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On Bounded Second Variation

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ABSTRACT

In this paper, we discuss various aspects of the problem of space-invariance, under compositions, of certain subclasses of the space of all continuously differentiable functions on an interval $[a,b]$. We present a result about integrability of products of the form $g\circ f\cdot f'\cdot f^{(k)}$ under suitable mild conditions and, finally, we prove that a Nemytskij operator S_g maps $BV''[a,b]$, a distinguished subspace of the space of all functions of second bounded variation, into itself if, and only if $g\in BV''_{loc}(\mathbb{R})$. A similar result is obtained for the space of all functions of bounded $(p,2)$-variation $(1\le p\le 1)$, $A_p^2[a,b]$.

Keywords: Function of Bounded Variation; Lipschitz Continuous Function; Absolutely Continuous Function; Nemytskij Operator

1. Introduction

Throughout this paper we use the following notations: if *g*, *f* are given functions, the expression $g\circ f$ stands for the composite function $g(f(t))$, whenever it is well-defined; $[a,b]$ denotes a compact interval in $\mathbb{R}$ (the field of all real numbers) and λ denotes the Lebesgue measure on $\mathbb{R}$. As usual, the set of all natural numbers will be denoted by $\mathbb{N}$.

Recall that a function $f:[a,b]\to\mathbb{R}$ is said to be of bounded variation on $[a,b]$ if the (total) variation of *f* on $[a,b]$

$$V(f)=V(f;[a,b]):=\sup\left\{\sum\left|f[I_n]\right|:\{I_n\}\right\}<\infty,$$

where $\{I_n\}$ denotes a (finite) partition of $[a,b]$ into non-overlapping intervals $[a_n,b_n]$ and

$$f[I_n]:=f(b_n)-f(a_n)$$

The class of all functions of bounded variation on $[a,b]$ is denoted as $BV[a,b]$. The renowned Jordan's theorem ([1]) states that a function $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ if, and only if, it is the diference of two monotone functions. In particular, every function in $BV[a,b]$ has left limit $f(x-)$ at every point $x\in(a,b]$ and right limit $f(x+)$ at every point $x\in[a,b)$; also, by the celebrated Lebesgue's Theorem (see e.g. [2, Theorem 1.2.8]) every function in $BV[a,b]$ is λ-*a.e.* diferentiable.

It is well known that the composition of two functions of bounded variation, say *g* and *f*, in general, need not be of bounded variation; in fact, not even if we choose the inner function well-enough behaved guarantees that the composition $g\circ f$ is of bounded variation.

For instance, if

$$g(x)=x^{\frac{1}{3}} \text{ and } f(x):=\begin{cases}0 & \text{if } x=0;\\ x^3\sin^3\dfrac{1}{x} & \text{if } x>0.\end{cases} \qquad (1.1)$$

then $f\in C^1[0,1]$, $g\in BV\left[f([0.1])\right]$ but $g\circ f$ is not of bounded variation.

However, the multiplication of $g\circ f$ by a derivative of *f*, which is of bounded variation, improves the properties of that composite function. Indeed, a proof of the following theorem can be found in [3, Theorem 5].

Theorem 1.1. ([3]). *Suppose that f has a derivative* $f^{(k)}$ *of order k everywhere on* $[a,b]$. *If* $f^{(k)}\in BV[a,b]$ *and if* $\psi\in BV[c,d]$, *where*

$$c:=\min_{[a,b]}f,\quad d:=\max_{[a,b]}f,$$

then the function $\psi\circ f\cdot f^{(k)}$ *is of bounded variation on* $[a,b]$; *moreover,*

$$\left\|\psi\circ f\cdot f^{(k)}\right\|_{[a,b]}\le(k+1)\|\psi\|_{[a,b]}\left\|f^{(k)}\right\|_{[a,b]}, \qquad (1.2)$$

where $\|\cdot\|_{[a,b]}$ is the norm on $BV[a,b]$ defined as

$$\|f\|_{[a,b]}:=\sup_{x\in[a,b]}|f(x)|+V(f;[a,b])$$

Let $\mathbb{X}$ be a subspace of $\mathbb{R}^{[a,b]}$. Given a function $g:\mathbb{R}\to\mathbb{R}$, the autonomous Nemytskij (or Superposition, see [4]) operator $S_g:\mathbb{X}\to\mathbb{R}^{[a,b]}$ generated by g, is defined as

$$S_g(f)(t):=g(f(t)),\, t\in[a,b]. \quad (1.3)$$

Given two linear spaces $\mathbb{X},\mathbb{Y}\subset\mathbb{R}^{[a,b]}$ and a function $g:\mathbb{R}\to\mathbb{R}$, a primary objective of research is to investigate under what conditions on the generating function the associated Nemytskij operator maps $\mathbb{X}$ into $\mathbb{Y}$. This problem is known as the *Superposition Operator Problem*.

