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Fock Spaces for the *q*-Dunkl Kernel

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ABSTRACT

In this work, we introduce a class of Hilbert spaces $\mathcal{F}_{q,\alpha}$ of entire functions on the disk $D\left(0,\frac{1}{1-q}\right)$, $0<q<1$, with reproducing kernel given by the *q*-Dunkl kernel $E_\alpha\left(z;q^2\right)$. The definition and properties of the space $\mathcal{F}_{q,\alpha}$ extend naturally those of the well-known classical Fock space. Next, we study the multiplication operator Q by z and the *q*-Dunkl operator $\Lambda_{q,\alpha}$ on the Fock space $\mathcal{F}_{q,\alpha}$; and we prove that these operators are adjoint-operators and continuous from this space into itself.

Keywords: Generalized *q*-Fock Spaces; *q*-Dunkl Kernel; *q*-Dunkl Operator; *q*-Translation Operators

1. Introduction

Fock space $\mathcal{F}$ (called also Segal-Bargmann space [1]) is the Hilbert space of entire functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ on $\mathbb{C}$ such that

$$\|f\|_{\mathcal{F}}^2:=\sum_{n=0}^{\infty}|a_n|^2\,n!<\infty.$$

This space was introduced by Bargmann in [2] and it was the aim of many works [1]. Especially, the differential operator $D=\mathrm{d}/\mathrm{d}z$ and the multiplication operator by z are densely defined, closed and adjoint-operators on $\mathcal{F}$ (see [2]).

In [3], Sifi and Soltani introduced a Hilbert space $\mathcal{F}_\alpha$ of entire functions on $\mathbb{C}$, where the inner product is weighted by the modified Macdonald function. On $\mathcal{F}_\alpha$ the Dunkl operator

$$\Lambda_\alpha f(z):=\frac{\mathrm{d}}{\mathrm{d}z}f(z)+\frac{2\alpha+1}{z}\left[\frac{f(z)-f(-z)}{2}\right],$$

$$\alpha>-1/2,$$

and the multiplication by z are densely defined, closed and adjoint-operators.

In this paper, we consider the *q*-Dunkl kernel:

$$E_\alpha\left(x;q^2\right):=\sum_{n=0}^{\infty}\frac{x^n}{b_n\left(\alpha;q^2\right)},$$

where $b_n\left(\alpha;q^2\right)$ are given later in Section 2. We discuss some properties of a class of Fock spaces associated to the *q*-Dunkl kernel and we give some applications.

In this work, building on the ideas of Bargmann and Cholewinski [4], we define the *q*-Fock space $\mathcal{F}_{q,\alpha}$ as the space of entire functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ on the disk $D\left(0,\frac{1}{1-q}\right)$ of center 0 and radius $\frac{1}{1-q}$, and such that

$$\|f\|_{\mathcal{F}_{q,\alpha}}^2:=\sum_{n=0}^{\infty}|a_n|^2\,b_n\left(\alpha;q^2\right)<\infty.$$

Let f and g be in $\mathcal{F}_{q,\alpha}$, such that $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}c_nz^n$, the inner product is given by

$$\langle f,g\rangle_{\mathcal{F}_{q,\alpha}}=\sum_{n=0}^{\infty}a_n\overline{c_n}b_n\left(\alpha;q^2\right).$$

The *q*-Fock space $\mathcal{F}_{q,\alpha}$ has also a reproducing kernel $\mathcal{K}_{q,\alpha}$ given by

$$\mathcal{K}_{q,\alpha}(w,z)=E_\alpha\left(\overline{w}z;q^2\right);\quad w,z\in D\left(0,\frac{1}{1-q}\right).$$

Then if $f\in\mathcal{F}_{q,\alpha}$, we have

$$\left\langle f,\mathcal{K}_{q,\alpha}(w,.)\right\rangle_{\mathcal{F}_{q,\alpha}}=f(w),\quad w\in D\left(0,\frac{1}{1-q}\right).$$

Using this property, we prove that the space $\mathcal{F}_{q,\alpha}$ is a Hilbert space and we give an Hilbert basis.

Next, using the previous results, we consider the multiplication operator Q by z and the *q*-Dunkl operator $\Lambda_{q,\alpha}$ on the Fock space $\mathcal{F}_{q,\alpha}$, and we prove that these

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operators are continuous from $\mathcal{F}_{q,\alpha}$ into itself, and satisfy:

$$\|\Lambda_{q,\alpha} f\|_{\mathcal{F}_{q,\alpha}} \le C_{q,\alpha} |f|_{\mathcal{F}_{q,\alpha}},$$
$$\|Qf\|_{\mathcal{F}_{q,\alpha}} \le C_{q,\alpha} |f|_{\mathcal{F}_{q,\alpha}},$$

where $C_{q,\alpha}$ is a constant independent of *f*.

