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The Manifolds with Ricci Curvature Decay to Zero*

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ABSTRACT

The paper quotes the concept of Ricci curvature decay to zero. Base on this new concept, by modifying the proof of the canonical Cheeger-Gromoll Splitting Theorem, the paper proves that for a complete non-compact Riemannian manifold *M* with Ricci curvature decay to zero, if there is a line in *M*, then the isometrically splitting $M = R \times N$ is true.

Keywords: Cheeger-Gromoll Theorem; Busemann Function; Complete Riemannian Manifold; Ricci Curvature Decay to Zero

1. Introduction

In 1971, J. Cheeger and D. Gromoll [1] proved the following classical:

Cheeger-Gromoll Splitting Theorem: Let M be a complete Riemannian manifold with

$$RicM \geq 0 .$$

If there is a line in *M*, then the isometrically splitting $M = R \times N$ is true.

The proof of Cheeger-Gromoll Splitting Theorem is based on the sub-harmonicity of the Busemann functions, we will give some details in what follows. Let *M* be a noncompact complete Riemannian manifold and

$$\gamma : [0, +\infty) \to M$$

be a ray of *M*. Busemann function B_γ is defined as

$$B_r(x) = \lim_{t\to\infty}\left(t - d\left(x, \gamma(t)\right)\right) \quad (1)$$

For a given point *x* in *M* and an arbitrary $t > 0$, let $\varsigma : [0, l] \to M$ be the normal shortest geodesic from *x* to $\gamma(t)$. Supposed that the ball $B_\varepsilon(0) \subset T_x M$ and the exponential mapping

$$\exp_x : B_\varepsilon(0) \to M$$

is embedding, so

$$\exp_x : B_\varepsilon(0) \to U = \exp_x B_\varepsilon(0)$$

is a differential homeomorphic mapping. Thus for any $y \in U$, there is $Y \in B_\varepsilon(0)$ such that $y = \exp_x(Y)$. Let $Y(s)$ be the parallel vector field along ς, $Y(0) = Y$, and

$$\varsigma_y(s) = \exp_{\varsigma(s)}\left(1 - \frac{1}{s}\right)Y(s).$$

Thus $\varsigma_y(s) : [0,1] \to M$ is a smooth curve which connects y with $\gamma(t)$. Moreover, $\varsigma_x = \varsigma$. Now let $\bar{g}_t(y) = L(\varsigma_y)$ and $g_t = t - \bar{g}_t$. Then g_t is a smooth function defined in *U*. It is easily to know that g_t supports

$$B_\gamma^t(y) = t - d\left(y, \gamma(t)\right), \; y \in U$$

at *x*, which means that

$$g_t(y) \leq B_\gamma^t(y), \; y \in U, \; g_t(x) = B_\gamma^t(x).$$

By [2], one has

$$\Delta g_t(x) = \int_0^l \left[-\frac{n-1}{l^2} + \left(1 - \frac{s}{l}\right)^2 Ric\left(\dot{\varsigma}(s), \dot{\varsigma}(s)\right)\right] ds . \quad (2)$$

The outline of the proof of Cheeger-Gromoll Theorem:

1) It is able to show that

$$SB_\gamma(x) = \lim_{t\to\infty} SB_\gamma^t(x) \geq \lim_{t\to\infty} \Delta g_t(x) \geq 0,$$

then, B_γ is subharmonic function, where *S* is a operator generalized from Laplace operator Δ, one can refer to [2] for details.

2) Since there is a line $L = \gamma^+ \cup \gamma^-$, by 1), one is able to prove that B_{γ^+} and B_{γ^-} are harmonic functions on *M*.

3) Let $\varphi = dB_{\gamma^+} = df$. By

$$\langle \Delta\varphi, \varphi \rangle = \sum_i \left|D_{X_i}\varphi\right|^2 + \frac{1}{2}\Delta|\varphi|^2 + Ric\left(\varphi^*, \varphi^*\right), \quad (3)$$

it can be proved that $grad\ f = (df)^*$ is a unite parallel

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vector field on M, where φ^* is the couple tangential vector field of φ induced by the Riemannian metric.

