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A Simple Proof That the Curl Defined as Circulation Density Is a Vector-Valued Function, and an Alternative Approach to Proving Stoke's Theorem

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ABSTRACT

This article offers a simple but rigorous proof that the curl defined as a limit of circulation density is a vector-valued function with the standard Cartesian expression.

Keywords: Curl; Circulation Density

1. Introduction

The standard mathematical presentation that the curl defined as a limit of circulation density is a vector-valued function with the standard Cartesian expression uses Stokes' Theorem. Most physics books use the multidimensional version of Taylor's Theorem to show this relationship works in the x, y plane by using linear approximations of $\boldsymbol{F}$, and simply assert that this special case extends to three dimensional space, [1, pp. 71-72]. This approach requires that $\boldsymbol{F}$ have continuity in the second partials, which is not necessary. Also, the assertion that the two dimensional case extends to three dimensions is not trivial. A more elementary proof is presented here, using only Green's Theorem on a right triangle [2, p. 1102], and the Integral Mean Value Theorem [2, p. 1071].

2. A Criticism of Existing Methods to Explain and Prove the Properties of the Curl

2.1. The Approach of Most Physicists

In this approach, we suppose we have a vector field $\boldsymbol{F}(x,y,z) = P(x,y,z)\boldsymbol{i} + Q(x,y,z)\boldsymbol{j} + R(x,y,z)\boldsymbol{k}$ and we are at a point p in space. The $curl\boldsymbol{F}(p)$ is then "defined" to be the vector such that for all unit vectors $\boldsymbol{n}$, the following equation is true:

$$curl\boldsymbol{F}(p)\cdot\boldsymbol{n} = \lim_{a\to 0}\frac{\oint_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}}{a} \qquad (1)$$

where C is a tiny loop or contour about p in the plane containing p with normal $\boldsymbol{n}$, and a is the area of the interior of the loop, see [3, p. 81] and [4]. The limit on the right hand side of (1) is given the name "circulation density of $\boldsymbol{F}$ at p in the direction of $\boldsymbol{n}$", or usually just "circulation density" it being understood that a unit normal vector $\boldsymbol{n}$ has been chosen. This definition has two fatal flaws.

2.1.1. Flaw 1

We don't even know if the limit on the right hand side of this equation exists. Indeed, it looks to be very dubious as to whether it exists. As a crude thought experiment, if the loop was a circle of radius r and if the tangential component of $\boldsymbol{F}$ is 1, then in (1) we would be looking at

$$\lim_{r\to 0}\frac{2\pi r}{\pi r^2} = \lim_{r\to 0}\frac{2}{r}$$

which, of course, does not exist. This is not a counterexample, since the tangential component of $\boldsymbol{F}$ always being 1 excludes this $\boldsymbol{F}$ from being integrable in (1), but it points out that the limit does not obviously exist. Also, the area of the loop going to zero does not force the loop to collapse about point p.

2.1.2. Flaw 2

Even if the limit on the right hand side of (1) exists, this definition asserts that the limit produces the existence of a fixed vector $curl\boldsymbol{F}(p)$. The limit is just a scalar that depends on $\boldsymbol{n}$. This produces an infinite number of equations, one for each $\boldsymbol{n}$. Unless you can show linearity of the limit with respect to $\boldsymbol{n}$, you can not solve this system. You can't just assert the existence of this property.

Nobel Prize winning physicist, Edward Purcell, points out this second flaw in his Berkeley Series text book

Electricity and Magnetism [1, p. 70], where he says that it can be shown that the above equation does indeed define a vector but then adds, "... we shall not do so here". He cites no reference where it is shown that this definition proves that the vector $curl\boldsymbol{F}(p)$ exists. For those thinking that the above equation defines a vector, he creates a similarly defined quantity, $\text{squrl}\boldsymbol{F}(p)$, by

$$\text{squrl}\boldsymbol{F}(p)\cdot\boldsymbol{n}=\lim_{a\to 0}\left(\frac{\oint_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}}{a}\right)^2$$

and then asks the reader to show that $\text{squrl}\boldsymbol{F}(p)$ is not a vector [1, problem 2.32 in p. 85].

