Scientific Research

Sufficient Conditions of Optimality for Convex Differential Inclusions of Elliptic Type and Duality

Elimhan N. Mahmudov[1,2]

[1]Department of Industrial Engineering, Faculty of Management, Istanbul Technical University, Istanbul, Turkey
[2]Azerbaijan National Academy of Sciences, Institute of Cybernetics, Baku, Azerbaijan
Email: elimhan22@yahoo.com

ABSTRACT

This paper deals with the Dirichlet problem for convex differential (P_C) inclusions of elliptic type. On the basis of conjugacy correspondence the dual problems are constructed. Using the new concepts of locally adjoint mappings in the form of Euler-Lagrange type inclusion is established extremal relations for primary and dual problems. Then duality problems are formulated for convex problems and duality theorems are proved. The results obtained are generalized to the multidimensional case with a second order elliptic operator.

Keywords: Differential Inclusion; Dirichlet; Locally Adjoint; Sufficient Conditions; Duality

1. Introduction

The present paper is devoted to an optimal control problems described by so-called discrete and differential inclusions of elliptic type. A lot of problems in economic dynamics, as well as classical problems on optimal control in vibrations, chemical engineering, heat, diffusion processes, differential games, and so on, can be reduced to such investigations with ordinary and partial differential inclusions [1-15]. We refer the reader to the survey papers [11,16-20]. The present paper is organized as follows.

In Section 2 first are given some suitable definitions, supplementary notions and results considered by author in [18,19]. Then a certain extremal Dirichlet's problem is formulated for so-called elliptic differential (P_C) inclusions with Laplace's operator and with second order elliptic operator in the multidimensional case. In the reviewed results for optimality the arisen adjoint inclusions using the locally adjoint multivalued (LAM) functions are stated in the Euler-Lagrange form [9,18,19]. It turn out that such form of optimality conditions automatically implies the Weierstrass-Pontryagin maximum condition. Apparently it happens because the LAM is more applicable apparat in different type of problems governed by differential inclusions [16-20].

In Section 3 the main problem is to formulate and study the dual problems to the stated problems with convex structures. Convexity is a crucial marker in classifying optimization problems, and it's often accompanied by interesting phenomena of duality. It is well known

that duality theory by virtue of its applications is one of the central directions in convex optimality problems. In mathematical economics duality theory is interpreted in the form of prices, in mechanics the potential energy and complementary energy are in a mutually dual relation, the displacement field and the stress field are solutions of the direct and the dual problems, respectively.

To establish the dual problem we use the duality theorems of operations of addition and infimal convolution of convex functions. Here a remarkable specific feature of second order elliptic partial differential inclusions in comparisons with ordinary ones is that they admit valuable results in the case of multidimensional domain. Our approach to establish duality theory for continuous problem is based on the passage to the formal limit from duality problem in approximating problem. But to avoid difficult and fatiguing calculations we omit it and announce only dual problem constructed for continuous problems (P_C) and then (P_M). Consequently construction of duality problem in our paper is an unforeseen part of the "iceberg". Further it is shown that direct and duality problems are connected to each other by the duality relations. The proved duality theorems allow one to conclude that a sufficient condition for an extremum is an extremal relation for the primary and dual problems. It means that if some pair of feasible solutions $(u(.),u^*(.))$ satisfy this relation, then $u(.)$ and $u^*(.)$ are solutions of the primary and dual problem, respectively. We note that a considerable part of the investigations of Ekeland and Temam [7] for simple variational problem is devoted to such problems. Besides there are similar results for ordinary differential

inclusions in [17-19]. Some duality relations and optimality conditions for an extremum of different control problems with partial differential inclusions can be found in [18,19]. At the end of Section 3 we consider a linear optimal control problem of elliptic type.

Furthermore, observe that in elliptic differential inclusions for simplicity of the exposition the solution is taken in the space of classical solutions. Apparently by passing to more general function spaces of generalized solutions the most natural approach for elliptic differential inclusions is the use of single-valued selections of a multivalued mapping [1,3,8,9].