4) By de Rham Partition Theorem, one has that $M = R \times N$.

Cheeger-Gromoll Splitting Theorem and its proof are so excellent that there is few generalization can be found, the only result we known is in Cai's paper [3], in which a local splitting theorem was got. In order to narrate our main result, we quote the following.

Definition 1: Let M be a noncompact complete Riemannian manifold. Suppose that there are two continuous functions $h(s)$, $f(s)$, $s \in [0,+\infty)$, satisfying

$$\int_0^\infty |f(s)| \mathrm{d}s < \infty, \int_0^\infty |h(s)| \mathrm{d}s = 0, \tag{4}$$

such that for any normal shortest geodesic $\varsigma : [0,l] \to M$, for any $s \in [0,l]$

$$\left| Ric(\dot{\varsigma}(s), \dot{\varsigma}(s)) \right| \le f(s), Ric(\dot{\varsigma}(s), \dot{\varsigma}(s) \ge h(s)), \tag{5}$$

then we say M is with Ricci curvature decay to zero.

From the definition, according to (4), (5), it is clear of that

$$\lim_{s \to \infty} Ric(\dot{\varsigma}(s), \dot{\varsigma}(s)) = 0.$$

A simple example of $h(s)$ satisfying (4) is defined as $h(s) = \frac{\sin s}{(k+1)^2}$, $s \in [2k\pi, (2k+2)\pi]$, $k = 0,1,2,\cdots$ Then we have:

Theorem 1: Let M be with Ricci curvature decay to zero. If there is a line included in M, then isometrically splitting $M = R \times N$ is still true.

By the way, when one discusses the relationship between a kind of curvature and topology of a Riemannian manifold, he generally assume that the sign of the kind of curvature is fixed, for examples, in [1] and [5], the authors assumed that the manifolds is with nonnegative Ricci curvature. If without this assumption, the corresponding problem seems more difficult, this is the reason we write out this short paper, though it is not easy to construct a manifold with Ricci curvature decay to zero for the time being.

2. The Proof of Theorem 1

Our argument follows closely that of Cheeger-Gromoll Splitting Theorem, but we should overcome some difficulties, especially in how to prove *grad f* is a unite parallel vector field of M.

Lemma 1: Let M be a complete noncompact Riemannian manifold with Ricci curvature decay to zero. Then the Busemann functions on M are subharmonic.

Proof: Let $\gamma : [0.+\infty) \to M$ be a ray in M. Just like above discussion, we can construct a smooth g_t supports

$$B_\gamma^t(y) = t - d(y, \gamma(t)), y \in U$$

at x. So

$$\begin{aligned} \Delta g_t(x) &= \int_0^l \left[-\frac{n-1}{l^2} + \left(1 - \frac{s}{l}\right)^2 Ric(\dot{\varsigma}(s), \dot{\varsigma}(s)) \right] \mathrm{d}s \\ &\ge -\frac{n-1}{l} + \int_0^l \left(1 - \frac{s}{l}\right)^2 h(s) \mathrm{d}s. \end{aligned} \tag{6}$$

By the assumption (4) and L'Hospital Rule, it is obvious that

$$\lim_{l \to \infty} \int_0^l \left(1 - \frac{s}{l}\right)^2 h(s) \mathrm{d}s = 0, \tag{7}$$

By (6) and (7), we have

$$SB_\gamma(x) = \lim_{t \to \infty} SB_\gamma^t(x) \ge \lim_{t \to \infty} \Delta g_t(x) \ge 0, \tag{8}$$

which means that B_γ is subharmonic.

