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Making Holes in the Hyperspace of Subcontinua of Some Continua

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ABSTRACT

Let X be a metric continuum. Let $A \in C(X)$, A is said to make a hole in $C(X)$, if $C(X)-\{A\}$ is not unicoherent. In this paper, we characterize elements $A \in C(X)$ such that A makes a hole in $C(X)$, where X is either a smooth fan or an Elsa continuum.

Keywords: Continuum; Elsa Continuum; Fan; Hyperspace; Property b); Unicoherence; Whitney Map

1. Introduction

A connected topological space Z is *unicoherent* if whenever $Z = A \cup B$, where A and B are connected and closed subsets of Z, the set $A \cap B$ is connected. Let Z be a unicoherent topological space and let z be an element of Z. We say that z *makes a hole in* Z if $Z-\{z\}$ is not unicoherent. A *compactum* is a nondegenerate compact metric space. A *continuum* is a connected compactum with metric d. Given a continuum X, the hyperspace of all nonempty subcontinua of X is denoted by $C(X)$ and it is considered with the Hausdorff metric. It is known that the hyperspace $C(X)$ is unicoherent (see [1, Theorem 19.8, p. 159]).

In the papers [2] and [3] the author present some partial solution to the following problem.

Problem. Let $\mathcal{H}(X)$ be a hyperspace of X such that $\mathcal{H}(X)$ is unicoherent. For which elements, $A \in \mathcal{H}(X)$, does A make a hole in $\mathcal{H}(X)$?

In the current paper we present the solution to that problem when X is either a smooth fan or an Elsa continuum and $\mathcal{H}(X) = C(X)$.

2. Preliminary

We use $\mathbb{N}$ and $\mathbb{R}$ to denote the set of positive integers and the set of real numbers, respectively. Let Z be a topological space and let A be a subset of Z. We denote $int(A)$ the interior of A in Z. An *arc* is any homeomorphic space to the closed unit interval $[0,1]$. Let p, q in a topological space Z, $[p,q]$ will denote an arc, where p and q are the end points of $[p,q]$. A *free arc* in a continuum X is an arc $[p,q]$ such that $[p,q]-\{p,q\}$ is open in X. A point z in a connected topological space Z is a *cut point of* (*non-cut point of*) Z provided that $Z-\{z\}$ is disconnected (is connected). A *map* is a continuous function. A map $f: Z \to S^1$, where Z is a connected topological space and S^1 is the unit circle in the Euclidean plane $\mathbb{R}^2$, *has a lifting* if there exists a map $h: Z \to \mathbb{R}$ such that $f = \exp \circ h$, where exp is the map from $\mathbb{R}$ onto S^1 defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$. A connected topological space Z *has property* b) if each map $f: Z \to S^1$ has a lifting.

By an *end point of* X, we mean an end point in the classical sense, which means a point p of X that is a non-cut point of any arc in X that contains p. A subspace Y of a topological space Z is a *deformation retract* of Z if there exists a map $H: Z \times I \to Z$ such that, for each $x \in Z$, $H(x,0) = x$, $H(Z \times \{1\}) = Y$ and, for each $y \in Y$, $H(y,1) = y$. We say that a topological space Z is *contractible* if there exists $z \in Z$, such that $\{z\}$ is a deformation retract of Z. It is known that each contractible normal topological space has property b), and so it is unicoherent (see [4, Theorems 2 and 3, pp. 69 and 70]).

3. Smooth Fans

A point p of a continuum X is a *ramification point* provided that p is a point which is a common end point of three or more arcs in X that are otherwise disjoint. A *fan* is an arcwise connected, hereditarily unicoherent continuum with exactly one ramification point (*hereditarily unicoherent* means each subcontinuum is unicoherent). The ramification point of a fan will be called the *vertex* of the fan. If X is a fan and $x, y \in X$, then

$[x,y]$ denotes the unique arc joining x and y. A fan X with vertex v is said to be *smooth* provided that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that it converges to a point $x \in X$, then the sequence $\{[v,x_n]\}_{n=1}^{\infty}$ converges to $[v,x]$ in $C(X)$.

To establish some notation, let X be a smooth fan with vertex v and let $E(X)=\{e_i : i \in \Delta\}$ be its end-points set, where Δ is an infinity indexing set. It follows from definition of smoothness that the set:

$$N[C(X)] = \{[v,x] : x \in X\}$$

is a natural homeomorphic copy of X in $C(X)$. By the smoothness of X, we have that the set:

$$T[C(X)] = \bigcup_{i\in\Delta} C([v,e_i])$$

is a closed subspace of $C(X)$. Furthermore, each hyperspace $C([v,e_i])$ is a 2-cell and $C([v,e_i]) \cap C([v,e_j]) = \{v\}$ for each $i,j \in \Delta$ which are different. The set of all elements of $C(X)$ such that it contains v will be denoted by $C(\{v\},X)$.