Then, we prove that these operators are adjoint-operators on $\mathcal{F}_{q,\alpha}$:

$$\langle Qf, g\rangle_{\mathcal{F}_{q,\alpha}} = \langle f, \Lambda_{q,\alpha} g\rangle_{\mathcal{F}_{q,\alpha}}; \quad f, g \in \mathcal{F}_{q,\alpha}.$$

Lastly, we define and study on the Fock space $\mathcal{F}_{q,\alpha}$, the *q*-translation operators:

$$T_z f(w) := E_\alpha(z\Lambda_{q,\alpha}; q^2) f(w); \quad w, z \in D\left(0, \frac{1}{1-q}\right),$$

and the generalized multiplication operators:

$$M_z f(w) := E_\alpha(zQ; q^2) f(w); \quad w, z \in D\left(0, \frac{1}{1-q}\right).$$

Using the continuous properties of $\Lambda_{q,\alpha}$ and Q we deduce also that the operators T_z and M_z, for $|z| < \frac{1}{(1-q)(2-q^{2\alpha+1})}$, are continuous from $\mathcal{F}_{q,\alpha}$ into itself, and satisfy:

$$\|T_z f\|_{\mathcal{F}_{q,\alpha}} \le E_\alpha(C_{q,\alpha}|z|; q^2)\|f\|_{\mathcal{F}_{q,\alpha}},$$
$$\|M_z f\|_{\mathcal{F}_{q,\alpha}} \le E_\alpha(C_{q,\alpha}|z|; q^2)\|f\|_{\mathcal{F}_{q,\alpha}}.$$

2. Preliminaries and the *q*-Fock Spaces $\mathcal{F}_{q,\alpha}$

Let a and q be real numbers such that $0 < q < 1$; the *q*-shifted factorial are defined by

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{i=0}^{n-1}(1 - aq^i), \quad n = 1, 2, \cdots, \infty.$$

Jackson [5] defined the *q*-analogue of the Gamma function as

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \ne 0, -1, -2, \cdots$$

It satisfies the functional equation

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x), \quad \Gamma_q(1) = 1,$$

and tends to $\Gamma(x)$ when q tends to 1^-. In particular, for $n = 1, 2, \cdots$, we have

$$\Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n}.$$

The *q*-derivative $D_q f$ of a suitable function *f* (see [6]) is given by

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \ne 0,$$

and $D_q f(0) = f'(0)$ provided $f'(0)$ exists.

If *f* is differentiable then $D_q f(x)$ tends to $f'(x)$ as $q \to 1^-$.

Taking account of the paper [3] and the same way, we define the *q*-Dunkl kernel by

$$E_\alpha(x;q^2) := I_\alpha(x;q^2) + \frac{x}{(1+q)[\alpha+1]_{q^2}} I_{\alpha+1}(x;q^2),$$

where $I_\alpha(x;q^2)$ is the *q*-modified Bessel function [7,8] given by

$$I_\alpha(x;q^2) := \Gamma_{q^2}(\alpha+1)\sum_{n=0}^{\infty}\frac{x^{2n}}{(1+q)^{2n}\Gamma_{q^2}(n+1)\Gamma_{q^2}(n+\alpha+1)}.$$

Furthermore, the Dunkl kernel $E_\alpha(x;q^2)$ can be expanded in a power series in the form

$$E_\alpha(x;q^2) := \sum_{n=0}^{\infty}\frac{x^n}{b_n(\alpha;q^2)}, \tag{1}$$

where

$$b_{2n}(\alpha;q^2) := \frac{(1+q)^{2n}\Gamma_{q^2}(n+1)\Gamma_{q^2}(n+\alpha+1)}{\Gamma_{q^2}(\alpha+1)},$$

and

$$b_{2n+1}(\alpha;q^2) := \frac{(1+q)^{2n+1}\Gamma_{q^2}(n+1)\Gamma_{q^2}(n+\alpha+2)}{\Gamma_{q^2}(\alpha+1)}.$$

If we put $U_n := \frac{1}{b_n(\alpha;q^2)}$, then

$$\frac{U_n}{U_{n+1}} \to \frac{1}{(1-q)^2}, \quad q \to 1^-.$$

Thus, the *q*-Dunkl kernel $E_\alpha(x;q^2)$ is defined on $D\left(0, \frac{1}{(1-q)^2}\right)$ and tends to the Dunkl kernel $E_\alpha(x)$ as $q \to 1^-$.

We consider the *q*-Dunkl operator operator $\Lambda_{q,\alpha}$ defined by

$$\Lambda_{q,\alpha} f(x) := D_q f(x) + \frac{[2\alpha+1]_q}{x}\left[\frac{f(qx) - f(-qx)}{2}\right],$$

where

$$[2\alpha+1]_q := \frac{1-q^{2\alpha+1}}{1-q}.$$

The q-Dunkl operator tends to the Dunkl operator Λ_α as $q \to 1^-$.

Lemma 1. *The function* $E_\alpha(\lambda.;q^2)$, $\lambda \in D\left(0, \frac{1}{1-q}\right)$, *is the unique analytic solution of the q-problem*:

$$\Lambda_{q,\alpha} y(x) = \lambda y(x), \quad y(0) = 1. \qquad (2)$$

Proof. Searching a solution of (2) in the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$D_q y(x) = \sum_{n=1}^{\infty} a_n [n]_q x^{n-1}.$$

Replacing in (2), we obtain

$$\sum_{n=1}^{\infty} a_n \left[[n]_q + q^n [2\alpha+1]_q \left(\frac{1-(-1)^n}{2} \right) \right] x^n = \lambda \sum_{n=1}^{\infty} a_{n-1} x^n.$$

Thus,

$$a_n \left[[n]_q + q^n [2\alpha+1]_q \left(\frac{1-(-1)^n}{2} \right) \right] = \lambda a_{n-1}, \quad n = 1, 2, \cdots.$$

Using the fact that $[2n+1]_q + q^{2n+1}[2\alpha+1]_q = [2n+2\alpha+2]_q$, we deduce that

$$a_{2n} = \frac{\lambda}{[2n]_q} a_{2n-1} \text{ and } a_{2n+1} = \frac{\lambda}{[2n+2\alpha+2]_q} a_{2n}.$$