Recently (see e.g. [5]), the Superposition Operator Problem have been studied extensively in various spaces of differentiable functions related to the spaces $BV[a,b]$ and $AC[a,b]$.

In this paper, we discuss various aspects of the the Superposition Operator Problem when the spaces $\mathbb{X},\mathbb{Y}\subset C^1[a,b]$ and are somehow related to the space $BV[a,b]$. We prove a version of Theorem 1.1 about the integrability of products of the form $g\circ f\cdot f'\cdot f^{(k)}$ when g is an integrable function and $f^{(k)}$ is continuous, and obtain an estimation of the norm $\left\|g\circ f\cdot f'\cdot f^{(k)}\right\|_{L_1[a,b]}$. Finally, we prove two results in which we give necessary and sufficient conditions for the autonomous Nemytskij Operator to map the space of all functions of second bounded variation into itself and the class of functions of bounded $(p,2)$-variation into itself.

2. Some Function Spaces

In this section we recall some definitions and state some results which will be needed for the further development of this work.

- We will use the notation $BV(M;[a,b])$ to denote the space of all bounded functions f such that $f^{-1}[\alpha,\beta]$ can be expressed as a union of M subintervals of $[a,b]$, for all $[\alpha,\beta]\subseteq[c,d]$, where

$$c:=\min_{[a,b]} f,\ d:=\max_{[a,b]} f.$$

  M. Josephy proved in [6] that for all $M\in\mathbb{N}$ the class of all bounded functions in $BV(M;[a,b])$ is contained in $BV[a,b]$.
- A function $F:[a,b]\to\mathbb{R}$ is said to be Lipschitz continuous on $[a,b]$ iff

$$L(F):=\sup\left\{\frac{|F(x)-F(y)|}{|x-y|}: x,y\in[a,b],\ x\neq y\right\}<\infty.$$

  The class of all Lipschitz continuous functions in $\mathbb{R}^{[a,b]}$ is denoted as $Lip[a,b]$ and the functional

$$\|f\|:=\max\left\{\|f\|_\infty, L(f)\right\}$$

  defines a norm on it.
- Recall that a function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be *absolutely continuous* on $[a,b]$ if, given $\epsilon>0$, there exists some $\delta>0$ such that

$$\sup\left\{\sum\left|f[I_n]\right|:\{I_n\}\right\}<\epsilon,$$

  whenever $\{I_n=[a_n,b_n]\}$ is a finite collection of mutually disjoint subintervals of $[a,b]$ with

$$\sum_{i=1}^{n}|b_n-a_n|<\delta.$$

The class of all absolutely continuous functions on $[a,b]$, which is actually an algebra, is denoted as $AC[a,b]$.

Definition 2.1. (Luzin *N* property). A real-valued function defined on a compact interval $I\subset\mathbb{R}$ is said to satisfy the Luzin *N* property (or simply, property *N*) if it carries sets of λ-measure zero into sets of λ-measure zero.

It is easy to see that the composition of two functions that have property *N* also has property *N*. The class of all continuous functions that satisfy property *N* on an interval $[a,b]$ will be denoted as $N[a,b]$.

The following characterization of absolutely continuous functions is well known (cf. [2, Chapter 7]).

Proposition 2.2. *The following statements on a function* $f:[a,b]\to\mathbb{R}$ *are equivalent*:

1) f *is absolutely continuous*,
2) $f\in BV[a,b]\cap C[a,b]$ *and satisfies property N*,
3) f' *exists* λ*-a.e., is integrable on* $[a,b]$ *and*

$$f(x)=f(a)+\int_a^x f'(t)\,\mathrm{d}t.$$

The equivalence (a) $\Leftrightarrow$ (b) is known as the Banach-Zareckiĭ theorem. The functional

$$\|f\|_{AC}:=|f(a)|+\|f'\|_{L_1}$$

defines a norm on $AC[a,b]$; in fact, $\|f'\|_{L_1}=V(f;[a,b])$.

