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Classification of Rational Homotopy Type for 8-Cohomological Dimension Elliptic Spaces

Mohamed Rachid Hilal[1], Hassan Lamane[1], My Ismail Mamouni[2*]
[1]Faculté des Sciences Aïn Chock, Casablanca, Morocco
[2]Centre Pédagogique Régional, Rabat, Morocco
Email: {rhilali, hlamanee}@hotmail.com, *mamouni.myismail@gmail.com

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ABSTRACT

The different methods used to classify rational homotopy types of manifolds are in general fascinating and various (see [1,7,8]). In this paper we are interested to a particular case, that of simply connected elliptic spaces, denoted *X*, by discussing its cohomological dimension. Here we will the discuss the case when $\dim H^*(X;\mathbb{Q}) = 8$ and $\chi(X) = 0$.

Keywords: Rational Homotopy Theory; Elliptic Spaces; Classification; Rational Homotopy Type; Minimal Model of Sullivan

1. Introduction

Let us first recall some basic definitions of rational homotopy theory. A simply connected space *X* is called *elliptic*, if both of $H^*(X;\mathbb{Q})$ and $\pi_*(X)\otimes\mathbb{Q}$ are finite dimension, and that its *cohomological Euler-Poincar characteristic* is given as $\chi_c(X) := \sum_{k\geq 0}(-1)^k \dim H^k(X;\mathbb{Q})$. We will fix this throughout this paper. The space is called *rational* if $\pi_*(X)$ is a $\mathbb{Q}$-vector space. If it is not, by [4], we can associate a rational simply connected space, denoted $X_{\mathbb{Q}}$, verifying

$$\begin{array}{lll} H^*(X_{\mathbb{Q}};\mathbb{Q}) & \mathbb{Q}H^*(X,\mathbb{Q}) & \text{asalgebras,} \\ \pi_*(X_{\mathbb{Q}}) & \mathbb{Q}\pi_*(X)\otimes\mathbb{Q} & \text{asvectorspaces.} \end{array}$$

The rational homotopy type of *X* is defined as the homotopy type of its *rationalization* $X_{\mathbb{Q}}$. Our purpose in this paper to give a complete classification this rational homotopy type when $\dim H^*(X;\mathbb{Q})$ and $\chi_c(X) = 0$.

2. Preliminaries

The rational homotopy theory was founded in the the end of the sixties by Daniel Quillen and Denis Sullivan. One of the technical gadget of this theory is the *minimal model of Sullivan*, it is a free $\mathbb{Q}$-commutative differential graded algebra $(\Lambda V, d)$ associated to any simply connected CW complex *X* of finite type [3]. Here $V = \oplus_{i\geq 2}V^i$ is $\mathbb{Q}$-graded vector space with $\dim V^i < \infty$ and *d* a decomposable differential; that means $dV^i \subset \left(\Lambda^{\geq 2}V\right)^{i+1}$ (*d* does not have a linear part) and that $d^2 = 0$. It is well known that the minimal model $(\Lambda V, d)$ determines the rational homotopy type of *X*, in the sense that

$$\begin{array}{lll} H^*(X;\mathbb{Q}) & \simeq H^*(\Lambda V, d) & \text{asalgebras} \\ \pi_*(X)\otimes\mathbb{Q} & \simeq V & \text{asvectorspaces.} \end{array}$$

For example, the minimal model of an even sphere $\mathbb{S}^{2n}$ is of the form $(\Lambda\{x,y\},d)$ with $|x| = 2n$, $|y| = 4n-1$, $dx = 0$, $dy = x^2$ and $H^*(\mathbb{S}^{2n};\mathbb{Q}) \cong \mathbb{Q}[x]/x^2$, while the minimal model of an odd sphere $\mathbb{S}^{2n+1}$ is of the form $(\Lambda\{x,y\},d)$ with $|y| = 2n+1$, $dy = 0$. It will be utile for our proofs, to recall the reader this simple properties. For a homogeneous element *x* of ΛV, $|x|$ denotes its degree, which verifies the following:

- $xy = (-1)^{|x||y|}yx$;
- $d(xy) = (dx)y + (-1)^{|x|}xdy$ (Leibniz formula).

In particular $x^2 = 0$, when $|x|$ is odd and $xy = yx$ when $|x|$ is even.