We get

$$a_{2n+1} = \frac{\lambda^2}{[2n]_q [2n+2\alpha+2]_q} a_{2n-1}.$$

Since $[2n]_q = (1+q)[n]_{q^2}$, we deduce

$$a_{2n+1} = \frac{\lambda^2}{(1+q)^2 [n]_{q^2} [n+\alpha+1]_{q^2}} a_{2n-1}.$$

This proves that

$$a_{2n+1} = \frac{\lambda^{2n+1} \Gamma_{q^2}(\alpha+1)}{(1+q)^{2n+1} \Gamma_{q^2}(n+1) \Gamma_{q^2}(n+\alpha+2)},$$

and

$$a_{2n} = \frac{\lambda^{2n} \Gamma_{q^2}(\alpha+1)}{(1+q)^{2n} \Gamma_{q^2}(n+1) \Gamma_{q^2}(n+\alpha+1)}.$$

Therefore,

$$y(x) = \Gamma_{q^2}(\alpha+1) \sum_{n=0}^{\infty} \frac{(\lambda x)^{2n}}{(1+q)^{2n} \Gamma_{q^2}(n+1) \Gamma_{q^2}(n+\alpha+1)}$$
$$+ \Gamma_{q^2}(\alpha+1) \sum_{n=0}^{\infty} \frac{(\lambda x)^{2n+1}}{(1+q)^{2n+1} \Gamma_{q^2}(n+1) \Gamma_{q^2}(n+\alpha+2)}$$
$$= I_\alpha(\lambda x; q^2) + \frac{\lambda x}{(1+q)[\alpha+1]_{q^2}} I_{\alpha+1}(\lambda x; q^2),$$

which completes the proof of the lemma. □

Lemma 2. *The constants* $b_n(\alpha;q^2)$, $n \in \mathbb{N}$ *satisfy the following relations*:

1) $b_{n+1}(\alpha;q^2)$
$$= \left[[n+1]_q + q^{n+1} [2\alpha+1]_q \left(\frac{1+(-1)^n}{2} \right) \right] b_n(\alpha;q^2),$$

2) $b_{2n+1}(\alpha;q^2) = (1+q)[\alpha+1]_{q^2} b_{2n}(\alpha+1;q^2)$,

3) $b_n(\alpha;q^2)$
$$:= \frac{(1+q)^n \Gamma_{q^2}([n/2]+1) \Gamma_{q^2}\left(\left[\frac{n+1}{2} \right] + \alpha + 1 \right)}{\Gamma_{q^2}(\alpha+1)},$$

where $[n/2]$ is the integer part of $n/2$.

Lemma 3. *For* $k \in \mathbb{N}$, *we have*

$$\Lambda_{q,\alpha} z^k = \frac{b_k(\alpha;q^2)}{b_{k-1}(\alpha;q^2)} z^{k-1}, \quad k \geq 1.$$

Proof. Since

$$E_\alpha(\lambda z; q^2) := \sum_{k=0}^{\infty} \frac{(\lambda z)^k}{b_k(\alpha;q^2)},$$

then from Equation (2) we obtain

$$\sum_{k=1}^{\infty} \frac{\Lambda_{q,\alpha} z^k}{b_k(\alpha;q^2)} \lambda^k = \sum_{k=1}^{\infty} \frac{z^{k-1}}{b_{k-1}(\alpha;q^2)} \lambda^k.$$

This clearly yields the result. □

Definition 1. Let $\alpha \geq -1/2$. The q-Fock space $\mathcal{F}_{q,\alpha}$ is the prehilbertian space of entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $D\left(0, \frac{1}{1-q}\right)$, such that

$$\|f\|^2_{\mathcal{F}_{q,\alpha}} := \sum_{n=0}^{\infty} |a_n|^2 b_n(\alpha;q^2) < \infty, \qquad (3)$$

where $b_n(\alpha;q^2)$ is given by (1).

The inner product in $\mathcal{F}_{q,\alpha}$ is given for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} c_n z^n$ by

$$\langle f,g\rangle_{\mathcal{F}_{q,\alpha}}=\sum_{n=0}^{\infty}a_n\overline{c_n}b_n\left(\alpha;q^2\right). \quad (4)$$

Remark 1. If $q\to 1^-$, the space $\mathcal{F}_{q,\alpha}$ agrees with the generalized Fock space associated to the Dunkl operator (see [3]).

Proposition 1. *For* $f,g\in\mathcal{F}_{q,\alpha}$ *, we have*

$$\langle f,g\rangle_{\mathcal{F}_{q,\alpha}}=f\left(\Lambda_{q,\alpha}\right)\tilde{g}(0),\quad \tilde{g}(z)=\overline{g(\bar{z})}.$$

Proof. Given $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{F}_{q,\alpha}$ and $g(z)=\sum_{n=0}^{\infty}c_nz^n\in\mathcal{F}_{q,\alpha}$. Since from Lemma 3,

$$\Lambda_{q,\alpha}^n z^k=\frac{b_k\left(\alpha;q^2\right)}{b_{k-n}\left(\alpha;q^2\right)}z^{k-n},\quad k\ge n, \quad (5)$$

we can write

$$g(z)=\sum_{n=0}^{\infty}\frac{\Lambda_{q,\alpha}^n g(0)}{b_n\left(\alpha;q^2\right)}z^n. \quad (6)$$

Using (4) and (6), we get

$$\langle f,g\rangle_{\mathcal{F}_{q,\alpha}}=\sum_{n=0}^{\infty}a_n\overline{\Lambda_{q,\alpha}^n g(0)}=\sum_{n=0}^{\infty}a_n\Lambda_{q,\alpha}^n\tilde{g}(0).$$

Thus

$$\langle f,g\rangle_{\mathcal{F}_{q,\alpha}}=f\left(\Lambda_{q,\alpha}\right)\tilde{g}(0),$$

which gives the desired result. □

The following theorem proves that $\mathcal{F}_{q,\alpha}$ is a reproducing kernel space.

