A Note on the (Faith-Menal) Counter Example

R. H. Sallam
Mathematics Department, Faculty of Science, Helwan University, Cairo, Egypt
Email: rsallams@hotmail.com

Received September 1, 2011; revised October 15, 2011; accepted October 22, 2011

ABSTRACT

Faith-Menal counter example is an example (unique) of a right John’s ring which is not right Artinian. In this paper we show that the ring T which considered as an example of a right Johns ring in the (Faith-Menal) counter example is also Artinian. The conclusion is that the unique counter example that says a right John’s ring can not be right Artinian is false and the right Noetherian ring with the annihilator property rl(A) = A may be Artinian.

Keywords: John’s Ring; Artinian and Noetherian Rings

1. Introduction

A ring R is called right John’s ring if it is right Noetherian and every right ideal A of R is a right annihilator i.e. rl(A) = A for all right ideals A of R.

John’s ([1], Theorem 1) by using a result of Kurshan ([2], Theorem 3.3), showed that a right Noetherian ring is right Artinian provided that every right ideal is a right annihilator.

Ginn [3] showed that Kurshan result was false. Ginn’s example does not provide a counter example to John’s theorem. Therefore the validity of John’s theorem was doubtful.

Faith-Menal counter example proved that there is an example (unique) of a right John’s ring which is not right Artinian.

Here in this paper we prove the false of the Faith-Menal counter example by proving that the considered non-Artinian right John’s ring is in fact right Artinian. So the John’s theorem may be true see [1].

All rings considered in this paper are associative rings with identity.

We recall the Faith-Menal counter example in Section 1 and we prove that it is false in Section 2.

2. Section 1: The Counter Example

Example 8.16 (Faith-Menal) [4]. Let D be any countable, externally closed division ring over a field F, and let R = D ⊗_F (x). Then T(R, D) is a non-Artinian right John’s ring.

Proof

Cohn shows that R is simple, principle right ideal domain that is right V ring (Theorems 8.4.5 & 5.5.5 [4]) and D is an R-R bimodule such that DR is the unique simple right R module. Hence T(R, D) is a right John’s ring by Theorem 8.15 (in this book). But T(R, D) is not Artinian because if it were then R would also right Artinian and hence a field which is a contradiction. Here

T = R ⊕ D and T/(0, d) ≅ R

For more information about this example see [4].

3. Section 2: A Note on the Counter Example

Theorem

The right John’s ring T(R, D) defined in the counter example is Artinian.

Proof

Recall the following (Exercises 10.7 [5]):

Let φ: D → R be a ring homomorphism and let M be a right R-module (or left R-module) then

1) Via φ M is right D-module;
2) If MD is Artinian or Noetherian then so is MR;
3) If R is finite dimensional algebra (via φ) over a field D then the following is equivalent:
   a) MR is Artinian and Noetherian
   b) MR is finitely generated
   c) MD is finite dimension

1) Consider the ring homomorphism φ: D → R defined by φ(d) = d ⊗_F 1. Every R-module homomorphism is a D-homomorphism via φ. Since D is a division ring so it will be semisimple ring and hence every right D-module is semisimple (Corollary 8.2.2 [3]).

2) If the ring T = R ⊕ D is John’s ring (as in the counter example above) then it is Noetherian and hence R and D are Noetherian. As D is right Noetherian ring then every finitely generated right D module is Noetherian (6.1.3 [6]).

3) Every finitely generated right D-module M is semisimple so it is right Artinian.
4) Since every finitely generated right D-module M is Artinian and Noetherian then M is Artinian and Noetherian as an R-module.

5) Now since R is simple principle ideal domain then R is a finite dimensional k-algebra where k is a subring of the center of R with identity 1_R and hence R is a finite dimensional algebra over the field Z(D) the center of D.

Applying the above equivalence we get that every finitely generated right R-module is Artinian and hence R is right Artinian and this imply that T is also Artinian.

4. Conclusion

The conclusion is that the unique example that says a right John’s ring can not be right Artinian is false.

REFERENCES