Let $A \in C(X)$. We say that A is a *simple arc* if A is an arc such that $A \cap E(X) = \varnothing$ and, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of $C(X)$ satisfying the following properties:

1) $A = \lim A_n$ and
2) for each $n \in \mathbb{N}$,
   a) $v \notin A_n$,
   b) $int(A_n) \neq \varnothing$ and
   c) $A \cap A_n \neq \varnothing$.

Since X is embeded in the Cantor fan (see [5]), we can regard X as embedded in the Euclidean plane $\mathbb{R}^2$ such that $v=(0,0)$ and each $[v,e_i]$ is a convex arc, where $e_i \in E(X)$. Note that for $r=0$, $re_i = v$ for each $e_i \in E(X)$. Throughout this section h will denote the map from $X \times [0,1]$ onto $T[C(X)]$ defined by $h(x,t) = [tx,x]$. We assume in this section that if $[a,b] \subset [v,e_i]$, then the distance between v and a is less than the distance between v and b.

Lemma 3.1. *Let* X *be a smooth fan with vertex* v*. If* $[a,b]$ *is an arc contained in* $[v,e_{i_0}]$*, where* $e_{i_0} \in E(X)$*, then*:

1) If $int([a,b]) = \varnothing$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of X such that $[v,b] = \lim[v,x_n]$ and, for each $n \in \mathbb{N}$, $x_n \notin [v,e_{i_0}]$.

2) If $e_{i_0} \notin [a,b]$ and $int([a,b]) \neq \varnothing$, then $[b,e_{i_0}]$ is a free arc in X.

Proof. The proof of (1) is easy.

In order to prove (2), we suppose that $[b,e_{i_0}]$ is not a free arc in X. Then there exists $y_0 \in [b,e_{i_0}] - \{b,e_{i_0}\}$ such that $y_0 \notin int([b,e_{i_0}])$. Hence, $y_0 \notin int([v,e_{i_0}])$. Then, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ of $X - [v,e_{i_0}]$ such that $y_0 = \lim y_n$. Since X a smooth fan, $[v,y_0] = \lim[v,y_n]$. Notice that $[a,b] \subset [v,b] \subset [v,y_0]$. Let $z_0 \in int([a,b])$. There exists a sequence $\{z_n\}_{n=1}^{\infty}$ of X such that $z_0 = \lim z_n$ and, for each $n \in \mathbb{N}$, $z_n \in [v,y_n]$. Clearly $v \notin int([a,b])$. Hence, $z_0 \neq v$. Let $\varepsilon > 0$ be such that $v \notin B_\varepsilon(z_0)$ and $B_\varepsilon(z_0) \subset [a,b]$. Let $n_0 \in \mathbb{N}$ be large enough such that $z_{n_0} \in B_\varepsilon(z_0) \subset [a,b] \subset [v,e_{i_0}]$. Thus, $z_{n_0} \in [v,e_{i_0}] \cap [v,y_{n_0}]$. Since X is a fan and $y_{n_0} \in X - [v,e_{i_0}]$, $z_{n_0} = v$, this is a contradiction. □

Since the Hilbert cube, Q, is homogeneous (see [1, Theorem 11.9.1, p. 93]) and $Q - \{(1,1,1,\cdots)\}$ is contractible, we have the following result.

Lemma 3.2. *Let* $q \in Q$*. Then* $Q - \{q\}$ *has property b).*

Theorem 3.3. *Let* X *be a smooth fan with vertex* v*. If* A *is a subcontinuum of* X *such that* $v \in A$ *and, for each* $e_i \in E(X)$*,* $A \not\subseteq [v,e_i]$*, then* A *does not make a hole in* $C(X)$*.*

Proof. We are going to prove that $C(\{v\},X) - \{A\}$ is a deformation retract of $C(X) - \{A\}$. Notice that, for each $B \in T[C(X)]$, there exists $(x_B,t_B) \in X \times [0,1]$ such that $h(x_B,t_B) = B$. We define

$$H = (B,t)\begin{cases} B, & \text{if } B \in C(\{v\},X), \\ [(t-1)t_B x_B, x_B], & \text{if } B \in T[C(X)]. \end{cases}$$