Remark 2.3. *The same functions given in* (1.1) *show that the class of all absolutely continuous functions is not closed under compositions.*

- In the year 1908, de La Vallée Poussin ([7]), introduced the notion of bounded second variation. The class of all functions of bounded second variation on an interval $[a,b]$ is denoted by $BV^2[a,b]$ and is characterized by the following result due to F. Riesz ([8]):

Proposition 2.4. *A real valued function f is in the class* $BV^2[a,b]$ *if, and only if, there is a function* $\tilde{f}\in BV[a,b]$ *such that*

$$f(x)=f(a)+\int_a^x \tilde{f}(x)\,\mathrm{d}x. \quad (2.1)$$

In this case, the relation

$$\|f\|_{BV^2[a,b]} := |f(a)| + |\tilde{f}(a)| + V(\tilde{f};[a,b]) \qquad (2.2)$$

defines a norm on $BV^2[a,b]$.

Definition 2.5. Using the notation of (2.1) we define

$$BV''[a,b] := \left\{ f \in BV^2[a,b] : \tilde{f} \in BV[a,b] \cap C[a,b] \right\}.$$

Clearly, $BV''[a,b]$ is a linear subspace $BV^2[a,b]$. Notice also that by the Fundamental Theorem of Calculus if $f \in BV''[a,b]$ then f is differentiable on $[a,b]$ and $f' = \tilde{f}$. In fact, $BV''[a,b]$ if, and only if, $f \in C^1[a,b]$ and $f' \in BV[a,b]$.

- In 1997, N. Merentes, in [9], introduced the notion of function of bounded $(p,2)$-variation, for $1 < p < \infty$. The class of all functions in $\mathbb{R}^{[a,b]}$ of bounded $(p,2)$-variation is denoted by $A_p^2[a,b]$ and its characterized as follows:

Proposition 2.6. ([9]). *A function* $f \in \mathbb{R}^{[a,b]}$ *is in the class* $A_p^2[a,b]$ *if, and only if,* $f' \in AC[a,b]$ *and* $f'' \in L_p[a,b]$. *In this case the relation*

$$\|f\|_{A_p^2[a,b]} := |f(a)| + |f'(a)| + \|f''\|_{L_p[a,b]}$$

defines a norm on $A_p^2[a,b]$.

Clearly a continuously differentiable function is Lipschitz continuous and any Lipschitz continuous function is absolutely continuous. In fact, $p > 1$, the following chain of strict inclusions holds (see e.g., [5,10]):

$$\begin{aligned} A_p^2[a,b] &\subset BV''[a,b] \subset C^1[a,b] \subset Lip[a,b] \\ &\subset AC[a,b] \subset BV[a,b]. \end{aligned} \qquad (2.3)$$

3. Main Results

We begin this section by stating some fundamentals facts concerning compositions of functions on *BV* and *AC*. In these cases the intrinsic properties of the inner function (in the composition) will show to play also an important role. We recall that if *D* and *E* are given sets, *X* is a linear subspace of $\mathbb{R}^E$ and φ is a map from *D* to *E*, the *linear composition operator* $C_\varphi : X \to \mathbb{R}^D$ is defined by $C_\varphi(f) := f \circ \varphi$.

Remark 3.1.

1) *Although both classes BV and AC are not closed under composition, they do satisfy a weaker property in that respect. More precisely, it readily follows from a result given by M. Josephy in* [7, *Theorem* 3] *that if* $\varphi : [a,b] \to [c,d]$ *then, the operator* C_φ *maps* $BV[c,d]$ *into* $BV[a,b]$ *if, and only if,* $\varphi \in BV(M;[a,b])$ *for some M. From this, it readily follows that if* $\varphi \in BV(M;[a,b]) \cap AC[a,b]$ *then* C_φ *maps* $AC[c,d]$ *into* $AC[a,b]$. *The converse of this proposition is also true* (*see* [3]).