2.2. The Approach of Most Mathematicians

Mathematicians take as their definition the standard Cartesian formula,

$$curl\langle P,Q,R\rangle=\left\langle \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right\rangle. \quad (2)$$

Students find this formula mysterious and troubling. The formula should arise from the physical nature of the circulation density. It is the purpose of this paper to supply the motivation for (2).

3. Main Result

Given a point p and a unit vector $\boldsymbol{n}$, consider the plane containing p with normal $\boldsymbol{n}$. Form a tetrahedron by shifting the origin to the negative part of the line through p with direction $\boldsymbol{n}$. The plane intersects these new coordinate axes at $(X,0,0)$, $(0,Y,0)$, $(0,0,Z)$, see **Figure 1**. Call the triangle in the plane that connects these points T (even though $\boldsymbol{n}$ has all non-zero components, the proof below will work with minor modifications for $\boldsymbol{n}$ with some zero components).

Theorem 1 (curl of *F*) *Let* $\boldsymbol{F}:R^3\to R^3$ *have continuous partial derivatives. Given a unit vector* $\boldsymbol{n}$ *and a point p, let T be the triangle constructed as in* **Figure 1**. *The vector* $curl\boldsymbol{F}(p)$ *can be defined by*

$$curl\boldsymbol{F}(p)\cdot n=\lim_{\|T\|\to 0}\frac{\int_{\partial T}\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s}}{|T|}$$

where $|T|$ is the area of T and $\|T\|$ is the diameter of the smallest ball centered at P containing T.

Proof. The area of T is half of the parallelogram formed by the vectors $\langle X,0,-Z\rangle$ and $\langle 0,Y,-Z\rangle$. The area of the parallelogram is the length of the cross product, so that

$$\langle X,0,-Z\rangle\times\langle 0,Y,-Z\rangle=\langle YZ,XZ,XY\rangle$$

$$\left|\langle X,0,-Z\rangle\times\langle 0,Y,-Z\rangle\right|=2|T|$$

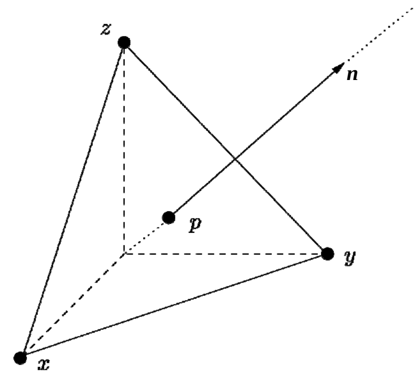

Figure 1. Local coordinate system.

and

$$\boldsymbol{n}=\left\langle \frac{YZ}{2|T|},\frac{XZ}{2|T|},\frac{XY}{2|T|}\right\rangle. \quad (3)$$

Let T_{xy} be the right-triangular patch in the xy plane with vertices $(X,0,0)$, $(0,0,0)$, $(0,Y,0)$. Using Green's Theorem and Integral Mean Value Theorem, we have that for some $(x^*,y^*,0)\in T_{xy}$,

$$\begin{aligned}\oint_{\partial T_{xy}}\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}&=\int_{\partial T_{xy}}F_1(x,y,0)\mathrm{d}x+F_2(x,y,0)\mathrm{d}y\\&=\iint_{T_{xy}}\frac{\partial F_2}{\partial x}(x,y,0)-\frac{\partial F_1}{\partial y}(x,y,0)\mathrm{d}A\\&=\left(\frac{\partial F_2}{\partial x}(x^*,y^*,0)-\frac{\partial F_1}{\partial y}(x^*,y^*,0)\right)|T_{xy}|\\&=\left(\frac{\partial F_2}{\partial x}(x^*,y^*,0)-\frac{\partial F_1}{\partial y}(x^*,y^*,0)\right)\frac{XY}{2}\\&=:c_3\frac{XY}{2}.\end{aligned}$$