$\chi_\pi(X) := \sum_{k\geq 0}(-1)^k \dim V^k$ is called the *homotopic Euler-Poincar characteristic* of *X*. In [5], S. Halperin have shown the following:

$$\begin{array}{lll} \chi_c \geq 0 & \text{and} & \chi_\pi \leq 0 \\ \chi_c > 0 & \Leftrightarrow & \chi_\pi = 0 \\ & \Leftrightarrow & H^{\text{odd}}(\Lambda V, d) = 0 \end{array} \tag{1}$$

One other notion that we will use throughout this paper is the formal dimension of *X*, given as $fd(X) := \max\{n, H^n(X;\mathbb{Q}) \neq 0\}$. We know from [5] that, when $a_1,\cdots,a_n$ are the elements of an homogene-

*Corresponding author.

ous basis of V,

$$fd(X)=\sum_{|a_i|\text{odd}}|a_i|-\sum_{|a_i|\text{even}}(|a_i|-1). \tag{2}$$

Our proofs are essentialy based on this equality combined with an other equality established by J. Friedlander and S. Halperin in [2], that

$$\begin{aligned}\sum_{|a_i|\text{even}}|a_i| &\le fd(X)\\ \sum_{|a_i|\text{odd}}|a_i| &\le 2\,fd(X)-1.\end{aligned} \tag{3}$$

Finally, let us recall that $H^*(X;\mathbb{Q})$ satisfies the Poincar duality, that means that the multiplication $H^k(X;\mathbb{Q})\times H^{n-k}(X;\mathbb{Q})\to H^n(X;\mathbb{Q})\cong\mathbb{Q}\mu$ is a non degenerate bilinear form (here $n=fd(X)$ and μ denotes the so called *fundamental class* of $H^*(X;\mathbb{Q})$). For the reader interested by more details about the rational homotopy theory, we recommend the basic reference [3].

3. The Main Theorem

In all the remainder of this paper, X denotes a simply connected elliptic space with $\dim H^*(X;\mathbb{Q})=8$, $\chi_c=0$ and $(\Lambda V,d)$ will denotes it minimal model. Put $\{1,\alpha_1,\cdots,\alpha_6,\mu\}$ a basis for $H^*(X;\mathbb{Q})$ with the condition that $|\alpha_i|\le|\alpha_{i+1}|$ and that $a_i\in\Lambda V$ with $[a_i]=\alpha_i$. The following table summarizes the classification of its rational homotopy type.

Rational homotopy type of X	Legend
$(\mathbb{S}^{2k+1})^3$	$fd(X)=3(2k+1)$
$\mathbb{S}^n\times\mathbb{S}^n\times\mathbb{S}^{2n}$	$fd(X)=4n$ and n is odd
$(\mathbb{S}^n\times\mathbb{S}^n\times\mathbb{S}^{2n})\#\mathbb{S}^{2n}_{(2)}$	$fd(X)=4n$ and n is odd
$\mathbb{S}^{2k+1}\times\mathbb{S}^{2k+1}\times\mathbb{S}^{2(k+p)}$	$fd(X)=2(2k+1)+2(k+p)$
$\mathbb{S}^{2n}\otimes_{d_2}\mathbb{S}^n\times\mathbb{S}^{2n})\times\mathbb{S}^{2(n+p)+1}$	$fd(X)=4n+2(n+p)+1$
$\mathbb{S}^{2n}\times\mathbb{S}^{2n}\times\mathbb{S}^{2(n+p)+1}$	$fd(X)=4n+2(n+p)+1$
$\mathbb{S}^{2n}\times(\mathbb{S}^{2(n+p)+1})^2$	
$\mathbb{S}^{2n}\times\mathbb{S}^k\times\mathbb{S}^k$	$k\ge 2n+2$
$\mathbb{S}^{2n+1}\times Y_\lambda$	$\lambda\in\mathbb{Q}^*$
$\mathbb{S}^{2n}\times\mathbb{S}^{2k+1}\times\mathbb{S}^{2(n+k)+1}$	
$\mathbb{S}^{2n+1}\times\mathbb{S}^{2(n+k)+1}\otimes_d\mathbb{S}^{2k}$	
$\mathbb{S}^n_{(3)}\times\mathbb{S}^{2k+1}$	
$\mathbb{S}^{2k_1+1}\times\mathbb{S}^{2k_2+1}\times\mathbb{S}^{2k_3+1}$	
E	E : the total space of the fiber bundle with $\mathbb{S}^{2p+1}\times\mathbb{S}^{2q+1}$ as base space