Clearly H is a map. Then, $C(\{v\},X) - \{A\}$ is a deformation retract of $C(X) - \{A\}$. Since Q is homeomorphic to $C(\{v\},X)$ (see [6, Theorem 3.1, p. 282]), $C(\{v\},X) - \{A\}$ has property b) (see Lemma 3.2). Therefore $C(X) - \{A\}$ has property b) (see [2, Proposition 9, p. 2001]). □

Lemma 3.4. *Let* X *be a smooth fan with vertex* v *and let* $[a,b] \in C([v,e_{i_0}])$ *be a simple arc contained in* X*, for some* $e_{i_0} \in E(X)$*. Then*

$$int([b,e_{i_0}]) = [b,e_{i_0}] - \{b\}.$$

Proof. Since $[a,b]$ is a simple arc, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of $C(X)$ that satisfies the required properties of the definition. Notice that, for each $n \in \mathbb{N}$, $A_n \in T[C(X)]$ and $A_n \subset [v,e_{i_0}]$. Given $n \in \mathbb{N}$, let a_n, $b_n \in [v,e_{i_0}]$ such that $A_n = [a_n,b_n]$.

We need to prove the following claim.

Claim. $\left[b,e_{i_0}\right]$ is a free arc in X.

Let $n\in\mathbb{N}$. First, we suppose that there exists $n_0\in\mathbb{N}$ such that $b_{n_0}\in[v,b]$. Since $int\left(\left[a_{n_0},b_{n_0}\right]\right)\neq\varnothing$ and $e_{i_0}\notin\left[a_{n_0},b_{n_0}\right]$, $\left[b_{n_0},e_{i_0}\right]$ is a free arc (see (2) of Lemma 3.1). Hence, $\left[b,e_{i_0}\right]$ is a free arc in X.

Now, we assume that, for each $n\in\mathbb{N}$, $b_n\in\left[b,e_{i_0}\right]-\{b\}$. Let $y\in\left[b,e_{i_0}\right]-\left\{b,e_{i_0}\right\}$. Notice that $b=\lim b_n$ and $d(b,y)>0$. Then there exists $n_0\in\mathbb{N}$, such that $d\left(b,b_{n_0}\right)<d(b,y)$. Since $\left[v,e_{i_0}\right]$ is a convex arc of $\mathbb{R}^2$, we have that $y\in\left[b_{n_0},e_{i_0}\right]-\left\{b_{n_0},e_{i_0}\right\}$. Since $int\left(A_{n_0}\right)\neq\varnothing$, $\left[b_{n_0},e_{i_0}\right]$ is a free arc in X (see (2) of Lemma 3.1). Thus, $\left[b_{n_0},e_{i_0}\right]-\left\{b_{n_0},e_{i_0}\right\}$ is an open subset of X such that $y\in\left[b_{n_0},e_{i_0}\right]-\left\{b_{n_0},e_{i_0}\right\}\subset\left[b,e_{i_0}\right]-\left\{b,e_{i_0}\right\}$. Hence $\left[b,e_{i_0}\right]-\left\{b,e_{i_0}\right\}$ is an open subset of X. This proves the claim.

By Claim, $\left[b,e_{i_0}\right]$ is a free arc in X. Since $\left[v,e_{i_0}\right]=[v,a]\cup[a,b]\cup\left[b,e_{i_0}\right]$, $b\notin int\left(\left[b,e_{i_0}\right]\right)$. Suppose that $e_{i_0}\notin int\left(\left[b,e_{i_0}\right]\right)$. Then there exists a sequence $\{y_m\}_{m=1}^{\infty}$ in X such that $e_{i_0}=\lim y_m$ and, for each $m\in\mathbb{N}$, $y_m\notin\left[b,e_{i_0}\right]$. Since $\left[v,e_{i_0}\right]=[v,a]\cup[a,b]\cup\left[b,e_{i_0}\right]$, $d\left(b,e_{i_0}\right)>0$ and $e_{i_0}=\lim y_m$, we may assume that, for each $m\in\mathbb{N}$, $y_m\notin\left[v,e_{i_0}\right]$. Since X is a smooth fan, $\left[v,e_{i_0}\right]=\lim[v,y_m]$. Let $z_0\in\left[b,e_{i_0}\right]-\left\{b,e_{i_0}\right\}\subset\left[v,e_{i_0}\right]$. There exists a sequence $\{z_m\}_{m=1}^{\infty}$ such that, $z_0=\lim z_m$ and, for each $m\in\mathbb{N}$, $z_m\in[v,y_m]$. Since $\left[b,e_{i_0}\right]-\left\{b,e_{i_0}\right\}$ is an open set in X and $z_0\in\left[b,e_{i_0}\right]-\left\{b,e_{i_0}\right\}$, there exists $m_0\in\mathbb{N}$ such that $z_{m_0}\in\left[b,e_{i_0}\right]-\left\{b,e_{i_0}\right\}\subset\left[v,e_{i_0}\right]$. Then $z_{m_0}\in\left[v,e_{i_0}\right]\cap[v,y_m]=\{v\}$, this is a contradiction. Therefore $int\left(\left[b,e_{i_0}\right]\right)=\left[b,e_{i_0}\right]-\{b\}$. □

