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On BCL$^+$-Algebras

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ABSTRACT

This paper presents the BCL$^+$-algebras, which is derived the fundamental properties. Results are generalized with version of BCL-algebras [5], using some unusual for a binary relation * and a constant 1 (one) in a non-empty set *X*, one may take different axiom systems for BCL$^+$-algebras.

Keywords: BCL-Algebra; BCL$^+$-Algebra; Logic Algebra

1. Introduction

The BCK/BCI/BCH-algebra (see [1-4]) has been a major issue, but BCL-algebra (see [5]) is a new algebra structure—and we started to grasp the properties. This paper presents the BCL$^+$-algebras, we show that under our formulation, the BCL$^+$-algebra is a variant of a BCL-algebra. We can define by taking some axioms and important properties in this way for the BCL$^+$-algebras.

A BCL-algebra may be defined as a non-empty set *X* with a binary relation * and a constant 0 (zero) satisfying the following axioms:

Definition 1.1. [5] An algebra $(X;*,0)$ of type $(2,0)$ is said to be a BCL-algebra if and only if for any $x,y,z \in X$, the following conditions:

1) BCL-1: $x*x=0$;
2) BCL-2: $x*y=0$ and $y*x=0$ imply $x=y$;
3) BCL-3:

$$(((x*y)*z)*((x*z)*y))*((z*y)*x)=0.$$

Such set *X* in Definition 1.1 is called the underlying set of a BCL-algebra $(X;*,0)$, which needs the following theorem:

Theorem 1.1. [5] Algebra $(X;*,0)$ of type $(2,0)$ is a BCL-algebra if and only if it satisfies the following conditions: for all $x,y,z \in X$,

1) BCL-1: $x*x=0$;
2) BCL-2: $x*y=0$ and $y*x=0$ imply $x=y$;
3) $((x*y)*z)*((x*z)*y)=(z*y)*x$.

2. Main Result

The BCL$^+$ product, denoted by *. We call the binary operation * on *X* the * product on *X*, and the constant 1(one) of *X* the unit element of *X*. For brevity we often write *X* instead of $(X;*,1)$. We begin with the following definition:

Definition 2.1. An algebra $(X;*,1)$ is called a BCL$^+$-algebra if it satisfies the following laws hold: for any $x,y,z \in X$,

1) BCL$^+$-1: $x*x=1$;
2) BCL$^+$-2: $x*y=1$ and $y*x=1$ imply $x=y$;
3) BCL$^+$-3:

$$((x*y)*z)*((x*z)*y)=(z*y)*x.$$

Such definition, clearly, the BCL$^+$-algebra is a generalization of the BCL-algebra, imply a BCL-algebra is a BCL$^+$-algebra, however, the converse is not true. We illustrate with the next theorem.

Theorem 2.1. A BCL$^+$-algebra is existent.

Proof. The proof of this Theorem 2.1 is not difficult and uses only example. Let $X=\{0,1,2,3\}$. Define an operation * on *X*, which are given in **Table 1**.

Then $(X;*,1)$ is a proper BCL$^+$-algebra. It is easy to verify that there are

BCI-1:

$$\begin{aligned}&((2*3)*(2*1))*(1*3)\\&=(1*1)*3\\&=1*3\\&=3\neq 0;\end{aligned}$$

BCI-2:

$$\begin{aligned}&(2*(2*3))*3\\&=(2*1)*3\\&=1*3\\&=3\neq 0;\end{aligned}$$

BCH-3: 1) The left side of the equation is

$$(2*3)*1=1*1=1;$$

Table 1. BCL$^+$ operation.

*	0	1	2	3
0	1	0	0	0
1	0	1	3	3
2	3	1	1	1
3	1	3	3	1

2) The right side of the equation is

$$(2*1)*3 = 1*3 = 3.$$

In the expression we see that $1 \neq 3$.

BCL-3:

$$\left(\left((2*3)*1\right)*\left((2*1)*3\right)\right)*\left((1*3)*2\right)$$
$$= \left((1*1)*(1*3)\right)*(3*2)$$
$$= (1*3)*3$$
$$= 3*3$$
$$= 1 \neq 0.$$

BCL$^+$-3: 1) The left side of the equation is

$$\left((2*3)*1\right)*\left((2*1)*3\right)$$
$$= (1*1)*(1*3)$$
$$=1*3$$
$$=3;$$

2) The right side of the equation is

$$(1*3)*2 = 3*2 = 3.$$

In the expression we see that BCL$^+$-3 is valid. In fact, it is not difficult to verity that BCL$^+$-1 and BCL$^+$-2 are valid.

