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A Note on the (Faith-Menal) Counter Example

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ABSTRACT

Faith-Menal counter example is an example (unique) of a right John's ring which is not right Artinian. In this paper we show that the ring T which considered as an example of a right Johns ring in the (Faith-Menal) counter example is also Artinian. The conclusion is that the unique counter example that says a right John's ring can not be right Artinian is false and the right Noetherian ring with the annihilator property rl(A) = A may be Artinian.

Keywords: John's Ring; Artinian and Noetherian Rings

1. Introduction

A ring R is called right John's ring if it is right Noetherian and every right Ideal A of R is a right annihilator *i.e.* rl(A) = A for all right ideals A of R.

John's ([1], Theorem 1) by using a result of Kurshan ([2], Theorem 3.3), showed that a right Noetherian ring is right Artinian provided that every right ideal is a right annihilator.

Ginn [3] showed that Kurshan result was false. Ginn's example does not provide a counter example to John's theorem. Therefore the validity of John's theorem was doubtful.

Faith-Menal counter example proved that there is an example (unique) of a right John's ring which is not right Artinian.

Here in this paper we prove the false of the Faith-Menal counter example by proving that the considered non-Artinian right john's ring is in fact right Artinian. So the John's theorem may be true see [1].

All rings considered in this paper are associative rings with identity.

We recall the Faith-Menal counter example in Section 1 and we prove that it is false in Section 2.

2. Section 1: The Counter Example

Example 8.16 (Faith-Menal) [4]. Let D be any countable, extentially closed division ring over a field F, and let $R = D \otimes_F F(x)$. Then T(R, D) is a non-Artinian right John's ring.

Proof

Cohn shows that R is simple, principle right ideal domain that is right V ring (Theorems 8.4.5 & 5.5.5 [4]) and D is an R-R bimodule such that D_R is the unique simple right R module. Hence T(R, D) is a right John's ring by Theorem 8.15 (in this book). But T(R, D) is not Artinian because if it were then R would also right Artinian and hence a field which is a contradiction.

Here

$$T = R \oplus D \quad \text{and} \quad T/\langle 0, d\rangle \cong R$$

For more information about this example see [4].

3. Section 2: A Note on the Counter Example

Theorem

The right John's ring T(R, D) defined in the counter example is Artinian.

Proof

Recall the following (Exercises 10.7 [5]):

Let φ: D→R be a ring homomorphism and let M be a right R-module (or left R-module) then

1) Via φ M is right D-module;

2) If M_D is Artinian or Noetherian then so is M_R;

3) If R is finite dimensional algebra (via φ) over a field D then the following is equivalent:

a) M_R is Artinian and Noetherian

b) M_R is finitely generated

c) M_D is finite dimension

1) Consider the ring homomorphism $\varphi: D \to R$ defined by $\varphi(d) = d \otimes 1_F$. Every R-module homomorphism is a D-homomorphism via φ. Since D is a division ring so it will be semisimple ring and hence every right D-module is semisimple (Corollary 8.2.2 [3]).

2) If the ring $T = R \oplus D$ is John's ring (as in the counter example above) then it is Noetherian and hence R and D are Noetherian. As D is right Noetherian ring then every finitely generated right D module is Noetherian (6.1.3 [6]).

3) Every finitely generated right D-module M is semisimple so it is right Artinian.

4) Since every finitely generated right D-module M is Artinian and Noetherian then M is Artinian and Noetherian as an R-module.

5) Now since R is simple principle ideal domain then R is a finite dimensional k-algebra where k is a subring of the center of R with identity 1_R and hence R is a finite dimensional algebra over the field Z (D) the center of D.

Applying the above equivalence we get that every finitely generated right R-module is Artinian and hence R is right Artinian and this imply that T is also Artinian.

4. Conclusion

The conclusion is that the unique example that says a right John's ring can not be right Artinian is false.

REFERENCES

[1] B. Johns, "Annihilator Conditions in Noetherian Rings," *Journal of Algebra*, Vol. 49, No. 1, 1977, pp. 222-224. doi:10.1016/0021-8693(77)90282-4

[2] R. P. Kurshan, "Rings Whose Cyclic Modules Have Finitely Generated Socle," *Journal of Algebra*, Vol. 15, No. 3, 1970, pp. 376-386. doi:10.1016/0021-8693(70)90066-9

[3] S. M. Ginn, "A Counter Example to a Theorem of KurshAn," *Journal of Algebra*, Vol. 40, No. 1, 1976, pp. 105-106. doi:10.1016/0021-8693(76)90090-9

[4] W. K. Nicholson and M. F. Yousif, "Quasi-Frobenius Rings," *Series Cambridge Tracts in Mathematics*, No. 158, 2003.

[5] F. W. Anderson and K. R. Fuller, "Rings and Categories of Module," Springer Verlag, New York, 1991.

[6] F. Kasch, "Modules and Rings, London Mathematical Society Monographs," Vol. 17, Academic Press, New York, 1982.