Theorem 3.5. *Let X be a smooth fan with vertex $\{v\}$. If $A\in C(X)$ is a simple arc, then A makes a hole in $C(X)$.*

Proof. We may assume that $A=[a,b]\in C\left(\left[v,e_{i_0}\right]\right)$, where $e_{i_0}\in E(X)$ and $t_0\in[0,1]$ such that $a=t_0b$. Let:

$$\mathcal{A}=\left(C(\{v\},X)\cup\left(T\left[C\left(X-int\left(\left[b,e_{i_0}\right]\right)\right)\right]\right)-\{A\}\right)$$

and

$$\mathcal{B}=h\left(\left[b,e_{i_0}\right]\times[0,1]\right)-\{A\}.$$

By Lemma 3.4, $X-int\left(\left[b,e_{i_0}\right]\right)$ is a smooth fan. Then $\left(T\left[C\left(X-int\left(\left[b,e_{i_0}\right]\right)\right)\right]\right)-\{A\}$ is a connected and closed subset of $C(X)-\{A\}$. So, $\mathcal{A}$ is a connected and closed subset of $C(X)-\{A\}$.

Notice that $\mathcal{B}$ is homeomorphic to $\left[b,e_{i_0}\right]\times[0,1]-\{(b,t_0)\}$. Since $\left[b,e_{i_0}\right]\times[0,1]-\{(b,t_0)\}$ is a connected subset of $X\times[0,1]-\{(b,t_0)\}$, we have that $\mathcal{B}$ is a connected subset of $C(X)-\{A\}$. Clearly $\mathcal{B}$ is a closed subset of $C(X)-\{A\}$.

Notice that $C(X)-\{[a,b]\}=\mathcal{A}\cup\mathcal{B}$ and:

$$\mathcal{A}\cap\mathcal{B}=h\left((\{b\}\times[0,1])-\{(b,t_0)\}\right)\cup h\left(\left[b,e_{i_0}\right]\times\{0\}\right).$$

Let:

$$\mathcal{F}_1=\left(h\left((\{b\}\times[0,t_0])-\{(b,t_0)\}\right)\right)\cup h\left(\left[b,e_{i_0}\right]\times\{0\}\right)$$

and

$$\mathcal{F}_2=h\left((\{b\}\times[t_0,1])-\{(b,t_0)\}\right).$$

Clearly, $\mathcal{F}_1\cup\mathcal{F}_2=\mathcal{A}\cap\mathcal{B}$ is a separation of $\mathcal{A}\cap\mathcal{B}$. Then $C(X)-\{A\}$ is not unicoherent. □

Theorem 3.6. Let X be a smooth fan with vertex v, let $e_{i_0}\in E(X)$ and let $a\in\left[v,e_{i_0}\right]-\left\{e_{i_0}\right\}$. Then $\left[a,e_{i_0}\right]$ does not make a hole in $C(X)$.

Proof. Let

$$G:\left(C(X)-\left\{\left[a,e_{i_0}\right]\right\}\right)\times[0,1]\to C(X)-\left\{\left[a,e_{i_0}\right]\right\}$$

be defined by:

$$G(A,t)=\{ta:a\in A\}.$$

It is easy to prove that G is well defined. In order to show that G is continuous, we define $G':C(X)\times[0,1]\to C(X)$ by $G'(A,t)=\{ta:a\in A\}$. We prove that G is continuous. Let $\{(A_n,t_n)\}_{n=1}^{\infty}$ be a sequence in $C(X)\times[0,1]$ and $(A_0,t_0)\in C(X)\times[0,1]$ such that $(A_0,t_0)=\lim(A_n,t_n)$. We suppose that there exists $B\in C(X)$ such that $B=\lim G'(A_n,t_n)$. We will show $B=G'(A_0,t_0)$. Let $b\in B$. Consider two sequences $\{b_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ of X such that