Theorem 1. *The function* $\mathcal{K}_{q,\alpha}$ *given for* $w,z\in D\left(0,\frac{1}{1-q}\right)$, *by*

$$\mathcal{K}_{q,\alpha}(w,z)=E_\alpha\left(\bar{w}z;q^2\right),$$

is a reproducing kernel for the *q*-Fock space $\mathcal{F}_{q,\alpha}$, that is:

1) For all $w\in D\left(0,\frac{1}{1-q}\right)$, the function $z\to\mathcal{K}_{q,\alpha}(w,z)$ belongs to $\mathcal{F}_{q,\alpha}$.

2) For all $w\in D\left(0,\frac{1}{1-q}\right)$ and $f\in\mathcal{F}_{q,\alpha}$, we have

$$\left\langle f,\mathcal{K}_{q,\alpha}(w,.)\right\rangle_{\mathcal{F}_{q,\alpha}}=f(w).$$

Proof. 1) Since

$$\mathcal{K}_{q,\alpha}(w,z)=\sum_{n=0}^{\infty}\frac{\bar{w}^n}{b_n\left(\alpha;q^2\right)}z^n;\quad z,w\in D\left(0,\frac{1}{1-q}\right), \quad (7)$$

then from (3), we deduce that

$$\left\|\mathcal{K}_{q,\alpha}(w,.)\right\|_{\mathcal{F}_{q,\alpha}}^2=\sum_{n=0}^{\infty}\frac{|w|^{2n}}{b_n\left(\alpha;q^2\right)}=E_\alpha\left(|w|^2;q^2\right)<\infty,$$

which proves 1).

2) If $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{F}_{q,\alpha}$, from (4) and (7), we deduce

$$\left\langle f,\mathcal{K}_{q,\alpha}(w,.)\right\rangle_{\mathcal{F}_{q,\alpha}}=\sum_{n=0}^{\infty}a_nw^n=f(w),\quad w\in D\left(0,\frac{1}{1-q}\right).$$

This completes the proof of the theorem. □

Remark 2. From Theorem 1 2), for $f\in\mathcal{F}_{q,\alpha}$ and $w\in D\left(0,\frac{1}{1-q}\right)$, we have

$$\begin{aligned}|f(w)|&\le\left\|\mathcal{K}_{q,\alpha}(w,.)\right\|_{\mathcal{F}_{q,\alpha}}\|f\|_{\mathcal{F}_{q,\alpha}}\\&=\left[E_\alpha\left(|w|^2;q^2\right)\right]^{1/2}\|f\|_{\mathcal{F}_{q,\alpha}}.\end{aligned} \quad (8)$$

Proposition 2. The space $\mathcal{F}_{q,\alpha}$ equipped with the inner product $\langle .,.\rangle_{\mathcal{F}_{q,\alpha}}$ is an Hilbert space; and the set $\left\{\xi_n\left(.;q^2\right)\right\}_{n\in\mathbb{N}}$ given by

$$\xi_n\left(z;q^2\right)=\frac{z^n}{\sqrt{b_n\left(\alpha;q^2\right)}},\quad z\in D\left(0,\frac{1}{1-q}\right),$$

forms an Hilbert basis for the space $\mathcal{F}_{q,\alpha}$.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{F}_{q,\alpha}$. We put

$$f=\lim_{n\to\infty}f_n,\text{ in }\mathcal{F}_{q,\alpha}.$$

From (8), we have

$$\left|f_{n+p}(w)-f_n(w)\right|\le\left[E_\alpha\left(|w|^2;q^2\right)\right]^{1/2}\left\|f_{n+p}-f_n\right\|_{\mathcal{F}_{q,\alpha}}.$$

This inequality shows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to *f*. Since the function $w\to\left[E_\alpha\left(|w|^2;q^2\right)\right]^{1/2}$ is continuous on $D\left(0,\frac{1}{1-q}\right)$, then $\{f_n\}_{n\in\mathbb{N}}$ converges to *f* uniformly on all compact set of $D\left(0,\frac{1}{1-q}\right)$. Consequently, f is an entire function on $D\left(0,\frac{1}{1-q}\right)$, then f belongs to the space $\mathcal{F}_{q,\alpha}$.

On the other hand, from the relation (4), we get

$$\left\langle\xi_n\left(.;q^2\right),\xi_m\left(.;q^2\right)\right\rangle_{\mathcal{F}_{q,\alpha}}=\delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker symbol.

This shows that the family $\left\{\xi_n\left(.;q^2\right)\right\}_{n\in\mathbb{N}}$ is an orthonormal set in $\mathcal{F}_{q,\alpha}$.

