_{1,3}) = \begin{pmatrix} q & q(q-1) & \dfrac{1-q-tq(q-1)^2}{t}\\ -tq(q-1)^2 & tq\left[tq^2(q^2-q+1)-(q-1)^3\right] & m\\ tq(1-q) & tq(q-1)(1-q+tq^2) & n \end{pmatrix},$$

where

$$m = -1+q\left[2-2q+q^2+t(q-1)^4+q^2(1-q)(1+q(q-1))t^2\right]$$

and

$$n = 1+q(q-1)\left[1+t(q-1)(-1+q-tq^2)\right].$$

Specializing the indeterminates q and t to nonzero complex numbers gives a representation $K(q,t): P_3 \to GL(3,\mathbb{C})$ which is irreducible if and only if

$$t^2q^3 \ne 1,\ tq^3 \ne 1,\ t\ne -1,\ q\ne 1,\ tq\ne 1 \text{ and } tq^2 \ne -1.$$

For more details, see [2].

We now specialize the indeterminate t by one. We then write $K(q)$ for $K(q,1)$. So, $K(q)$ is irreducible if and only if

$$q^3 \ne 1 \text{ and } q^2 \ne -1.$$

By replacing t by one, the matrices of the generators become as follows:

$$\boldsymbol{K}(q)(A_{1,2}) = \begin{pmatrix} q^4 & 0 & 0\\ q^3(q-1) & q & q(1-q)\\ q(q-1) & 1-q & 1-q+q^2 \end{pmatrix},$$

$$\boldsymbol{K}(q)(A_{2,3}) = \begin{pmatrix} 1-q+q^2 & q(1-q) & q^3(q-1)\\ 1-q & q & q^4(q-1)\\ 0 & 0 & q^4 \end{pmatrix}$$

and

$$\boldsymbol{K}(q)(A_{1,3}) = \begin{pmatrix} q & q(q-1) & 1-q-q(q-1)^2\\ -q(q-1)^2 & q\left[q^2(q^2-q+1)-(q-1)^3\right] & m\\ q(1-q) & q(q-1)(1-q+q^2) & n \end{pmatrix},$$

where

$$m=-1+q\left[2-2q+q^2+(q-1)^4+q^2(1-q)\left(1+q(q-1)\right)\right]$$

and

$$n=1+q(q-1)\left[1+(q-1)\left(-1+q-q^2\right)\right].$$

In our work, we consider the tensor product of two irreducible complex specializations of Krammer's representations of the pure braid group, namely, $K(q_1)\otimes K(q_2)$.

3. Sufficient Conditions for Irreducibility

In this section, we find sufficient conditions that guarantee the irreducibility of the tensor product of two ireducible complex specializations of Krammer's representations of the pure braid group on three strings, P_3.

Theorem 1. *For* $(q_1,q_2)\in\left(\mathbb{C}^*\right)^2$, *the tensor product of two irreducible complex specializations of Krammer's representations* $K(q_1)\otimes K(q_2):P_3\to GL\left(\mathbb{C}^3\otimes\mathbb{C}^3\right)$ *is irreducible if*

$$q_1^4q_2^4\neq 1,\ q_1^4\neq q_2^4,\ q_1^2q_2^4\neq 1,\ q_1^4q_2^2\neq 1,\ q_1^2\neq q_2^4,$$
$$q_1^4\neq q_2^2,\ q_1^4\neq 1,\ q_2^4\neq 1,\ q_1^3\neq 1,\ q_2^3\neq 1.$$

Proof. For simplicity, we write A_{ij} to mean $A_{ij}(q_1)\otimes A_{ij}(q_2)$. Let us diagonalize the matrix $\boldsymbol{A}_{23}$ by an invertible matrix $\boldsymbol{T}$, and conjugate the matrix $\boldsymbol{A}_{12}$ by $\boldsymbol{T}$. The invertible matrix $\boldsymbol{T}$ is given by

$$\boldsymbol{T}=\begin{pmatrix}
1 & -q_1 & a & -q_2 & q_1q_2 & -aq_2 & c & -cq_1 & ac\\
1 & -q_1 & a & 1 & -q_1 & a & d & -dq_1 & \dfrac{q_1^2q_2\left(-1+q_2+q_2^2\right)}{(1+q_1)^2(1+q_2)^2}\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -q_1 & a\\
1 & 1 & b & -q_2 & -q_2 & \dfrac{-q_1\left(-1+q_1+q_1^2\right)q_2}{(1+q_1)^2} & c & c & \dfrac{q_1\left(-1+q_1+q_1^2\right)q_2^2}{(1+q_1)^2(1+q_2)^2}\\
1 & 1 & b & 1 & 1 & b & d & d & e\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & b\\
0 & 0 & 1 & 0 & 0 & -q_2 & 0 & 0 & c\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & d\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

where

$$a=\frac{q_1^2}{(1+q_1)^2},\ b=-1+q_1+\frac{1}{(1+q_1)^2},\ c=\frac{q_2^2}{(1+q_2)^2},$$
$$d=-1+q_2+\frac{1}{(1+q_2)^2}\text{ and }e=\frac{q_1\left(-1+q_1+q_1^2\right)q_2\left(-1+q_2+q_2^2\right)}{(1+q_1)^2(1+q_2)^2}.$$

Diagonalizing $\boldsymbol{A}_{23}$ by $\boldsymbol{T}$, we get that

$$\boldsymbol{A}_{23}=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & q_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & q_1^4 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & q_2^2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & q_1^2q_2^2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & q_1^4q_2^2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & q_2^4 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1^2q_2^4 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1^4q_2^4
\end{pmatrix}.$$