$b = \lim b_n$ and, for each $n \in \mathbb{N}$, $b_n \in G'(A_n, t_n)$, $a_n \in A_n$ and $b_n = t_n a_n$. Taking subsequences if necessary, we may assume that there exists $a_0 \in X$ such that $a_0 = \lim a_n$. Then $a_0 \in A_0$. Moreover, $t_0 a_0 = \lim t_n a_n = b$ and, so $b \in G'(A_0, t_0)$. This proves that $B \subset G'(A_0, t_0)$. Now, let $t_0 a_0 \in G'(A_0, t_0)$. Then $a_0 \in A_0$. Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in X such that $a_0 = \lim a_n$ and, for each $n \in \mathbb{N}$, $a_n \in A_n$. So $t_0 a_0 = \lim t_n a_n$. Since, for each $n \in \mathbb{N}$, $t_n a_n \in G'(A_n, t_n)$, $t_0 a_0 \in B$. Thus $B = G'(A_0, t_0)$.

Hence, G is a map. So $\{v\}$ is a deformation retract of $C(X) - \{[a, e_{i_0}]\}$.

Then $C(X) - \{[a, e_{i_0}]\}$ is contractible. Therefore $C(X) - \{[a, e_{i_0}]\}$ has property b) (see [2, Proposition 9, p. 2001]). □

Theorem 3.7. *Let X be a smooth fan with vertex v, let $e_{i_0} \in E(X)$ and let $[a,b] \in C([v, e_{i_0}])$ such that $e_{i_0} \notin [a,b]$ and $[b, e_{i_0}]$ is not a free arc of X. Then $[a,b]$ does not make a hole in $C(X)$.*

Proof. In light of Proposition 9 of [2, p. 2001], it suffices to prove that there exist two connected, closed subsets $\mathcal{D}$ and ε of $C(X) - \{[a,b]\}$ which have property b) and the intersection of them is connected.

We may assume that there exists $t_0 \in [0,1]$ such that $t_0 b = a$.

We consider two cases.

Case 1. $t_0 = 0$.

Then $[a,b] = [v,b]$. Let $\mathcal{D} = T[C(X)] - \{[a,b]\}$ and $\varepsilon = C(\{v\}, X) - \{[a,b]\}$. Clearly $\mathcal{D}$ has property b). By Theorem 3.1 of [6, p. 282], $C(\{v\}, X)$ is a Hilbert cube. By Lemma 3.2, ε has property b). Notice that $(\mathcal{D} \cap \varepsilon) - \{[a,b]\} = N[C(X)] - \{[a,b]\}$. Clearly $N[C(X)] - \{[a,b]\}$ is homeomorphic to $X - \{b\}$. Since $X - \{b\}$ is connected, $(\mathcal{D} \cap \varepsilon) - \{[a,b]\}$ is connected. By Proposition 8 of [2], $C(X) - \{[a,b]\} = (\mathcal{D} \cup \varepsilon) - \{[a,b]\}$ has property b).

Case 2. $t_0 > 0$. Consider the following sets:

$$\mathcal{D} = h\left(X \times [t_0, 1] - \{(b, t_0)\}\right)$$

and

$$\varepsilon = C(\{v\}, X) \cup h\left(X \times [0, t_0] - \{(b, t_0)\}\right).$$

Clearly $\mathcal{D}$ and ε are connected, closed subsets of $C(X) - \{[a,b]\}$ and $\mathcal{D} \cap \varepsilon = h\left(X \times \{t_0\} - \{(b, t_0)\}\right)$.

Notice that $\mathcal{D} \cap \varepsilon$ is homeomorphic to $X - \{b\}$. So, since $X - \{b\}$ is connected, $\mathcal{D} \cap \varepsilon$ is connected.

Now, we are going to prove that $\mathcal{D}$ and ε have property b). If we define $H : \mathcal{D} \times [0,1] \to \mathcal{D}$ by $H(h(x,t), s) = h(x, t + (1-t)s)$, we have $h(X \times \{1\})$ is a deformation retract of $\mathcal{D}$. Since $h(X \times \{1\})$ is contractible, $h(X \times \{1\})$ has property b) (see [2, Proposition 9, p. 2001]). Hence, $\mathcal{D}$ has property b) (see [2, Proposition 9, p. 2001]).