A BCL$^+$-algebra $(X;*,1)$ is a partially ordered relation $\leq$ on X, now we obtain the following definition:

Definition 2.2. Suppose that $(X;*,1)$ is a BCL$^+$-algebra, the ordered relation if

$$x \leq y \text{ if and only if } x*y=1, \quad \text{for all } x, y \in X, \tag{2.1}$$

then $(X;\leq)$ is partially ordered set and $(X;*,1)$ is an algebra of partially ordered relation.

Corollary 2.1. Let every $x \in X$. Then 1(one) is maximal element in a BCL$^+$-algebra $(X;*,1)$ such that

$$1 \leq x \text{ imply } x = 1. \tag{2.2}$$

Definition 2.3. A BCL$^+$-algebra X is called proper BCL$^+$-algebra if X is not a BCL-algebra.

Example 2.1. Let $X = \{0,a,b,c,1\}$. We define an operation * on X by **Table 2**.

In fact, it is not difficult to verify that $(X;*,1)$ is a BCL$^+$-algebra.

Table 2. BCL$^+$ operation.

*	0	*a*	*b*	*c*	1
0	1	0	0	0	0
a	*a*	1	1	*c*	1
b	*b*	*a*	1	*c*	1
c	*c*	*b*	*c*	1	1
1	0	1	1	1	1

Theorem 2.2. Assume that $(X;*,1)$ is any a BCL$^+$-algebra. Then the following hold: for any $x, y, z \in X$,

1) $\left(x*(x*y)\right)*y = 1$;
2) $x*1 = x$ imply $x = 1$;
3) $\left((x*y)*(x*z)\right)*(z*y) = 1$;
4) BCL$^+$-2: $x*y=1$ and $y*x=1$ imply $x=y$.

Proof. *Necessity*. By BCL$^+$-1 and 3), we obtain

$$\left(x*(x*y)\right)*y = \left((x*1)*(x*y)\right)*(y*1) = 1. \tag{2.3}$$

So, 1) holding.

By the same reasons, we derive

$$x*1 = (x*1)*1 = \left((x*1)*(x*1)\right)*(1*1) = 1. \tag{2.4}$$

Hence, 2) holding.

Sufficiency. It only needs to show BCL$^+$-1. Substituting y for 1(one) in 1), we have

$$\left(x*(x*1)\right)*1 = 1. \tag{2.5}$$

Replacing $x*1$ by y and x by z in 3), it follows

$$\left(\left(x*(x*1)\right)*(x*x)\right)*\left(x*(x*1)\right) = 1. \tag{2.6}$$

Using 2) and BCL$^+$-1, we get

$$\left(\left(x*(x*1)\right)*1\right)*\left(x*(x*1)\right) = 1. \tag{2.7}$$

Clearly, an application of (2.5) to (2.7) can give

$$1*\left(x*(x*1)\right) = 1. \tag{2.8}$$

Comparing (2.4) with (2.7) and using BCL$^+$-2, we get

$$x*(x*1) = 1. \tag{2.9}$$

Also, by 2) and 1), the following holds:

$$(x*1)*x = \left(x*(x*x)\right)*x = 1. \tag{2.10}$$

Combining (2.9) and (2.10) with 4) create $x*1 = x$. So Theorem 2.2 is valid.

Theorem 2.3. An algebra $(X;*,1)$ is a BCL$^+$-algebra if and only if it satisfies the following conditions: for all $x, y, z \in X$,

1) BCL$^+$-1: $x*x=1$;
2) BCL$^+$-2: $x*y=1$ and $y*x=1$ imply $x=y$;

3) $(((x*y)*z)*((x*z)*y))*((z*y)*x)=1$;

4) $x*(1*y)=x$.

Proof. The proof is routine. *Necessity*. To prove 1). By BCL$^+$-3.

$$\begin{aligned} x*x &= (x*x)*1 \\ &= ((1*x)*x)*((1*x)*x)=1. \end{aligned} \tag{2.11}$$

Then 1) holding.

Sufficiency. Substituting $x*1$ for y and x for z in 3), by BCL$^+$-3 and 1), it follows

$$\begin{aligned} &(((x*(x*1))*x)*((x*x)*(x*1)))*((x*(x*1))*x) \\ &=((1*x)*(1*x))*(1*x). \end{aligned} \tag{2.12}$$

Also, substituting $1*x$ for x in (2.11), by BCL$^+$-3 and 1), we have

$$\begin{aligned} &((1*x)*(1*x))*(1*x) \\ &=((x*x)*x)*((x*x)*x) \\ &=(1*x)*(1*x)=x*x=1. \end{aligned} \tag{2.13}$$

Using Theorem 2.2 with 4), we obtain

$$x*1=x*(1*x)=x. \tag{2.14}$$

Hence $(X;*,1)$ is a BCL$^+$-algebra.

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