2. Necessary Concepts and Problems Statements

Throughout this section and the next sections we use special notation conventional in the [18,19]. Let R^n be the n-dimensional Euclidian space, (u_1, u_2) is a pair of elements $u_1, u_2 \in R^n$ and $\langle u_1, u_2 \rangle$ is their inner product. A multivalued mapping $F : R^n \to P(R^n)$ ($P(R^n)$ denotes the family of all subsets of R^n) is convex if its graph $gph\, F = \{(u,v) : v \in F(u)\}$ is a convex subset of R^{2n}. It is convex-valued if $F(u)$ is a convex set for each $u \in dom\, F = \{u : F(u) \neq \varnothing\}$. Let us introduce the notations:

$$M(u, v^*) = \sup_{v}\{\langle v, v^* \rangle : v \in F(u)\},$$

$$F(u, v^*) = \{v \in F(u) : \langle v, v^* \rangle = M(u, v^*)\},$$

$$v^* \in R^n$$

For convex F we let $M(u, v^*) = -\infty$ if $F(u) = \varnothing$. Obviously the function M and the sets $F(u, v^*)$ can be interpreted as Hamiltonian function and argmaximum sets, respectively.

Definition 2.1. *For a convex mapping F a multivalued mapping from R^n into R^n defined by*

$$F^*(v^*, (u,v)) = \{u^* : (u^*, -v^*) \in K^*_{gph\, F}(u,v)\}$$

is called the locally adjoint mapping (LAM) to F at the point $(u,v) \in gph\, F$, where $K^*_{gph\, F}(u,v)$ is the dual to the basic cone $K_{gph\, F}(u,v)$. We refer to [1,6,8,9] for various definitions in this direction.

It is clear that for a convex F the Hamiltonian is concave on u and convex on v^* function. Let us denote

$$H(u^*, v^*) = \inf\{\langle u, u^* \rangle - \langle v, v^* \rangle : (u,v) \in gph\, F\}.$$

It is clear that by the conjugacy correspondence of convex analysis [4],[6-9]:

$$H(u^*, v^*) = \inf_{u}\{\langle u, u^* \rangle - M(u, v^*)\}$$

$$= -\left(-M(., v^*)\right)^*(-u^*).$$

Corollary 2.1.
The inclusion $u^* \in F^*(v^*(u,v))$ and equality $H(u^*, v^*) = \langle u, u^* \rangle - M(u, v^*)$ are equivalent.

In Section 3 we study the following problem for elliptic differential inclusion with homogeneous boundary value conditions:

$$\text{minimize} \quad J(u(.)) := \iint_{\mathbb{R}} g(u(x), x)\mathrm{d}x,$$

subject to

$$\Delta u(x) \in F(u(x), x), \; x \in R, \tag{1}$$

and

$$u(x) = 0, \; x \in B, \tag{2}$$

where Δ is a Laplace's operator, $F(., x) : R^n \to P(R^n)$ is multivalued mapping for all $x = (x_1, x_2)$ in the bounded region $\mathbb{R} \subset R^1 \times R^1$, a closed piecewise-smooth simple curve B is its boundary, $g : R^n \times \mathbb{R} \to R^1$ is a continuous convex function on u and $\mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2$. We label this continuous problem (P_C) and call it Dirichlet problem for elliptic differential inclusions. The problem is to find a solution $\tilde{u}(x)$ of the boundary value problem (1), (2) that minimizes the cost functional $J(u(.))$. Here, a feasible solution is understood to be a classical solution for simplicity of the exposition.