In order to prove that ε has property b), note that $C(\{v\}, X)$ is a deformation retract of ε. By Theorem 3.1 of [6, p. 282], $C(\{v\}, X)$ is homeomorphic to a Hilbert cube. Thus, $C(\{v\}, X)$ has property b). Hence, ε has property b) (see Proposition 9 of [2, p. 2001]). Therefore $C(X) - \{[a,b]\} = \mathcal{D} \cup \varepsilon$ has property b). □

Classification

Theorem 3.8. *Let X be a smooth fan with vertex v and $A \in C(X)$. Then A makes a hole in $C(X)$ if and only if A a simple arc.*

Proof. Let $A \in C(X)$ be such that A makes a hole in $C(X)$. By Theorem 3 of [2, p. 2001] and by Theorem 3.3, A is an arc $[p,q]$. By Theorems 3.6 and 3.7, $[p,q] \subset [v, e_{i_0}] - \{v, e_{i_0}\}$ for some $e_{i_0} \in E(X)$, and $[q, e_{i_0}]$ is a free arc in X. In order to prove that $[p,q]$ is a simple arc, let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences in $[p,q] - \{p,q\}$ and $[q, e_{i_0}] - \{q\}$, respectively, such that $p = \lim a_n$ and $q = \lim b_n$. Then $[p,q] = lim[a_n, b_n]$ and, for each $n \in \mathbb{N}$, $v \notin [a_n, b_n]$, $int([a_n, b_n]) \neq \varnothing$ and $[p,q] \cap [a_n, b_n] \neq \varnothing$. Therefore A is a simple arc.

The sufficiency follows from Theorem 3.5.

4. Elsa Continua

A compactification of $[0, \infty)$ with an arc as the remainder is called an *Elsa continuum*. The Elsa continua was defined by S. B. Nadler Jr., in [7]. A particular example of an Elsa continuum is the familiar sin(1/x)-continuum. There are uncountably many topologically different Elsa continua, the different topological types being a consequence of different ways $[0, \infty)$ “patterns into” the remainder of the compactification [8, p. 184]. Let X be a continuum. A *Whitney map* for $C(X)$ is a continuous function $\mu : C(X) \to [0,1]$ that satisfies the following two conditions:

1) for any A, $B \in C(X)$ such that $A \subset B$ and $A \neq B$, $\mu(A) < \mu(B)$,

2) $\mu(\{x\}) = 0$ for each $x \in X$ and c) $\mu(X) = 1$.

A *Whitney block* in $C(X)$, respectively a *Whitney level* in $C(X)$, is a set of the form $\mu^{-1}([s,t])$, respectively $\mu^{-1}(t)$, where $0 \leq s \leq t \leq 1$. It is known that Whitney maps always exist (see [1, Theorem 13.4, p. 107]). Moreover, Whitney blocks and Whitney levels in

$C(X)$ are continua (see [1, Theorem 19.9, p. 160]).

Throughout this section $X = I \cup R$ will denote a Elsa continuum, where I is the remainder of X and R is homeomorphic to the half-ray $[0,\infty)$.

Lemma 4.1. *Let $\mu : C(X) \to \mathbb{R}$ be a Whitney map, let $t \in (0, \mu(I)]$ and let $A_0 \in \mu^{-1}(t) \cap C(I)$. Then $\mu^{-1}([0,t]) - \{A_0\}$ has property b).*

Proof. We consider $\mu_1 = \mu|_{C(I)}$. It is easy to prove that $\mathcal{A} = \mu_1^{-1}([0,t]) - \{A_0\}$ has property b).

Let $\mathcal{B} = \mathcal{A} \cup \mu^{-1}(0)$. Since $\mu^{-1}(0)$ has property b) (see [9, 12.66, p. 269]) and $\mathcal{A} \cap \mu^{-1}(0) = \mu_1^{-1}(0)$, $\mathcal{B}$ has property b) (see [2, Proposition 8, p. 2001]).

Let $f : \mu^{-1}([0,t]) - \{A_0\} \to S^1$ be a map. Then there exists a map $h_0 : \mathcal{B} \to \mathbb{R}$ such that $\exp \circ h_0 = f|_{\mathcal{B}}$.

Given $A \in \mu^{-1}((0,t]) \cap C(R)$, it is an arc contained in R and it is determined by its end point, i_A, lying near to the end point of R. Let α_A be an order arc in $C(X)$ from i_A to A. Since α has property b), there exists a map $h_A : \alpha \to \mathbb{R}$ such that $\exp \circ h_A = f|_\alpha$ and $h_A(\{i_A\}) = h_0(\{i_A\})$.