Legend: 1) In [6], I. M. James has introduced the concept of reduced product when X is a based space. He put $X_{(1)}:=X$ and $X_{(p)}:=X\times\cdots\times X/(x_1,\cdots,*,\cdots,x_{p-1})\sim(*,x_1,\cdots,x_{p-1})$.

2) From this construction applied to an even sphere $\mathbb{S}^n$ arises the James sphere $\mathbb{S}^n_{(p)}$, satisfying $H^*(\mathbb{S}^n_{(p)};\mathbb{Q})\simeq\mathbb{Q}[a]/(a^{p+1})$. The use of the denotation $\mathbb{S}^n_{(p)}$ means implicitly that n is supposed to be even.

As the most of our proofs will be by contradiction, we will mark such proofs by (by contradiction) in its beginning and by (QED) when its end. In the spirit and desire to simplify the lecture of this paper, we will subdivide it on many propositions, lemmas and theorems. The first one is that:

Lemma 1 *There exists* $i\in\{1,\cdots,5\}$ *such that* $|\alpha_i|<|\alpha_{i+1}|$.

Proof. Suppose that $|\alpha_i|=|\alpha_{i+1}|=n$, then $\chi_c=1+(-1)^{fd(X)}+5\times(-1)^n\ne 0$.

3.1. The Case Where $|\alpha_1|=|\alpha_2|=|\alpha_3|<|\alpha_4|=|\alpha_5|=|\alpha_6|$

Proposition 2 *If* $|\alpha_1|=|\alpha_2|=|\alpha_3|<|\alpha_4|=|\alpha_5|=|\alpha_6|$, *then X has the rational homotopy type (r.h.t) of* $(\mathbb{S}^{2k+1})^3$, *with* $3(2k+1)=fd(X)$ *and* $k\in\mathbb{N}^*$.

Proof. Since $|\alpha_1|=|\alpha_2|=|\alpha_3|$, then $a_1,a_2,a_3\in V$. We distinguish two cases:

1) $|\alpha_1|$ is odd. Then $\alpha_1^2=\alpha_2^2=\alpha_3^2=0$. Let E be the vector space spanned by $\alpha_1\alpha_2,\alpha_1\alpha_3,\alpha_2\alpha_3$.

- If $\dim E=3$, we can take $\{\alpha_1\alpha_2,\alpha_1\alpha_3,\alpha_2\alpha_3\}$ as a basis of E. Let $\{b_1,\cdots,b_n\}$ an homogeneous basis of a complement of $\mathbb{Q}\{a_1,a_2,a_3\}$ in V with $|b_1|\le\cdots\le|b_n|$ and $db\in\Lambda^{\ge 2}\{a_1,a_2,a_3\}$, therefore $db_1=0$ and $\Lambda\{[a_1],[a_2],[a_3]\}\oplus\mathbb{Q}[b_1]\subset H^*(\Lambda V,d)$, what implies that $\dim H^*(\Lambda V,d)\ge 9$. So the minimal model of X is $(\Lambda\{a_1,a_2,a_3\},d)$ with $da_i=0$ and $|a_i|=2k+1$. This is exactly the minimal model of $(\mathbb{S}^{2k+1})^3$.
- If $1\le\dim E\le 2$, the there exist $\lambda_1,\lambda_2\in\mathbb{Q}$ such that $\lambda_1\alpha_1\alpha_2+\lambda_2\alpha_1\alpha 3+\alpha_2\alpha_3=0$, and then $da_4=\lambda_1a_1a_2+\lambda_2a_1a3+a_2a_3$. According to the Poincar duality, we have $\alpha_1\alpha_2\alpha_3=\mu$, so $d(a_1a_4)=a_1a_2a_3$ and $[a_1a_2a_3]=\mu=0$. This is impossible.