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an element of $\mathcal{F}_{q,\alpha}$ such that

$$\left\langle f,\xi_n\left(.;q^2\right)\right\rangle_{\mathcal{F}_{q,\alpha}}=0,\quad \forall n\in\mathbb{N}.$$

From the relation (4), we deduce that

$$a_n=0,\quad \forall n\in\mathbb{N}.$$

This completes the proof. □

Remark 3. 1) The set $\left\{E_\alpha\left(\overline{w}.;q^2\right), w\in D\left(0,\frac{1}{1-q}\right)\right\}$ is dense in $\mathcal{F}_{q,\alpha}$.

2) For all $z,w\in D\left(0,\frac{1}{1-q}\right)$, we have

$$E_\alpha\left(w\overline{z};q^2\right)=\left\langle E_\alpha\left(\overline{z}.;q^2\right),E_\alpha\left(\overline{w}.;q^2\right)\right\rangle_{\mathcal{F}_{q,\alpha}}.$$

3. Operators on the Fock Spaces $\mathcal{F}_{q,\alpha}$

On $\mathcal{F}_{q,\alpha}$, we consider the multiplication operators Q and N_q given by

$$Qf(z):=zf(z),$$

$$N_q f(z):=zD_q f(z)=\frac{f(z)-f(qz)}{1-q}.$$

We denote also by $\Lambda_{q,\alpha}$ the q-Dunkl operator defined for entire functions on $D\left(0,\frac{1}{1-q}\right)$.

We write

$$\left[\Lambda_{q,\alpha},Q\right]=\Lambda_{q,\alpha}Q-Q\Lambda_{q,\alpha}.$$

Then by straightforward calculation we obtain.

Lemma 4. $\left[\Lambda_{q,\alpha},Q\right]=B_q+W_{q,\alpha}$, where

$$B_q f(z):=f(qz),$$

$$W_{q,\alpha}:=\frac{[2\alpha+1]_q}{2}\left[(q-1)B_q+(q+1)B_{-q}\right].$$

Remark 4. The Lemma 4 is the analogous commutation rule of [3]. When $q\to 1^-$, then $\left[\Lambda_{q,\alpha},Q\right]$ tends to $I+(2\alpha+1)B$, where I is the identity operator and B is the parity operator given by $Bf(z):=f(-z)$.

Lemma 5. *If $f\in\mathcal{F}_{q,\alpha}$ then $B_q f$, $N_q f$ and $W_{q,\alpha}f$ belong to $\mathcal{F}_{q,\alpha}$, and*

1) $\left\|B_q f\right\|_{\mathcal{F}_{q,\alpha}}=\left\|B_{-q} f\right\|_{\mathcal{F}_{q,\alpha}}\le\|f\|_{\mathcal{F}_{q,\alpha}}$,

2) $\left\|N_q f\right\|_{\mathcal{F}_{q,\alpha}}\le\frac{1}{1-q}\|f\|_{\mathcal{F}_{q,\alpha}}$,

3) $\left\|W_{q,\alpha} f\right\|_{\mathcal{F}_{q,\alpha}}\le[2\alpha+1]_q\|f\|_{\mathcal{F}_{q,\alpha}}$.

Proof. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n\in\mathcal{F}_{q,\alpha}$, then

$$B_q f(z)=f(qz)=\sum_{n=0}^{\infty} a_n q^n z^n,$$

$$N_q f(z)=\frac{f(z)-f(qz)}{1-q}=\sum_{n=0}^{\infty} a_n [n]_q z^n,$$

and from (3), we obtain

$$\left\|B_q f\right\|^2_{\mathcal{F}_{q,\alpha}}=\sum_{n=0}^{\infty}|a_n|^2 q^{2n} b_n\left(\alpha;q^2\right)$$
$$\le\sum_{n=0}^{\infty}|a_n|^2 b_n\left(\alpha;q^2\right)=\|f\|^2_{\mathcal{F}_{q,\alpha}},$$

and

$$\left\|N_q f\right\|^2_{\mathcal{F}_{q,\alpha}}=\sum_{n=0}^{\infty}|a_n|^2\left([n]_q\right)^2 b_n\left(\alpha;q^2\right).$$

Using the fact that $[n]_q\le\frac{1}{1-q}$, we deduce

$$\left\|N_q f\right\|^2_{\mathcal{F}_{q,\alpha}}\le\frac{1}{(1-q)^2}\sum_{n=0}^{\infty}|a_n|^2 b_n\left(\alpha;q^2\right)=\frac{1}{(1-q)^2}\|f\|^2_{\mathcal{F}_{q,\alpha}}.$$

On the other hand from 1) we deduce that

$$\left\|W_{q,\alpha} f\right\|_{\mathcal{F}_{q,\alpha}}$$
$$\le\frac{[2\alpha+1]_q}{2}\left[(1-q)\left\|B_q f\right\|_{\mathcal{F}_{q,\alpha}}+(q+1)\left\|B_{-q} f\right\|_{\mathcal{F}_{q,\alpha}}\right]$$
$$\le[2\alpha+1]_q\|f\|_{\mathcal{F}_{q,\alpha}},$$

which completes the proof of the Lemma. □

We now study the continuous property of the operators $\Lambda_{q,\alpha}$ and Q on $\mathcal{F}_{q,\alpha}$.

Theorem 2. *If $f\in\mathcal{F}_{q,\alpha}$ then $\Lambda_{q,\alpha}f$ and Qf belong to $\mathcal{F}_{q,\alpha}$, and we have*

1) $\left\|\Lambda_{q,\alpha} f\right\|_{\mathcal{F}_{q,\alpha}}\le C_{q,\alpha}\|f\|_{\mathcal{F}_{q,\alpha}}$,

2) $\left\|Qf\right\|_{\mathcal{F}_{q,\alpha}}\le C_{q,\alpha}\|f\|_{\mathcal{F}_{q,\alpha}}$, where

$$C_{q,\alpha}:=\left([2\alpha+1]_q+\frac{1}{1-q}\right)^{1/2}.$$

Proof. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n\in\mathcal{F}_{q,\alpha}$.