We define $h : \mu^{-1}([0,t]) - \{A_0\} \to \mathbb{R}$ by

$$h(A)\begin{cases} h_0(A), & \text{if } A \in \mathcal{B}, \\ h_A(A), & \text{if } A \in \mu^{-1}T([0,t]) \cap C(R). \end{cases}$$

In order to prove that h is continuous, let $\{B\}_{n=1}^{\infty}$ be a sequence of $\mu^{-1}([0,t]) - \{A_0\}$ such that $lim B_n = B_0$ for some $B_0 \in \mu^{-1}([0,t]) - \{A_0\}$. We consider two cases.

Case 1. For each $n \in \mathbb{N}$, $B_n \in \mathcal{B}$.

Since $\mathcal{B}$ is a closed subset of $\mu^{-1}([0,t]) - \{A_0\}$, $B_n \in \mathcal{B}$. Then $lim h(B_n) = h(B_0)$.

Case 2. For each $n \in \mathbb{N}$, B_n is an arc contained in R.

We consider two subcases.

Subcase 1. $B_0 \subset R$.

Let $p \in R$ be such that $\bigcup_{n=0}^{\infty} B_n \subset [p,q]$, where q denotes the end point of R. Then $\mu|_{C([p,q])}$ is a Whitney map for $C([p,q])$. Since $(\mu|_{C([p,q])})^{-1}(0)$ is an arc, it has property b). By Lemma 4 of [10, p. 254], $(\mu|_{C([p,q])})^{-1}([0,t])$ has property b). Then there exists a map $g : (\mu|_{C([p,q])})^{-1}([0,t]) \to \mathbb{R}$ such that $\exp \circ g = f|_{C([p,q])}$ and $g(\{i_{B_1}\}) = h_0(\{i_{B_1}\})$. Notice that $h_0|_{(\mu|C([p,q]))^{-1}(0)}$ and $g|_{(\mu|C([p,q]))^{-1}(0)}$ are liftings of $f|_{(\mu|C([p,q]))^{-1}(0)}$ and $g(\{i_{B_1}\}) = h_0(\{i_{B_1}\})$. Then $h_0|_{(\mu|C([p,q]))^{-1}(0)} = g|_{(\mu|C([p,q]))^{-1}(0)}$.

Given $n \in \mathbb{N} \cup \{0\}$. Notice that h_{B_n} *y* $g|_{\alpha_{Bn}}$ are lift-liftings of $f|_{\alpha_{Bn}}$ and $h_{B_n}(i_{B_n}) = h_0(i_{B_n}) = g|_{\alpha_{Bn}}(i_{B_n})$. Then $h_{B_n} = g|_{\alpha_{B_n}}$. Hence,

$$\begin{aligned} h(B_0) &= h_{B_0}(B_0) = g(B_0) = \lim g(B_n) \\ &= \lim h_{B_n}(B_n) = \lim h(B_n). \end{aligned}$$

Subcase 2. $B_0 \subset I$.

We can consider that, for any $n, m \in \mathbb{N}$, $i_{B_n} \neq i_{B_m}$, if $n \neq m$.

Since $B_0 = \lim B_n$ and X is a compact space, we may assume that there exists $i_{B_0} \in B_0$ such that $i_{B_0} = \lim i_{B_n}$. We can suppose, taking subsequence if it is necessary, that there exists a subcontinuum α_{B_0} of $C(X)$ such that $\alpha_{B_0} = \lim \alpha_{B_n}$. It is easy to show that α_{B_0} is either an order arc from b_0 to B_0 or a one point-set.

Given $n \in \mathbb{N} \cup \{0\}$, we have $\mu|_{\alpha_{Bn}}$ is an homeomorphism between $[0, \mu(B_n)]$ and α_{B_n}. Let $g_n : [0, \mu(B_n)] \to \alpha_{B_n}$ be such that $g_n = (\mu|_{\alpha_{Bn}})^{-1}$.

By Lemma 3.1 of [7, p. 330], we can assume that X is a subset of $\mathbb{R}^2$. Let

$$D = X \times \{0\} \cup \left(\bigcup_{n=0}^{\infty} \left(\{i_{B_n}\} \times [0, \mu(B_n)] \right) \right).$$

Notice that D is a subset of the Euclidian space $\mathbb{R}^3$ and $X \times \{0\}$ is a deformation retract of D. Then D has property b).