2) $|\alpha_1|$ is even. Then $fd(X)$ and $|\alpha_4|$ are odd, because that $\chi_c=1+(-1)^{fd(X)}+3+(-1)^{|\alpha_4|}=0$. Therefore $\alpha_1^2=\alpha_2^2=\alpha_3^2=\alpha_1\alpha_2=\alpha_1\alpha_3=\alpha_2\alpha_3=0$, and there exist tree generators b_i of ΛV with even degrees such that $db_i=a_i^2$, $i=1\cdots 3$. Then $\sum_{|a_i|\text{even}}|a_i|\ge 3|a_4|+3|b_1|\ge 3|a_4|+6|a_1|-3>2\,fd(X)-1$.

This is impossible.

3.2. The Case Where $|\alpha_1| = |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| = |\alpha_6|$

Lemma 3 *If* $|\alpha_1| = |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| = |\alpha_6|$, *then* $fd(X)$ *and* $|\alpha_3|$ *are even,* $|\alpha_1|$ *is odd, and* $\alpha_1\alpha_2 \neq 0$.

Proof. First, because of the Poincar duality, we have $fd(X) = |\alpha_1| + |\alpha_6| = 2|\alpha_3|$ is even, and $|\alpha_1|$, $|\alpha_6|$ have the same parity. Hence $0 = \chi_c = 2 + 4(-1)^{|\alpha_1|} + 2(-1)^{|\alpha_3|}$, $|\alpha_1|$ is odd and $|\alpha_3|$ is even.

(*By contradiction*) Suppose now that $\alpha_1\alpha_2 = 0$, since $\alpha_1^2 = \alpha_2^2 = 0$ then $\{a_i, i = 1\cdots 4\} \subset V$. Otherwise, the Poincar duality let us to suppose that $\alpha_1\alpha_6 = \alpha_2\alpha_5 = \mu$ and to conclude that $a_5, a_6 \notin \Lambda\{a_i, i = 1\cdots 4\}$ and that a_5, a_6 are also generators of $(\Lambda V, d)$. So

$$\sum_{|a_i|\text{odd}} |a_i| \geq |\alpha_1| + |\alpha_2| + |\alpha_5| + |\alpha_6|$$
$$= 4\, fd(X) > 2\, fd(X) - 1.$$

This is impossible (QED).

Lemma 4 *If* $|\alpha_1| = |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| = |\alpha_6|$, *then there exists an homogeneous generator b of* ΛV, *satisfying* $db = a_4^2 - \lambda a_1 a_2 a_4$, *where* $\lambda \in \mathbb{Q}^*$.

Proof. Since $\alpha_1\alpha_2 \neq 0$, we can assume that $a_1 a_2 = a_3$ and that $a_4 \in V$, then $[a_3 a_4] = \alpha_1\alpha_2\alpha_4 = \mu$. Otherwise $[a_4^2] = \lambda[a_1 a_2 a_4]$, then there exists an homogeneous generator *b* of ΛV, such $db = a_4^2 - \lambda a_1 a_2 a_4$.

Lemma 5 *If* $|\alpha_1| = |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| = |\alpha_6|$, *then* $(\Lambda V, d) \cong (\Lambda(x_1, x_2, x_3, y), D)$ *with*:

- $Dx_1 = Dx_2 = Dy = 0$,
- $Dy = y^2 - \lambda x_1 x_2 y$ where $\lambda \in \mathbb{Q}^*$,
- $|x_1| = |x_2| = |a_1|$ et $|y| = |a_4|$.

Proof. We have $H^*(\Lambda V, d) \cong H^*(\Lambda W, D)$ where $\Lambda W = \Lambda(x_1, x_2, x_3, y)$. We define the algebra homomorphism $\Psi : (\Lambda W, D) \to (\Lambda V, d)$ as $\Psi(x_1) = a_1$, $\Psi(x_2) = a_2$, $\Psi(x_3) = b$ and $\Psi(y) = a_4$. Ψ is into because it transforms the basis $\{x_1, x_2, x_3, y\}$ of *W* on a the linearly independent family $\{a_1, a_2, a_4, b\}$. Let $V_0 = \Psi(W)$ and $V = V_0 \oplus V_1$, since $H^*(\Lambda V, d) = H^*(\Lambda V_0, d)$ then $dV_1 \subset \Lambda V_0 \otimes \Lambda V_1 \setminus \{0\}$. Assume that $V_1 \neq 0$ and consider $c \in V_1$ such that $|c| = \min\{|x|, x \in V_1, x \neq 0\}$, then $dc = \omega_1 b + \omega_2$ where $\omega_1, \omega_2 \in \Lambda(a_1, a_2, a_4)$. As $0 = d^2 c = \omega_1 db = \omega_2(a_4^2 - \lambda a_1 a_2 a_4)$, then $\omega_1 = 0$. We have to discuss two cases:

- $dc = (\lambda_1 a_1 + \lambda_2 a_2) a_4^n; n \geq 2$. In this case $|c| = (2n+1)|a_1| - 1$ and therefore
$$\sum_{|a_i|\text{even}} |a_i| \geq |b| + |c| \geq 4|a_1| + (2n+1)|a_1| - 2$$
$$\geq 9|a_1| - 2 > fd(X).$$
This is impossible.
- $dc = \lambda_3 a_1 a_2 a_4^m$, $m \geq 1$. In this case, $|c| = (2m+3)|a_1| - 1$ and $d(a_1 c) = 0$, then $|a_1 c| > 5|a_1| - 1 > fd(X)$, and so $[a_1 c] = 0$. Let $\beta = \beta_1 + \cdots + \beta_k \in \Lambda V$, such $d\beta = a_1 c$ where $\beta_i \in \Lambda^i V$ and $|\beta_1| = (2m+3)|a_1| - 2$ is odd, in particular. But $\beta_1 \neq 0$, because if not we will have
$$\sum_{|a_i|\text{odd}} |a_i| \geq |a_1| + |a_2| + |c| + |\beta_1| \geq (4m+7)|a_1| - 3$$
$$\geq 11|a_1| - 3 > 2\, fd(X) - 1.$$
This is impossible.

Proposition 6 *If* $|\alpha_1| = |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| = |\alpha_6|$, *then X have one of the following r.h.t*:

- $\mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^{2n}$, where *n* is odd and $fd(X) = 4n$.
- $(\mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^{2n}) \# \mathbb{S}^{2n}_{(2)}$, where *n* is odd and $fd(X) = 4n$.

Proof. Let us recall that $fd(X)$ and $|\alpha_4|$ are even, and that $|\alpha_1|$ and $|\alpha_6|$ are odd.

- First case: $\alpha_4^2 = 0$. Since $\mu = \lambda_1\alpha_1\alpha_2\alpha_4$ and $\lambda_1 \neq 0$, then $\alpha_1\alpha_4 \neq 0$ and $\alpha_2\alpha_4 \neq 0$. Hence $\{1, \alpha_1, \alpha_2, \alpha_4, \alpha_1\alpha_2, \alpha_1\alpha_4, \alpha_2\alpha_4, \alpha_1\alpha_2\alpha_4\}$ is a basis for $H^*(X;\mathbb{Q})$, and therefore $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[a,b,c]/(a^2, b^2, c^2)$, *i.e.*, *X* has the r.h.t of $\mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^{2n}$.
- Second case: $\alpha_4^2 \neq 0$. Here $\alpha_1\alpha_4$ and $\alpha_2\alpha_4$ are both non null, because in the opposite case we will have $a_1 a_4 = db$ or $a_2 a_4 = db$ where *b* is a generator of ΛV, and in this cases $\sum_{|a_i|\text{odd}} |a_i| \geq |a_1| + |a_2| + |a_5| + |a_6| > 2\, fd(X) - 1$. This is impossible. Recap $\alpha_1^2 = \alpha_2^2 = 0$, $\alpha_1\alpha_2 \neq 0$, $\alpha_4 \neq 0$, $\alpha_1\alpha_4 \neq 0$, $\alpha_4\alpha_4 \neq 0$, this leads us to conclude that *X* have the r.h.t of $(\mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^{2n}) \# \mathbb{S}^{2n}_{(2)}$.

3.3. The Case Where $|\alpha_1| = |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| = |\alpha_6|$

Proposition 7 *If* $|\alpha_1| = |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| = |\alpha_6|$ *and if* $fd(X)$ *is even, then X have the r.h.t of* $\mathbb{S}^{2k+1} \times \mathbb{S}^{2k+1} \times \mathbb{S}^{2(k+p)}$ *with* $fd(X) = 2(k+1) + 2(k+p)$.