The proof of Theorem 1: Let M be a complete noncompact Riemannian manifold with Ricci curvature decay to zero. Then the Busemann functions on M are subharmonic. If there is a line

$$\gamma : (-\infty, +\infty) \to M,$$

then $\gamma^+ = \gamma|_{[0,+\infty)}$ and $\gamma^- = \gamma(-t), t \in [0,+\infty)$ are rays in M. It is easily to know that

$$B_{\gamma^+}(x) + B_{\gamma^-}(x) \le 0,$$

$$B_{\gamma^+}(\gamma(t)) + B_{\gamma^-}(\gamma(t)) = 0.$$

By maximum principle, B_{γ^+} and B_{γ^-} are harmonic functions on M. By canonical Weyl Theorem (c.f. [2]), we know that they are all smooth in M. In simplicity, we set $B = B_{\gamma^+}$. It is well known that

$$|dB| = |gradB| = 1. \tag{9}$$

Let

$$M_t = \{x | : x \in M, B(x) = t\}, t \in (-\infty, +\infty).$$

By (9), B has not critical point, this means M_t is smooth hypersurface of M.

Supposed that X and Y are tangential vector fields on M, the Hessian of B satisfies

$$D^2 B(X.Y) = D^2 B(Y, X)$$

$$\begin{aligned} \langle D_X gradB, Y \rangle &= D^2 B(X.Y) = D^2 B(Y, X) \\ &= \langle D_Y gradB, X \rangle \end{aligned}$$

In particular, by (9),

$$0 = 2\langle D_X gradB, gradB \rangle = 2\langle D_{gradB} gradB, X \rangle.$$

Since X is an arbitrary tangential vector field on M, we

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have

$$D_{gradB}\, gradB = 0\,.$$

This means the integral curves of *grad B* are the geodesics in *M*. It is clear that *grad B* is unit normal vector field of M_t. By reviewing the definition of mean curvature of the horizontal hypersurface M_t, we have

$$H = \frac{1}{n-1} tracD^2 B = \frac{1}{n}\Delta B = 0\,. \qquad (10)$$

Setting $N = gradB$, by 1.5.8 of [4],

$$H' \le -\frac{1}{n-1} Ric(N,N) - H^2\,,$$

which means that

$$Ric(N,N) \le 0\,, \qquad (11)$$

Let $\varsigma_l = \varsigma(s)$ be the shortest geodesic from x to $\gamma(t)$ as before, where l is the length of $\varsigma(s)$, without loss of the generality, we can assume that

$$\lim_{l\to\infty} \varsigma_l = \varsigma_x$$

is a ray emanating from x, which is asymptotic to γ. By [2,5], we know that

$$B(\varsigma_x(t)) = B(x) + t,\ \forall t \in [0,+\infty)\,,$$

which means that ς_x is a integral curve of $N = gradB$ and $\dot{\varsigma}_x = N$.

Now by (4), (5) and (6),

$$\begin{aligned} 0 &= \lim_{l\to\infty} \int_0^l \left(1-\frac{s}{l}\right)^2 h(s)\,ds \\ &\le \lim_{l\to\infty} \int_0^l \left(1-\frac{s}{l}\right)^2 Ric(\dot{\varsigma}_l(s), \dot{\varsigma}_l(s))\,ds \\ &= \lim_{l\to\infty} \int_0^l Ric(\dot{\varsigma}_l(s), \dot{\varsigma}_l(s))\,ds \\ &\le \lim_{l\to\infty} \int_0^l |f(s)|\,ds < \infty. \end{aligned} \qquad (13)$$

by Lebesgue Control Convergent Theorem, by (4), (5) and (13), we have

$$\int_0^\infty Ric(N,N)\,ds = 0\,, \qquad (14)$$

By (11), (14), we have

$$Ric(N,N) = 0\,.$$

By (3), it is easily know that $N = gradB$ is a unit parallel vector field of *M*. By de Rham Partition Theorem, we have $M = R \times N$. Thus we get Theorem 1.

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