2) *By the fundamental Theorem of Algebra and Rolle's Theorem, if f is a polynomial of degree M, then for all* $[a,b] \subset \mathbb{R}$ $f \in BV(M;[a,b]) \cap AC[a,b]$; *also, every monotone function* $\varphi \in \mathbb{R}^{[a,b]}$ *is in* $BV(M;[a,b])$. *for some* $M \in \mathbb{N}$.

In what follows we will observe more instances of a very remarkable phenomenon that often occurs in nonlinear functional analysis: is the case in which given two functions, say *g* and *f*, the multiplication of $g \circ f$ by a continuous derivative $(f^{(k)}, k \in \mathbb{N})$ of *f* improves the properties of the composition.

The following proposition is a corollary of Theorem 1.1. The result follows from the fact that the space $N[a,b]$ is an algebra with respect to pointwise multiplication (see [3]), and the Banach-Zareckii Theorem.

Proposition 3.2. (Burenkov). *If f has an absolutely continuous* k^{th}*-derivative* $f^{(k)}$ *on* $[a,b]$ *and if* $\psi \in AC(\mathbb{R})$, *then the function* $\psi \circ f \circ f^{(k)}$ *is also absolutely continuous on* $[a,b]$ *and inequality* (1.2) *holds.*

If $g \in L_1[a,b]$ similar considerations as those discussed above apply with respect to the integrability of products of the form $g \circ f \circ f'$ or even $|g|^p \circ f \circ f'$. Now, not even the fact that the function *g* is integrable and the function *f* is absolutely continuous guarantees that the product $g \circ f \circ f'$ is integrable; for instance (see [3]), let $g(0) := f(0) := 0$ and, for $x > 0$, let $g(x) := 1/\sqrt{x}$ and $f(x) := x^6(\sin 1/x^3 + 2)$, then $f \in AC[0,1]$, *g* is integrable in $f([0,1])$, but $g \circ f \circ f'$ is not integrable in $[0,1]$. In that respect, the following proposition is well known (see, for instance, [11, Theorem 3.54]):

Proposition 3.3. [Change of Variables] Let $g : [c,d] \to \mathbb{R}$ be an integrable function and let $f : [a,b] \to [c,d]$ be a function differentiable *λ-a.e.* in $[a,b]$. Then $g \circ f \cdot f'$ is integrable and

$$\int_{f(\alpha)}^{f(\beta)} g(t)\,\mathrm{d}t = \int_\alpha^\beta g(f(x)) f'(x)\,\mathrm{d}x$$

holds for all $\alpha, \beta \in [a,b]$ if, and only if, the function $G \circ f \in AC[0,1]$, where

$$G(z) := \int_c^z g(t)\,\mathrm{d}t, \ z \in [c,d]. \qquad (3.1)$$

Notice that *G* is an absolutely continuous function, which bring us back to the same situations considered above.

It turns out that, if *g* is an integrable function, multiplication by a continuous derivative of *f* improves the (integrability) properties of the product $g \circ f \cdot f'$. By analogy with an useful notion originated from the theory of partial differential equations, we might call this derivative an *integrating factor*. The following lemma provides a version of Theorem 1.1 when the outer function in the composition is an integrable function. The proposition might be of some interest in itself.

Lemma 3.4. *Suppose that* $g \in L_{1loc}(\mathbb{R})$ *and that* $f^{(k)} \in C[a,b]$ $(k \in \mathbb{N})$. *Then* $g \circ f \cdot f' \cdot f^{(k)} \in L_1[a,b]$;

moreover,

$$\left\| g\circ f\cdot f'\cdot f^{(k)}\right\|_{L_1([a,b])} \le k\left\| f^{(k)}\right\|_\infty \|g\|_{L_1(f([a,b]))}.$$

Proof. The continuity of $f^{(k)}$ implies that the open set $S:=\left\{x\in(a,b): f^{(k)}(x)\ne 0\right\}$ can be expressed as a countable union of component open intervals, say $S=\cup_{i=1}^{N}(a_i,b_i)$, where $N\in\mathbb{N}$ or $N=\infty$. Now, since $f^{(k)}\ne 0$ on S, to each $i\in\{1,\cdots,N\}$ corresponds a nonnegative integer $m_i\le k$ such that $[a_i,b_i]$ can be decomposed into at most m_i disjoint non-degenerated intervals, $\left[a_i^1,b_i^1\right],\left[a_i^2,b_i^2\right],\cdots,\left[a_i^{m_i},b_i^{m_i}\right]$, on which f is monotone.