Similarly for the other two right-triangular patches,

$$\begin{aligned}\oint_{\partial T_{yz}}\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}&=\left(\frac{\partial F_3}{\partial y}(0,y^*,z^*)-\frac{\partial F_2}{\partial z}(0,y^*,z^*)\right)\frac{YZ}{2}\\&=:c_1\frac{YZ}{2}\\\oint_{\partial T_{xz}}\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}&=\left(\frac{\partial F_3}{\partial x}(x^*,0,z^*)-\frac{\partial F_1}{\partial z}(x^*,0,z^*)\right)\frac{XZ}{2}\\&=:-c_2\frac{XZ}{2}.\end{aligned}$$

Reversing the orientation of ∂T_{xz}, all the paths along the coordinate axes cancel, see **Figure 2**, and

$$\begin{aligned}\oint_{\partial T}\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s}&=\oint_{\partial T_{yz}}\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s}-\oint_{\partial T_{xz}}\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s}+\oint_{\partial T_{xy}}\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s}\\&=c_1\frac{YZ}{2}+c_2\frac{XZ}{2}+c_3\frac{XY}{2}.\end{aligned}$$

Using (3), we then have

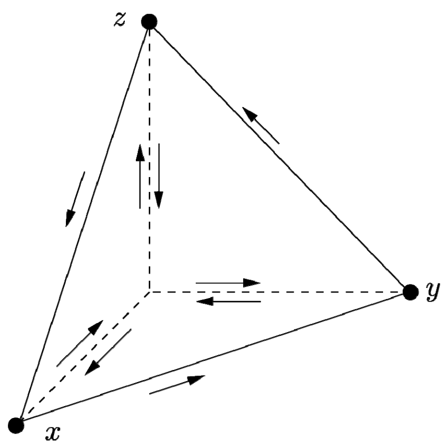

Figure 2. Traversing paths.

$$\oint_{\partial T} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \langle c_1, c_2, c_3 \rangle \cdot \boldsymbol{n} |T|$$

$$\frac{\oint_{\partial T} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}}{|T|} = \langle c_1, c_2, c_3 \rangle \cdot \boldsymbol{n}.$$

Taking the limit as $\|T\| \to 0$, the origin of the local coordinate system moves to *p* and $\langle c_1, c_2, c_3 \rangle \to curl\boldsymbol{F}(p)$ by the continuity of the partial derivatives.

By driving the radius of the ball containing the patch *T* to zero, we get that the limit of the circulation density is a fixed vector dot the normal, and that the expression of the fixed vector in cartesian coordinates is the standard expression for the $curl\boldsymbol{F}(p)$.

The last equation in the proof makes the following two propositions immediately obvious:

Proposition 1. The direction of maximum circulation density is in the direction of the curl.

Proposition 2. This maximum circulation density is in fact just the magnitude of the curl.

4. An Intuitive Proof of Stokes' Theorem

Theorem 2 (Stokes' Theorem) *Let* $\boldsymbol{F}: R^3 \to R^3$ *have continuous partial derivatives. Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve. Then*

$$\iint_S curl\boldsymbol{F} \cdot \boldsymbol{n}\mathrm{d}S = \oint_{\partial S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

Proof. Triangulate the surface *S*. Apply Theorem 1 to each of the triangular faces approximating *S*, and all interior paths cancel leaving an approximate boundary integral to the surface. Refine the approximation.

This idea can be made into a rigorous proof, but there is quite a bit of mathematical machinery that is necessary for the meaning of "refine the approximation" to be made precise. As a practical matter, any reasonable interpretation will suffice. For example, the area of the largest triangular face going to zero will suffice to refine the approximation [5].

REFERENCES

[1] E. Purcell, "Electricity and Magnetism, Berkeley Physics Course," 2nd Edition, McGraw Hill, New York, 1985.

[2] J. Stewart, "Calculus," 4th Edition, Brooks/Cole, Stamford, 1999.

[3] H. M. Schey, "Div, Grad, Curl, and All That: An Informal Text on Vector Calculus," 4th Edition, W. W. Norton, New York, 2005.

[4] E. Weisstein, "Wolfram Mathworld." http://mathworld.wolfram.com/Curl.html

[5] K. Hildebrandt, K. Polthier and M. Wardetzky, "On the Convergence of Metric and Geometric Properties of Polyhedral Surfaces," *Geometriae Dedicata*, Vol. 123, No. 1, 2005, pp. 89-112. doi:10.1007/s10711-006-9109-5