The subject of the research in Section 6 in the following multidimensional optimal control problem (P_M) for elliptic differential inclusions:
minimize

$$J(u(.)) := \int_G g(u(x), x)\mathrm{d}x,$$

subject to

$$Lu(x) \in F(u(x), x), \; x \in G, \tag{3}$$

and

$$u(x) = 0, \; x \in S \tag{4}$$

where $F(., x) : R^1 \to P(R^1)$ is a convex closed multivalued mapping for all n-dimensional vectors $x = (x_1, \cdots, x_n)$ in the bounded set $G \subset R^n$, a closed piecewise-smooth surface S is its boundary, $g : R^1 \times G \to R^1$ is a continuous and convex on u function, $\mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n$. L is a second-order elliptic operator:

$$Lu := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

$$a_{ij}(x) \in C^1(\overline{G}), b_i(x) \in C^1(\overline{G}), c(x) \in C(\overline{G})$$

where $\|a_{ij}(x)\|$ is a positively definite matrix, $\tilde{u}(x)$

and $C^1(\bar{G})$ are the spaces of continuous functions and functions having a continuous derivative in G, respectively.

A function $u(x)$ in $C^2(G) \cap C(\bar{G})$, that satisfies the inclusion (3) in G and the boundary condition (4) on S we call a classical solution of the problem posed, where $C^2(G)$ is the space of functions $u(x)$ having continuous all second-order derivatives. It is required to find a classical solution $\tilde{u}(x)$ of the boundary value problem (P_M) that minimizes the cost functional $J(u(.))$.

In the next theorem is referred sufficient conditions for optimality for problems (P_C) and (P_M) of Mahmudov [18].

Theorem 2.1. Assume that a continuous function g is convex with respect to u, and $F(.,x)$ is a convex mapping for all fixed x. Then for the optimality of the solution $\tilde{u}(x)$ among all feasible solutions in convex problem (P_C) it is sufficient that there exist a classical solution $u^*(x)$ such that the following condition:

a) $\Delta u^*(x) \in F^*\big(u^*(x),(\tilde{u}(x),\Delta\tilde{u}(x)),x\big)$
$$-\partial g(\tilde{u}(x),x),$$

$u^*(x)=0, \; x \in B,$

b) $\Delta\tilde{u}(x) \in F\big(\tilde{u}(x),u^*(x),x\big), \; x=(x_1,x_2)\in\mathbb{R}.$

For a problem (P_M) the Euler-Lagrange type inclusion (*a*) and argmaximum condition (*b*) consist of the following conditions, respectively:

i) $L^*u^*(x) \in F^*\big(u^*(x),(\tilde{u}(x),L\tilde{u}(x)),x\big)$
$$-\partial g(\tilde{u}(x),x),$$

ii) $L\tilde{u}(x) \in F\big(\tilde{u}(x),u^*(x),x\big), \; u^*(x)=0, x\in S$

where L^* is the operator adjoint to L.

3. On Duality in Elliptic Differential Inclusions

According to the definition in [4,8,9,18,19]

Let us denote

$J_*\big(u^*(x),z^*(x)\big)$
$$=\iint_{\mathbb{R}}\Big[H\big(\Delta u^*(x)+z^*(x),u^*(x),x\big)-g^*\big(z^*(x),x\big)\Big]dx$$

where H is a Hamiltonian function and $g^*(z^*,x)$ is conjugate function to function $g(\cdot,x)$ for every fixed $x\in R^1\times R^1$. Then the problem of determining the maximum

(P_D) $\underset{\substack{u^*(x),z^*(x),x\in\mathbb{R},\\u^*(x)=0,x\in B}}{\text{maximize}} J_*\big(u^*(x),z^*(x)\big)$

is called the dual problem to the primaryconvex problem (P_C). It is assumed that

$u^*(x)\in C^2(\mathbb{R})\cap C(\bar{\mathbb{R}}), \; z^*(x)\in C(\mathbb{R}).$

Theorem 3.1. Assume that $u(x), x\in\mathbb{R}$ is an arbitrarily feasible solution of the primary problem (P_C) and $\{u^*(x),z^*(x)\}$ is a feasible solution of the dual problem (P_D). Then the inequality $J(u(x))\geq J_*(u^*(x))$ is valid.