We define $f_1 : D \to \mu^{-1}([0,t_0])$ by

$$f_1(x,t) = \begin{cases} x & \text{if } t = 0, \\ g_n(t), & \text{if } (x,t) \in \{i_{B_n}\} \times [0, \mu(B_n)]. \end{cases}$$

It is easy to prove that f_1 is a map. Since D has property b), there exists a map $h_3 : D \to \mathbb{R}$ such that $\exp \circ h_3 = f \circ f$ and $h_3(i_{B_1}, 0) = h_0(f_1(i_{B_1}, 0))_1$. Then $h_3|_{X \times \{0\}} = h_0 \circ f_1|_{X \times \{0\}}$. Thus, given $n \in \mathbb{N} \cup \{0\}$, it can prove that $h_3|_{\{i_{Bn}\} \times [0,\mu(B_n)]} = h_{B_n} \circ f_1|_{\{i_{Bn}\} \times [0,\mu(B_n)]}$. Hence, $h(B_0) = \lim h(B_n)$.

This proves that $\mu^{-1}([0,t]) - \{A_0\}$ has property b).

Theorem 4.2. *Let $X = I \cup R$ be an Elsa continuum and let $A \in C(I)$. Then A does not make a hole in $C(X)$.*

Proof. In light of Proposition 8 of [2], it suffices to prove that there exist two connected and closed subsets $\mathcal{A}$ and $\mathcal{B}$ of $C(X) - \{A\}$, which have property b) and the intersection of them is connected.

Let $\mu : C(X) \to [0,1]$ be a Whitney map. Let

$t = \mu(A)$, $\mathcal{A} = \mu^{-1}([t,1]) - \{A\}$ and $\mathcal{B} = \mu^{-1}([0,t]) - \{A\}$. Clearly $\mathcal{A} \cup \mathcal{B} = C(X) - \{A\}$, $\mathcal{A}$ and $\mathcal{B}$ are connected and closed subsets of $C(X) - \{A\}$.

By Lemma 13 of [2, p. 2004], $\mathcal{A}$ has property b) and, by Lemma 4.1, $\mathcal{B}$ has property b).

In order to show that $\mathcal{A} \cap \mathcal{B}$ is connected, notice that $\mathcal{A} \cap \mathcal{B} = \mu^{-1}(\{t\}) - \{A\}$ and $A \in \mu^{-1}(\{t\}) \cap C(I)$. By Corollary 3 of [11, p. 386],

$\mu^{-1}(\{t\}) = (\mu^{-1}(\{t\}) \cap C(I)) \cup (\mu^{-1}(\{t\}) \cap C(R))$ and $\mu^{-1}(\{t\}) \cap C(R)$ approximates the whole continuum $\mu^{-1}(\{t\}) \cap C(I)$. Hence, $\mathcal{A} \cap \mathcal{B}$ is connected.

Theorem 4.3. *Let* $X = I \cup R$ *be an Elsa continua. If* $A \in C(X)$ *such that* A *is homeomorphic to* X, *then* A *does not make a hole in* $C(X)$.

Proof. In light of Proposition 2.4 of [3, p. 3], it suffices to prove that there exists a closed neighborhood $\mathcal{W}$ of A in $C(X)$ such that $\mathcal{W} - \{A\}$ has property b) and $bd_{C(X)}(\mathcal{W})$ is connected ($bd_{C(X)}(\mathcal{W})$ denotes the boundary of $\mathcal{W}$ in $C(X)$).

Let $\mu : C(X) \to [0,1]$ be a Whitney map. Let $\mathcal{W} = \mu^{-1}([\mu(I),1])$. Clearly $\mathcal{W}$ is a closed neighborhood of A. Since $bd_{C(X)}(\mathcal{W}) = \mu^{-1}(\mu(I))$, $bd_{C(X)}(\mathcal{W})$ is connected. By [12, Theorem 4.3, p. 217], $\mathcal{W}$ is a 2-cell. Moreover, A is an element of its manifold boundary (see [11, Lemma 2, p. 386]). Then $\mathcal{W} - \{A\}$ is contractible. Therefore $\mathcal{W} - \{A\}$ has property b) (see [2, Proposition 9, p. 2001]).

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Theorem 4.4. *Let* $X = I \cup R$ *be an Elsa continuum and let* $A \in C(X)$. *Then* A *makes a hole in* $C(X)$ *if and only if* A *is a free arc* pq *such that* $p, q \notin int(pq)$.

Proof. Let $A \in C(X)$ be such that A makes a hole in $C(X)$. By Theorem 3 of [2, p. 2001] and Theorems 4.2 and 4.3, A is an arc pq contained in R. So, A is a free arc in X. By Theorem 4 of [2, p. 2001], $p, q \notin int(pq)$.

The sufficiency follows from Theorem 1 of [2, p. 2001].

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