Now, being f' continuous on $[a,b]$, the Fundamental Theorem of Calculus implies that $f\in AC[a,b]$; likewise, the indefinite integral function G, defined by (3.1), is absolutely continuous, thus by Remark 3.1, the monotonicity of f on $\left[a_i^j,b_i^j\right]$ implies that $G\circ f\in AC[a,b]$; consequently, $g\circ f\cdot f'$ is integrable on this interval. Hence, since f' does not change sign on $\left[a_i^j,b_i^j\right]$, we must have

$$\int_{a_i^j}^{b_i^j}\left|g(f(t))\right|\cdot\left|f'(t)\right|\mathrm{d}t=\int_{\left\langle f\left(a_i^j\right),f\left(b_i^j\right)\right\rangle}\left|g(x)\right|\mathrm{d}x$$

where the notation $\langle\alpha,\beta\rangle$ stands for $[\alpha,\beta]$ if $\alpha<\beta$ or $[\beta,\alpha]$ otherwise.

Now, since $\left|f^{(k)}\right|$ is continuous, *the* (*generalized*) *mean value theorem for integrals* implies that, on each $\left[a_i^j,b_i^j\right]$, there is a point c_i^j such that

$$\int_{a_i^j}^{b_i^j}\left|g(f(t))\cdot f'(t)\right|\cdot\left|f^{(k)}\right|\mathrm{d}t =\left|f^{(k)}\left(c_i^j\right)\right|\int_{\left\langle f\left(a_i^j\right),f\left(b_i^j\right)\right\rangle}\left|g(x)\right|\mathrm{d}x.$$

Notice that the product $g\circ f\cdot f'\cdot f^{(k)}$ is a measurable function on $[a,b]$. Thus

$$\begin{aligned}
&\int_a^b\left|g(f(t))\cdot f'(t)\cdot f^{(k)}(t)\right|\mathrm{d}t\\
&=\sum_{i=1}^{N}\sum_{j=1}^{m_i}\left|f^{(k)}\left(c_i^j\right)\right|\int_{\left\langle f\left(a_i^j\right),f\left(b_i^j\right)\right\rangle}\left|g(x)\right|\mathrm{d}x\\
&\le\sum_{i=1}^{N}\sum_{j=1}^{m_i}\left\|f^{(k)}\right\|_\infty\int_{\left\langle f\left(a_i^j\right),f\left(b_i^j\right)\right\rangle}\left|g(x)\right|\mathrm{d}x\\
&\le\sum_{i=1}^{N}k\left\|f^{(k)}\right\|_\infty\int_{\left\langle f\left(a_i^j\right),f\left(b_i^j\right)\right\rangle}\left|g(x)\right|\mathrm{d}x\\
&=k\left\|f^{(k)}\right\|_\infty\|g\|_{L_1(f([a,b]))}.
\end{aligned}$$

The proof is complete. □

The Autonomous Nemytskij Operator on the Spaces $BV''[a,b]$ and $A_p^2[a,b]$

For convenience we state the next result as a single proposition. The proof of it is based in three separate results of M. Josephy [6] (see also [12]), N. Merentes [13] and N. Merentes and S. Rivas [14].

Proposition 3.5. Suppose $\mathcal{P}[a,b]:=BV[a,b]$, $AC[a,b]$ or $RBVp[a,b]$ ([14]). Then S_g maps $\mathcal{P}[a,b]$ into itself if, and only if, $g\in Lip_{loc}(\mathbb{R})$.

Now we present a result that gives a necessary and sufficient condition for the Nemytskij operator to map the space $BV''[a,b]$ into itself.

Theorem 3.6. S_g *maps* $BV''[a,b]$ *into itself if, and only if,* $g\in BV''_{loc}(\mathbb{R})$. *Moreover, in this case* S_g *is automatically bounded.*

Proof. Suppose that $g\in BV''_{loc}(\mathbb{R})$. Then, by Proposition 3.5, for all $f\in BV''[a,b]$, $g\circ f\in Lip[a,b]\subset AC[a,b]$ (since both g and f are Lipschitz continuous); thus, for λ-*a.e.* $x\in[a,b]$ $(g\circ f)'=\tilde{g}\circ f\cdot\tilde{f}$, and, by Theorem 1.1, with $\psi=\tilde{g}$ and $k=1$, we get $\tilde{g}\circ f\cdot\tilde{f}\in BV[a,b]$ and, since it is clearly continuous on $[a,b]$, it follows that $g\circ f\in BV''[a,b]$.