Proof. It is clear from the definitions of the functions H and g^* that the following inequalities hold:

$H\big(\Delta u^*(x)+z^*(x),u^*(x),x\big)$
$$\leq\big\langle\Delta u^*(x)+z^*(x),u(x)\big\rangle-\big\langle u^*(x),\Delta u(x)\big\rangle$$

$g^*\big(z^*(x),x\big)\geq\big\langle z^*(x),u(x)\big\rangle-g\big(u(x),x\big)$

Therefore;

$H\big(\Delta u^*(x)+z^*(x),u^*(x),x\big)-g^*\big(z^*(x),x\big)$
$$\leq\big\langle\Delta u^*(x),u(x)\big\rangle-\big\langle u^*(x),\Delta u(x)\big\rangle+g\big(u(x),x\big). \quad (5)$$

Then since $u^*(x)=0, u(x)=0, x\in B$, by the familiar Green theorem [21] we have

$$\iint_{\mathbb{R}}\Big[\big\langle\Delta u^*(x),u(x)\big\rangle-\big\langle u^*(x),\Delta u(x)\big\rangle\Big]dx$$
$$=\int_{B}\Bigg[\bigg\langle\frac{\partial u^*(x)}{\partial n},u(x)\bigg\rangle-\bigg\langle u^*(x),\frac{\partial u(x)}{\partial n}\bigg\rangle\Bigg]ds=0 \quad (6)$$

where n is outher normal for a curve B. Then integrating both sides of the inequality (5) due to (6) we obtain the required inequality.

Theorem 3.2. If the feasible solutions $\tilde{u}(x)$ and $\{u^*(x),z^*(x)\}$, $z^*(x)\in\partial g(\tilde{u}(x),x)$ satisfy the conditions of Theorem 2.1, then they are optimal solutions of the primary (P_C) and dual (P_D) problems, respectively, and their values are equal.

Proof. To proceed, first note that by Theorem 2.1 $\tilde{u}(x)$ is a solution of the primary problem (P_C). We need to prove that the pair $\{u^*(x),z^*(x)\}$ is a solution to problem (P_D). By Definition 2.1 of a LAM, the condition (*a*) of the Theorem 2.1 is equivalent to the inequality

$\big\langle\Delta u^*(x)+z^*(x),u-\tilde{u}(x)\big\rangle$
$$-\big\langle u^*(x),v-\Delta\tilde{u}(x)\big\rangle\geq 0,$$
$$(u,v)\in gphF(\cdot,\cdot,x).$$

The latter yields

$$\big(\Delta u^*(x)+z^*(x),u^*(x)\big)\in domH(\cdot,\cdot,x), \quad (7)$$

where

$$domH(\cdot,\cdot,x):=\big\{(u^*,v^*):H(u^*,v^*,x)>-\infty\big\}.$$

Further, since [4,6,8,9] $\partial g(u,x) \subset dom g^*(\cdot,x)$ it is clear that

$$z^*(x) \in dom g^*(\cdot,x). \tag{8}$$

Consequently, it can be concluded from (7.3), (7.4) that the indicated pair of functions $\{u^*(x), z^*(x)\}$ is a feasible solutions, *i.e.* the set of feasible solutions to (P_D) is nonempty. Let us now justify the optimality of the solution $\{u^*(x), z^*(x)\}$ to problem (P_D). By the Corollary 2.1

$$F^*(v^*,(u,v),x)$$
$$= \{u^* : H(u^*,v^*,x) = \langle u,u^* \rangle - M(u,v^*,x)\}.$$

Using this formula and the condition (a) of the Theorem 2.1 we get

$$H(\Delta u^*(x) + z^*(x), u^*(x), x)$$
$$= \langle \tilde{u}(x), \Delta u^*(x) + z^*(x) \rangle - M(\tilde{u}(x), u^*(x), x).$$