1) From Lemma 3,

$$\Lambda_{q,\alpha} f(z)=\sum_{n=1}^{\infty} a_n\frac{b_n\left(\alpha;q^2\right)}{b_{n-1}\left(\alpha;q^2\right)}z^{n-1}$$
$$=\sum_{n=0}^{\infty} a_{n+1}\frac{b_{n+1}\left(\alpha;q^2\right)}{b_n\left(\alpha;q^2\right)}z^n. \quad (9)$$

Then from (9), we get

$$\left\|\Lambda_{q,\alpha} f\right\|^2_{\mathcal{F}_{q,\alpha}}=\sum_{n=0}^{\infty}|a_{n+1}|^2\frac{b_{n+1}\left(\alpha;q^2\right)}{b_n\left(\alpha;q^2\right)}b_{n+1}\left(\alpha;q^2\right).$$

Using Lemma 2 1), we obtain

$$\left\|\Lambda_{q,\alpha} f\right\|_{\mathcal{F}_{q,\alpha}}^2 = \sum_{n=0}^{\infty}|a_n|^2\left[[n]_q + q^n[2\alpha+1]_q\left(\frac{1-(-1)^n}{2}\right)\right]b_n(\alpha;q^2). \tag{10}$$

Using the fact that $[n]_q \le \frac{1}{1-q}$, we obtain

$$\left\|\Lambda_{q,\alpha} f\right\|_{\mathcal{F}_{q,\alpha}} \le C_{q,\alpha}\left[\sum_{n=0}^{\infty}|a_n|^2 b_n(\alpha;q^2)\right]^{1/2} = C_{q,\alpha}\|f\|_{\mathcal{F}_{q,\alpha}}.$$

2) On the other hand, since

$$Qf(z) = \sum_{n=1}^{\infty} a_{n-1}z^n, \tag{11}$$

then

$$\|Qf\|_{\mathcal{F}_{q,\alpha}}^2 = \sum_{n=1}^{\infty}|a_{n-1}|^2 b_n(\alpha;q^2) = \sum_{n=0}^{\infty}|a_n|^2 b_{n+1}(\alpha;q^2).$$

By Lemma 2 1), we deduce

$$\|Qf\|_{\mathcal{F}_{q,\alpha}}^2 = \sum_{n=0}^{\infty}|a_n|^2\left[[n+1]_q + q^{n+1}[2\alpha+1]_q\left(\frac{1+(-1)^n}{2}\right)\right]b_n(\alpha;q^2). \tag{12}$$

Using the fact that $[n+1]_q \le \frac{1}{1-q}$, we obtain

$$\|Qf\|_{\mathcal{F}_{q,\alpha}} \le C_{q,\alpha}\|f\|_{\mathcal{F}_{q,\alpha}}. \qquad \square$$

We deduce also the following norm equalities.

Theorem 3. *If* $f \in \mathcal{F}_{q,\alpha}$ *then*

1) $$\left\|\Lambda_{q,\alpha} f\right\|_{\mathcal{F}_{q,\alpha}}^2 = \langle f, N_q f\rangle_{\mathcal{F}_{q,\alpha}} + \frac{[2\alpha+1]_q}{2}\langle f,(B_q - B_{-q})f\rangle_{\mathcal{F}_{q,\alpha}},$$

2) $$\|Qf\|_{\mathcal{F}_{q,\alpha}}^2 = \langle f, N_q f\rangle_{\mathcal{F}_{q,\alpha}} + \left\|B_{\sqrt{q}} f\right\|_{\mathcal{F}_{q,\alpha}}^2 + \frac{q[2\alpha+1]_q}{2}\langle f,(B_q + B_{-q})f\rangle_{\mathcal{F}_{q,\alpha}},$$

3) $$\|Qf\|_{\mathcal{F}_{q,\alpha}}^2 = \left\|\Lambda_{q,\alpha} f\right\|_{\mathcal{F}_{q,\alpha}}^2 + \left\|B_{\sqrt{q}} f\right\|_{\mathcal{F}_{q,\alpha}}^2 + \langle f, W_{q,\alpha} f\rangle_{\mathcal{F}_{q,\alpha}}.$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}_{q,\alpha}$.

1) From (10), we get

$$\left\|\Lambda_{q,\alpha} f\right\|_{\mathcal{F}_{q,\alpha}}^2 = \sum_{n=0}^{\infty}|a_n|^2\left[[n]_q + q^n[2\alpha+1]_q\left(\frac{1-(-1)^n}{2}\right)\right]b_n(\alpha;q^2)$$
$$= \langle f, N_q f\rangle_{\mathcal{F}_{q,\alpha}} + \frac{[2\alpha+1]_q}{2}\langle f,(B_q - B_{-q})f\rangle_{\mathcal{F}_{q,\alpha}}.$$