Conversely, assume S_g maps $BV''[a,b]$ into itself.

For any given pair of real numbers α, β with $\alpha<\beta$ denote by $f_{ab}^{\alpha\beta}$ the linear difeomorphism $f_{ab}^{\alpha\beta}:[a,b]\to[\alpha,\beta]$ defined as

$$f_{ab}^{\alpha\beta}(x):=m_{ab}^{\alpha\beta}(x-a)+\alpha,\quad\text{where}\quad m_{ab}^{\alpha\beta}:=\frac{\beta-\alpha}{b-a}.$$

Then, each $f_{ab}^{\alpha\beta}\in BV''[a,b]$ and therefore, for all $\alpha<\beta$: $S_g\circ f_{ab}^{\alpha\beta}\in BV''[a,b]$. Thus, by the first part of the proof we have

$$g=\left(S_g\circ f_{ab}^{\alpha\beta}\right)\circ\left(f_{ab}^{\alpha\beta}\right)^{-1}=\left(S_g\circ f_{ab}^{\alpha\beta}\right)\circ\left(f_{\alpha\beta}^{ab}\right)\in BV''[\alpha,\beta].$$

Hence, $g\in BV''_{loc}(\mathbb{R})$ and the proof is complete.

The conclusion about automatic continuity follows at once from (2.2) and (1.2). □

Now we present a similar result for the space $A_p^2[a,b]$. At this point, let us recall the following proposition (see, for instance [11, Theorem 3.44]):

Suppose g, f are functions defined on intervals and that $g\circ f$ *is well defined. If g, f and* $g\circ f$ *are* λ-*a.e. differentiable functions and g satisfies the property N then,*

$$(g\circ f)'(x)=g'(f(x))f'(x)\quad\text{for}\quad\lambda\text{-}a.e.\ x,$$

where $g'(f(x))f'(x)$ *is interpreted to be zero whenever* $f'(x)=0$.

Theorem 3.7. *Let* $p>1$. S_g *maps* $A_p^2[a,b]$ *into itself if, and only if,* $g\in A_{ploc}^2(\mathbb{R})$ *In this case* S_g *is automatically bounded.*

Proof. Suppose first that $g \in A^2_{ploc}(\mathbb{R})$ By Proposition 3.5, for all $f \in A^2_p[a,b]$, $g \circ f \in AC[a,b]$ (since g is Lipschitz continuous on $[a,b]$). Thus

$$(g \circ f)'(x) = g'(f(x))f'(x) \text{ for all } x \in [a,b],$$

and since $g' \in AC[a,b]$ (and in particular it satisfies property *N*), for λ-*a.e.* $x \in [a,b]$

$$(g \circ f)''(x) = g''(f(x))(f'(x))^2 + g'(f(x))f''(x). \quad (3.2)$$

Now, Since $g' \circ f$ is continuous, the second summand in the right hand side of (3.2) is in $L_p[a,b]$, and

$$\|g' \circ f \cdot f''\|_{L_p} \le \|g' \circ f\|_\infty \|f''\|_{L_p[a,b]}. \quad (3.3)$$

On the other hand,

$$\left|g''(f(x))(f'(x))^2\right|^p = \left|\left(|g''|^p \circ f\right)(x)(f'(x))^2\right| |f'(x)|^{2p-2}.$$

Hence, by Lemma 3.4

$$\left\|g'' \circ f \cdot (f')^2\right\|^p_{L_{p(f[a,b])}} \le \|f'\|^{2p-1}_\infty \|g''\|^p_{L_{p(f[a,b])}}. \quad (3.4)$$

From (3.3) and (3.4) it follows that S_g maps $A^2_p[a,b]$ into itself and that, in this case, S_g maps bounded sets on bounded sets.

The proof of the converse is similar to the one given for the the necessity of the condition in the proof of Theorem 3.6. □

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