Now based on the condition (c) of Theorem 2.1 we have the following equality

$$\langle \Delta \tilde{u}(x), u^*(x) \rangle = M(\tilde{u}(x), u^*(x), x).$$

Thus

$$H(\Delta u^*(x) + z^*(x), u^*(x), x)$$
$$= \langle \tilde{u}(x), \Delta u^*(x) + z^*(x) \rangle - \langle \Delta \tilde{u}(x), u^*(x) \rangle. \tag{9}$$

On the other hand the inclusion $z^*(x) \in \partial g(\tilde{u}(x), x)$ is equivalent with the equality

$$g^*(z^*(x), x) = \langle \tilde{u}(x), z^*(x) \rangle - g(\tilde{u}(x), x). \tag{10}$$

Then in view of (8)-(10) as in the proof of Theorem 3.1 it is not hard to show that $J(\tilde{u}(x)) = J_*(u^*(x), z^*(x))$. This completes the proof of the theorem.

Now let us formulate the dual problem to the convex problem (P_M) with homogeneous boundary conditions. In this case the duality problem consists in the following

$$(P_{MD}) \quad \underset{\substack{u^*(x), z^*(x), x \in G, \\ u^*(x)=0, x \in S}}{\text{maximize}} J_*(u^*(x), z^*(x)),$$

Here

$$J_*(u^*(x), z^*(x))$$
$$= \int_G \left[H(L'u^*(x) + z^*(x), u^*(x), x) - g^*(z^*(x), x) \right] dx$$

$$u^*(x) \in C^2(G) \cap C(\bar{G}), \ z^*(x) \in C(G),$$

$$x = (x_1, \cdots, x_n).$$

Now by replacing the Laplace operator Δ with the second order elliptic operator L and using the idea sug-

gested in the proofs of Theorems 3.1 and 3.2 it is easy to get the following theorem.

Theorem 3.3. If $\tilde{u}(x)$ and pair of functions $\{u^*(x), z^*(x)\}$, are feasible solutions to the primary convex problem (P_M) with homogeneous boundary value conditions and dual problem (P_{MD}), respectively, then $J(\tilde{u}(x)) \geq J_*(u^*(x), z^*(x))$. In addition, if the assertions (i), (ii) for sufficiency of optimality are valid here and $z^*(x) \in \partial g(\tilde{u}(x), x)$, then the values of the cost functionals are equal and $\{u^*(x), z^*(x)\}$ is solution of the dual problem (P_{MD}).

Let us consider the following example:

$$(P_{LD}) \quad \text{minimize } J(u(x)) = \iint_R g(u(x), x) dx$$

subject to $\Delta u(x) = Au(x) + Bw(x)$, $w(x) \in V$,

where A is $n \times n$ matrix, B is a rectangular $n \times r$ matrix, $V \subset R^r$ is a closed convex set and g is continuously differentiable function on u. It is required to find a controlling parameter $w(x) \in V$ such that the feasible solution corresponding to it minimizes $J(u(\cdot))$.

Let us introduce a convex mapping $F(u) = Au + BV$. By elementary calculations, it can be shown, that

$$H(u^*, v^*) = \inf_{u,w} \left\{ \langle u, u^* \rangle - \langle Au + Bw, v^* \rangle : w \in V \right\}$$

$$= \begin{cases} -M_V(B^*v^*), & u^* = A^*v^*, \\ -\infty, & u^* \neq A^*v^* \end{cases}$$

where $M_V(B^*w^*) = \sup_{w \in V} \langle w, B^*w^* \rangle$.

Then obviously the duality problem for primary problem (P_{LD}) has a form:

$$\text{maximize } J_*(u^*(x), z^*(x)),$$

$$\Delta u^*(x) + z^*(x) = A^*u^*(x), x \in R,$$

$$u^*(x) = 0, x \in B,$$

where

$$J_*(u^*(x), z^*(x)) =$$
$$-\iint_{\mathbb{R}} \left[M_V(B^*u^*(x) + g^*(z^*(x), x) \right] dx.$$

REFERENCES

[1] A. Kurzhanski, "Set-Valued Analysis and Differential Inclusions," Birkhäuser, Boston, 1993.