2) On the other hand, by (12) and using the fact that $[n+1]_q = [n]_q + q^n$, we obtain

$$\|Qf\|_{\mathcal{F}_{q,\alpha}}^2 = \sum_{n=0}^{\infty}|a_n|^2[n]_q b_n(\alpha;q^2) + \sum_{n=0}^{\infty}|a_n|^2 q^n b_n(\alpha;q^2) + \frac{q[2\alpha+1]_q}{2}\sum_{n=0}^{\infty}|a_n|^2 q^n\left(1+(-1)^n\right)b_n(\alpha;q^2)$$
$$= \langle f, N_q f\rangle_{\mathcal{F}_{q,\alpha}} + \left\|B_{\sqrt{q}} f\right\|_{\mathcal{F}_{q,\alpha}}^2 + \frac{q[2\alpha+1]_q}{2}\langle f,(B_q + B_{-q})f\rangle_{\mathcal{F}_{q,\alpha}}.$$

3) Follows directly from 1) and 2). $\square$

Proposition 3. *The operators* Q *and* $\Lambda_{q,\alpha}$ *are adjoint-operators on* $\mathcal{F}_{q,\alpha}$*; and for all* $f, g \in \mathcal{F}_{q,\alpha}$*, we have*

$$\langle Qf, g\rangle_{\mathcal{F}_{q,\alpha}} = \langle f, \Lambda_{q,\alpha} g\rangle_{\mathcal{F}_{q,\alpha}}.$$

Proof. Consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} c_n z^n$ in $\mathcal{F}_{q,\alpha}$. From (9) and (11),

$$\Lambda_{q,\alpha} g(z) = \sum_{n=0}^{\infty} c_{n+1}\frac{b_{n+1}(\alpha;q^2)}{b_n(\alpha;q^2)}z^n,$$

and

$$Qf(z) = \sum_{n=1}^{\infty} a_{n-1}z^n.$$

Thus from (4), we get

$$\langle Qf, g\rangle_{\mathcal{F}_{q,\alpha}} = \sum_{n=1}^{\infty} a_{n-1}\overline{c_n}b_n(\alpha;q^2) = \sum_{n=0}^{\infty} a_n\overline{c_{n+1}}b_{n+1}(\alpha;q^2) = \langle f, \Lambda_{q,\alpha} g\rangle_{\mathcal{F}_{q,\alpha}},$$

which gives the result. $\square$

In the next part of this section we study a generalized translation and multiplication operators on $\mathcal{F}_{q,\alpha}$. We begin by the following definition.

Definition 2. For $f \in \mathcal{F}_{q,\alpha}$ and $w, z \in D\left(0, \frac{1}{1-q}\right)$, we define:

- The q-translation operators on $\mathcal{F}_{q,\alpha}$, by

$$T_z f(w) := E_\alpha\left(z\Lambda_{q,\alpha}; q^2\right) f(w) = \sum_{n=0}^{\infty} \frac{\Lambda_{q,\alpha}^n f(w)}{b_n\left(\alpha; q^2\right)} z^n. \quad (13)$$

- The generalized multiplication operators on $\mathcal{F}_{q,\alpha}$, by

$$M_z f(w) := E_\alpha\left(zQ; q^2\right) f(w) = \sum_{n=0}^{\infty} \frac{Q^n f(w)}{b_n\left(\alpha; q^2\right)} z^n. \quad (14)$$

For $w, z \in D\left(0, \frac{1}{1-q}\right)$, the function $E_\alpha\left(.; q^2\right)$ satisfies the following product formulas:

$$T_z E_\alpha\left(.; q^2\right)(w) = E_\alpha\left(z; q^2\right) E_\alpha\left(w; q^2\right),$$

$$M_z E_\alpha\left(.; q^2\right)(w) = E_\alpha\left(wz; q^2\right) E_\alpha\left(w; q^2\right).$$

Remark 5. If $q \to 1^-$ in (13), we obtain the generalized translation operators given in ([9], page 102).

Proposition 4. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}_{q,\alpha}$ *and* $z, w \in D\left(0, \frac{1}{1-q}\right)$. *Then*

1) $T_z f(w) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \frac{b_n\left(\alpha; q^2\right)}{b_k\left(\alpha; q^2\right) b_{n-k}\left(\alpha; q^2\right)} w^{n-k} z^k,$

with

$$\frac{b_n\left(\alpha; q^2\right)}{b_k\left(\alpha; q^2\right) b_{n-k}\left(\alpha; q^2\right)} = \frac{c\left(n, k; q^2\right) \Gamma_{q^2}(\alpha+1) \Gamma_{q^2}\left(\left[\frac{n+1}{2}\right]+\alpha+1\right)}{\Gamma_{q^2}\left(\left[\frac{k+1}{2}\right]+\alpha+1\right) \Gamma_{q^2}\left(\left[\frac{n-k+1}{2}\right]+\alpha+1\right)},$$

where

$$c\left(n, k; q^2\right) = \frac{\Gamma_{q^2}\left([n/2]+1\right)}{\Gamma_{q^2}\left([k/2]+1\right) \Gamma_{q^2}\left(\left[\frac{n-k}{2}\right]+1\right)}.$$

2) $M_z f(w) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{a_{n-k}}{b_k\left(\alpha; q^2\right)} z^k\right] w^n.$

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}_{q,\alpha}$.