Conjugating $\boldsymbol{A}_{12}$ by $\boldsymbol{T}$, we get that $\boldsymbol{A}_{12} = \left(a_{ij}\right)$, where the entries of its first row and first column are given by

$$a_{11} = q_1^2\left(1+q_1^2\right)^2 q_2^2\left(1+q_2^2\right)^2 \Big/ \left(1+q_1\right)^2\left(1+q_2\right)^2,$$

$$a_{12} = -q_1^2\left(-1+q_1^2\left(-1+q_1+q_1^3\right)\right) q_2^2\left(1+q_2^2\right)^2 \Big/ \left(1+q_1\right)^2\left(1+q_2\right)^2,$$

$$a_{13} = q_1^2\left(-1+q_1^3\right)^2 q_2^2\left(1+q_2^2\right)^2 \Big/ \left(1+q_1\right)^4\left(1+q_2\right)^2,$$

$$a_{14} = -q_1^2\left(1+q_1^2\right)^2 q_2^2\left(-1+q_2^2\left(-1+q_2+q_2^3\right)\right) \Big/ \left(1+q_1\right)^2\left(1+q_2\right)^2,$$

$$a_{15} = q_1^2\left(-1+q_1^2\left(-1+q_1+q_1^3\right)\right) q_2^2\left(-1+q_2^2\left(-1+q_2+q_2^3\right)\right) \Big/ \left(1+q_1\right)^2\left(1+q_2\right)^2,$$

$$a_{16} = -q_1^2\left(-1+q_1^3\right)^2 q_2^2\left(-1+q_2^2\left(-1+q_2+q_2^3\right)\right) \Big/ \left(1+q_1\right)^4\left(1+q_2\right)^2,$$

$$a_{17} = q_1^2\left(1+q_1^2\right)^2 q_2^2\left(-1+q_2^3\right)^2 \Big/ \left(1+q_1\right)^2\left(1+q_2\right)^4,$$

$$a_{18} = -q_1^2\left(-1+q_1^2\left(-1+q_1+q_1^3\right)\right) q_2^2\left(-1+q_2^3\right)^2 \Big/ \left(1+q_1\right)^2\left(1+q_2\right)^4,$$

$$a_{19} = q_1^2\left(-1+q_1^3\right)^2 q_2^2\left(-1+q_2^3\right)^2 \Big/ \left(1+q_1\right)^4\left(1+q_2\right)^4,$$

$$a_{21} = -2\left(-1+q_1\right)q_1\left(1+q_1^2\right)\left(q_2+q_2^3\right)^2 \Big/ \left(1+q_1\right)^2\left(1+q_2\right)^2,$$

$$a_{31} = \left(-1+q_1\right)^2 q_2^2\left(1+q_2^2\right)^2 \Big/ \left(1+q_2\right)^2,$$

$$a_{41} = -2\left(q_1+q_1^3\right)^2\left(-1+q_2\right)q_2\left(1+q_2^2\right) \Big/ \left(1+q_1\right)^2\left(1+q_2\right)^2,$$

$$a_{51} = 4\left(-1+q_1\right)q_1\left(1+q_1^2\right)\left(-1+q_2\right)q_2\left(1+q_2^2\right) \Big/ \left(1+q_1\right)^2\left(1+q_2\right)^2,$$

$$a_{61} = -2\left(-1+q_1\right)^2\left(-1+q_2\right)q_2\left(1+q_2^2\right) \Big/ \left(1+q_2\right)^2,$$

$$a_{71} = q_1^2\left(1+q_1^2\right)^2\left(-1+q_2\right)^2 \Big/ \left(1+q_1\right)^2,$$

$$a_{81} = -2\left(-1+q_1\right)q_1\left(1+q_1^2\right)\left(-1+q_2\right)^2 \Big/ \left(1+q_1\right)^2,$$

$$a_{91} = \left(-1+q_1\right)^2\left(-1+q_2\right)^2.$$

Suppose, to get contradiction, that $K\left(q_1\right)\otimes K\left(q_2\right): P_3 \to GL\left(\mathbb{C}^3\otimes\mathbb{C}^3\right)$ is reducible, then there exists a proper nonzero invariant subspace S, where the dimension of S is between one and eight inclusively. By considering such an equivalent representation (after conjugation), we will easily see that this assumption leads to contradiction.

Since the eigen values of $\boldsymbol{A}_{23}$ are distinct by our hypothesis, it follows that $S = \left\langle e_{l_1}, e_{l_2}, \cdots, e_{l_s}\right\rangle$ where $l_\alpha \in \{1,\cdots,9\}$, $1 \le \alpha \le s \le 8$ and $e'_{l_\alpha}s$ are the standard unit vectors of $\mathbb{C}^9$.

Case 1. $e_1 \in \left\{e_{l_1}, e_{l_2}, \cdots, e_{l_s}\right\}$.

Since $1 \le \dim S \le 8$, it follows that there exists $e_i \notin \left\{e_{l_1}, e_{l_2}, \cdots, e_{l_s}\right\}$ where $i \in \{2,\cdots,9\}$. Since $e_1 \in S$, it follows that $A_{12}\left(e_1\right) \in S$ which implies that $a_{i1} = 0$, a contradiction since $a_{i1} \ne 0$ $\forall i \in \{2,\cdots,9\}$ by our hypothesis.

Case 2. $e_1 \notin \left\{e_{l_1}, e_{l_2}, \cdots, e_{l_s}\right\}$.

Since $1 \le \dim S \le 8$, it follows that there exists $e_i \in \left\{e_{l_1}, e_{l_2}, \cdots, e_{l_s}\right\}$ where $i \in \{2,\cdots,9\}$ and so

$A_{12}(e_1) \in S$ which implies that $a_{1i} = 0$, a contradiction since $a_{1i} \neq 0 \quad \forall i \in \{2, \cdots, 9\}$.

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