[2] J.-L. Lions, "Optimal Control of Systems Governed by Partial Differential Equations," Springer, Berlin, 1991.

[3] J. P. Aubin and A. Cellina, "Differential Inclusion," Springer-Verlag, Berlin, 1984.

[4] R. T. Rockafellar, "Convex Analysis," 2nd Edition, Princeton University, Hoboken, 1972.

APM

[5]   V. L. Makarov and A. M. Rubinov, "The Mathematical Theory of Economic Dynamics and Equilibrium," in Russian, Nauka, Moscow, 1973.

[6]   F. H. Clarke, "Optimization and Nomsmooth Analysis," John Wiley, New York, 1983.

[7]   I. Ekeland and R. Teman, "Analyse Convexe et Problems Variationelles," Dunod and Gauthier Villars, Paris, 1972.

[8]   A. D. Ioffe and V. M. Tikhomirov, "Theory of Extremal Problems," in Russian, Nauka, Moscow, 1974.

[9]   B. S. Mordukhovich, "Approximation Methods in Problems of Optimization and Control," in Russian, Nauka, Moscow, 1988.

[10] B. N. Pshenichnyi, "Convex Analysis and Extremal Problems," in Russian, Nauka, Moscow, 1980.

[11] R. P. Agarwal and D, O'Regan, "Fixed-Point Theory for Weakly Sequentially Upper-Semicontinuous Maps with Applications to Differential Inclusions," *Nonlinear Oscilations*, Vol. 5, No. 3, 2002, pp. 277-286. doi:10.1023/A:1022312305912

[12] R. Vinter, "Optimal Control," Birkhäuser, Boston, 2000.

[13] K. Wilfred, "Maxima and Minima with Applications, Practical Optimization and Duality," John Wiley and Sons, Inc., New York, 1999.

[14] W.-X. Zhong, "Duality System in Applied Mechanics and Optimal Control," Springer, New York, 1992.

[15] E. N. Mahmudov, "Approximation and Optimization of Discrete and Differential Inclusions," Elsevier, New York, 2011.

[16] E. N. Mahmudov, "Duality in the Problems of Optimal Control Described by First Order Partial Differential Inclusions," *Optimization*: *A Journal of Mathematical Programming and Operations Research*, Vol. 59, No. 4, 2010, pp. 589-599. doi:10.1080/02331930802434666

[17] E. N. Mahmudov, "Duality in Optimal Control Problems of Optimal Control Described by Convex Discrete and Differential Inclusions with Delay," *Automat Remote Control*, Vol. 48, No. 2, 1987, pp. 13-15.

[18] E. N. Mahmudov, "Necessary and Sufficient Conditions for Discrete and Differential Inclusions of Elliptic Type," *Journal of Mathematical Analysis and Applications*, Vol. 323, No. 2, 2006, pp. 768-789. doi:10.1016/j.jmaa.2005.10.069

[19] E. N. Mahmudov, "Locally Adjoint Mappings and Optimization of the First Boundary Value Problems for Hyperbolic Type Discrete and Differential Inclusions," *Nonlinear Analysis*: *Theory, Methods & Applications*, Vol. 67, No. 10, 2007, pp. 2966-2981. doi:10.1016/j.na.2006.09.054

[20] E. N. Mahmudov, "On Duality in Problems of Theory of Convex Difference Inclusions with Aftereffect," (*Russian*) *Differentsialnye Uravneniya*, Vol. 146, No. 8, 1987, pp. 1315- 1324.

[21] V. P. Mikhailov, "Partial Differantial Equations," in Russian, Nauka, Moscow, 1976.