1) From (13), we have

$$T_z f(w) = \sum_{n=0}^{\infty} \frac{\Lambda_{q,\alpha}^n f(w)}{b_n\left(\alpha; q^2\right)} z^n; \quad w, z \in D\left(0, \frac{1}{1-q}\right).$$

But from (5), we have

$$\Lambda_{q,\alpha}^n f(w) = \sum_{k=n}^{\infty} a_k \frac{b_k\left(\alpha; q^2\right)}{b_{k-n}\left(\alpha; q^2\right)} w^{k-n}.$$

Thus we obtain

$$T_z f(w) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \frac{b_n\left(\alpha; q^2\right)}{b_k\left(\alpha; q^2\right) b_{n-k}\left(\alpha; q^2\right)} w^{n-k} z^k.$$

On the other hand from Lemma 2 3), we get

$$\frac{b_n\left(\alpha; q^2\right)}{b_k\left(\alpha; q^2\right) b_{n-k}\left(\alpha; q^2\right)} = \frac{c\left(n, k; q^2\right) \Gamma_{q^2}(\alpha+1) \Gamma_{q^2}\left(\left[\frac{n+1}{2}\right]+\alpha+1\right)}{\Gamma_{q^2}\left(\left[\frac{k+1}{2}\right]+\alpha+1\right) \Gamma_{q^2}\left(\left[\frac{n-k+1}{2}\right]+\alpha+1\right)},$$

which gives the 1).

2) From (14), we have

$$M_z f(w) = \sum_{n=0}^{\infty} \frac{Q^n f(w)}{b_n\left(\alpha; q^2\right)} z^n; \quad w, z \in D\left(0, \frac{1}{1-q}\right).$$

But from (11), we have

$$Q^n f(w) = \sum_{k=n}^{\infty} a_{k-n} w^k.$$

Thus we obtain

$$M_z f(w) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{a_{n-k}}{b_k\left(\alpha; q^2\right)} z^k\right] w^n. \qquad \square$$

According to Theorem 2 we study the continuous property of the operators T_z and M_z on $\mathcal{F}_{q,\alpha}$.

Theorem 4. *If* $f \in \mathcal{F}_{q,\alpha}$ *and* $|z| < \frac{1}{(1-q)\left(2-q^{2\alpha+1}\right)}$, *then* $T_z f$ *and* $M_z f$ *belong to* $\mathcal{F}_{q,\alpha}$, *and we have*

1) $\|T_z f\|_{\mathcal{F}_{q,\alpha}} \le E_\alpha\left(C_{q,\alpha}|z|; q^2\right)\|f\|_{\mathcal{F}_{q,\alpha}}$,

2) $\|M_z f\|_{\mathcal{F}_{q,\alpha}} \le E_\alpha\left(C_{q,\alpha}|z|; q^2\right)\|f\|_{\mathcal{F}_{q,\alpha}}$,

where $C_{q,\alpha}$ are the constants of Theorem 2.

Proof. From (13) and Theorem 2 1), we deduce

$$\|T_z f\|_{\mathcal{F}_{q,\alpha}} \le \sum_{n=0}^{\infty} \left\|\Lambda_{q,\alpha}^n f\right\|_{\mathcal{F}_{q,\alpha}} \frac{|z|^n}{b_n\left(\alpha; q^2\right)} \le \sum_{n=0}^{\infty} \frac{\left(C_{q,\alpha}|z|\right)^n}{b_n\left(\alpha; q^2\right)} \|f\|_{\mathcal{F}_{q,\alpha}}.$$

Since $|z| < \frac{1}{(1-q)\left(2-q^{2\alpha+1}\right)}$, then $C_{q,\alpha}|z| < \frac{1}{(1-q)^2}$, and therefore

$$\|T_z f\|_{\mathcal{F}_{q,\alpha}} \le E_\alpha\left(C_{q,\alpha}|z|; q^2\right)\|f\|_{\mathcal{F}_{q,\alpha}},$$

which gives the first inequality, and as in the same way we prove the second inequality of this theorem. □

From Proposition 3 we deduce the following results.

Proposition 5. *For all* $f, g \in \mathcal{F}_{q,\alpha}$, *we have*

$$\langle M_z f, g\rangle_{\mathcal{F}_{q,\alpha}} = \langle f, T_{\bar{z}} g\rangle_{\mathcal{F}_{q,\alpha}},$$

$$\langle T_z f, g\rangle_{\mathcal{F}_{q,\alpha}} = \langle f, M_{\bar{z}} g\rangle_{\mathcal{F}_{q,\alpha}}.$$

We denote by R_z the following operator defined on $\mathcal{F}_{q,\alpha}$ by

$$R_z := T_{\bar{z}} M_z - M_{\bar{z}} T_z = E_\alpha\left(\bar{z}\Lambda_{q,\alpha}; q^2\right) E_\alpha\left(zQ; q^2\right) - E_\alpha\left(\bar{z}Q; q^2\right) E_\alpha\left(z\Lambda_{q,\alpha}; q^2\right).$$

Then, we prove the following theorem.

Theorem 5. *For all* $f \in \mathcal{F}_{q,\alpha}$, *we have*

$$\|M_z f\|^2_{\mathcal{F}_{q,\alpha}} = \|T_z f\|^2_{\mathcal{F}_{q,\alpha}} + \langle f, R_z f\rangle_{\mathcal{F}_{q,\alpha}}.$$

Proof. From Proposition 5, we get

$$\|M_z f\|^2_{\mathcal{F}_{q,\alpha}} = \langle f, T_{\bar{z}} M_z f\rangle_{\mathcal{F}_{q,\alpha}} = \langle f, (M_{\bar{z}} T_z + R_z) f\rangle_{\mathcal{F}_{q,\alpha}} = \|T_z f\|^2_{\mathcal{F}_{q,\alpha}} + \langle f, R_z f\rangle_{\mathcal{F}_{q,\alpha}}.$$ □

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