Proof. Because of the parity of $fd(X)$, the duality of Poincar and the fact that $\chi_c = 0$, then $|\alpha_1|$ and $|\alpha_3|$ are respectively odd and even, so $\alpha_1^2 = \alpha_2^2 = 0$. Assume that $\alpha_2\alpha_2 = 0$ and that $\alpha_5, \alpha_6 \in \Lambda(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, then there exist $P_1^i, P_2^i \in \Lambda(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such that $\alpha_i = P_1^i\alpha_1 + P_2^i\alpha_2$ for $i = 5, 6$. This implies the impossible situation that $\mu = \alpha_1\alpha_i = \alpha_2\alpha_i = 0$, but also that our second assumption is false. Thus necessarily a_5, a_6 are both generators of ΛV and that

$$\sum_{|a_i|\text{odd}} |a_i| \geq |a_1| + |a_2| + |a_5| + |a_6| \geq 2\, fd(X) > 2\, fd(X) - 1.$$

This another impossible situation implies that our first assumption is also false. Put $\alpha_1\alpha_3 \neq 0$, in this case $\{a_3, a_4\} \cap \mathbb{Q}\{a_1, a_2\} \neq \varnothing$, in particular a_j are generators of ΛV for $j = 3, 4$. The Poincar duality let us to write $\alpha_1\alpha_2\alpha_j = [a_1 a_2 a_j] = \mu$ and to conclude that $\alpha_1\alpha_j \neq 0$ and that $\alpha_2\alpha_j \neq 0$ and finally to write $\alpha_5 = \alpha_1\alpha_j$, $\alpha_6 = \alpha_2\alpha_j$. Recall that $\alpha_j^2 = 0$, because of the parity of the degree, then $(\Lambda V, d) \cong \Lambda(a, b, c, x), d)$ with $da = db = dc = 0$ and $dx = c^2$. This is the minimal model of $\mathbb{S}^{2k+1} \times \mathbb{S}^{2k+1} \times \mathbb{S}^{2(k+p)}$.

Proposition 8 *If* $|\alpha_1| = |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| = |\alpha_6|$ *and if* $fd(X)$ *is odd, then X has one of following r.h.t*:

- $\mathbb{S}^{2k+1} \times \mathbb{S}^{2k+1} \times \mathbb{S}^{2(k+p)}$ *with* $fd(X) = 2(k+1) + 2(k+p)$,
- $(\mathbb{S}^{2n} \otimes_{d_2} \mathbb{S}^{2n}) \times \mathbb{S}^{2(n+p)+1}$ *with* $fd(X) = 4n + 2(n+p) + 1$,
- $(\mathbb{S}^{2n} \times \mathbb{S}^{2n}) \times \mathbb{S}^{2(n+p)+1}$ *with* $fd(X) = 4n + 2(n+p) + 1$,

Proof. We will discuss three cases:

- First case: $|\alpha_1|$ is odd, then $|\alpha_5|$ is even. Suppose that $|\alpha_5| \in V^{\text{even}}$, then $\sum_{|a_i| \text{even}} |a_i| \geq |a_3| + |a_5| > |a_1| + |a_5| = fd(X)$. This is impossible. So $\{\alpha_3, \alpha_4\} \cap \mathbb{Q}\{\alpha_1, \alpha_2\}$ with $\alpha_1\alpha_2 \neq 0$ and a_j is an odd degree generator of ΛV for $j = 3$ or $j = 4$. A same justification as in the last proof let us to conclude that $\alpha_5 = \alpha_1\alpha_j$, $\alpha_6 = \alpha_2\alpha_j$ and that *X* have the r.h.t of $\mathbb{S}^{2k+1} \times \mathbb{S}^{2k+1} \times \mathbb{S}^{2(k+p)}$.
- Second case: $|\alpha_1|$ is even and $\alpha_1\alpha_2 = 0$. Since $|\alpha_5|$ is odd, then $\dim \mathbb{Q}\{\alpha_1^2, \alpha_2^2, \alpha_1\alpha_2\} \leq 1$. Assume for example that $\alpha_1^2 \neq 0$, then $\mu = \alpha_1^2\alpha_j$ with $j = 3$ or 4 and $|\alpha_j|$ is odd. Therefore $\alpha_1\alpha_j \neq 0$. Let suppose that $\alpha_1\alpha_j, \alpha_2\alpha_j$ are collinear and write $\alpha_2\alpha_j = \lambda\alpha_1\alpha_j$, then $\alpha_2^2\alpha_j = \lambda\alpha_2\alpha_1\alpha_j = 0$, so $a_5 \in V^{\text{odd}}$. Since that $\alpha_1\alpha_2 = 0$ and that $\alpha_1^2 = \beta\alpha_2^2$, then there exist two odd degree generators of ΛV, *b* and *c*, such that $db = a_1a_2$ and $dc = a_1^2 - \beta a_2^2$. We conclude that
$$\begin{aligned}\sum_{|a_i| \text{odd}} |a_i| &\geq |a_j| + |a_5| + |b| + |c| \geq |a_j| + |a_5| + 2(2|a_1| - 1) \\ &\geq 2\,fd(X) - 1 + |a_1| - 1 > 2\,fd(X) - 1\end{aligned}$$
(impossible). Put $\alpha_1\alpha_j = \alpha_6$ and $\alpha_2\alpha_j = \alpha_5$, then $\mu = \alpha_1^2\alpha_j = \alpha_2^2\alpha_j$ and $\alpha_1^2 \neq 0, \alpha_2^2 \neq 0$. The minimal model of *X* will be of the form $(\Lambda V, d) \cong (\Lambda(a_1, a_2, b_1, b_2, b_3), d)$ with $da_1 = da_2 = 0$, $db_1 = a_1a_2$, $db_2 = a_2^2 - \lambda a_1^2$, $db_3 = 0$ and $|a_1| = 2n$, $|b| = 2(n+p) + 1$, *i.e.*, $X \sim (\mathbb{S}^{2n} \otimes_{d_2} \mathbb{S}^{2n}) \times \mathbb{S}^{2(n+p)+1}$.
- Third case: $|\alpha_1|$ is even and $\alpha_1\alpha_2 \neq 0$. As in the first case, we can write $\alpha_5 = \alpha_1\alpha_j, \alpha_6 = \alpha_2\alpha_j$. Since $\dim \mathbb{Q}\{\alpha_1^2, \alpha_2^2, \alpha_1\alpha_2\} \leq 1$, then $\alpha_1^2 = \lambda_1\alpha_1\alpha_2$ and $\alpha_2^2 = \lambda_2\alpha_1\alpha_2$. Suppose that $\lambda_1 \neq 0$ and write $a_1^3 = da$, $a_1^2 - \lambda_1 a_1 a_2$ ($\alpha_1^3 = 0$), then $d\left(\frac{1}{\lambda_1}(a - a_1 b)\right)$, *i.e.*, $\alpha_1\alpha_2 = 0$. That contradicts the main hypothesis in our third case. Hence the minimal model of *X* will be of the form $(\Lambda V, d) \cong (\Lambda(a_1, a_2, b, b_1, b_2), d)$ with $da_1 = da_2 = db = 0$, $db_1 = a_1^2$, $db_2 = a_2^2$, *i.e.*, $X \sim \mathbb{S}^{2n} \times \mathbb{S}^{2n} \times \mathbb{S}^{2(n+p)+1}$.

3.4. The Case Where $|\alpha_1| < |\alpha_2| = |\alpha_3| < |\alpha_4| = |\alpha_5| < |\alpha_6|$

Proposition 9 *If* $|\alpha_1| < |\alpha_2| = |\alpha_3| < |\alpha_4| = |\alpha_5| < |\alpha_6|$ *and* $fd(X)$ *is even, then x have one the r.h.t of* $\mathbb{S}^{2n} \times \left(\mathbb{S}^{2(n+p)+1}\right)^2$.

Proof. As $fd(X)$ is even and $\chi_c = 0$, then $|\alpha_1|$ and $|\alpha_2|$ are respectively even and odd. Suppose (by contradiction) that $\alpha_1\alpha_2$ or $\alpha_1\alpha_3$ is null (for example $\alpha_1\alpha_2 = 0$). The duality of Poincar insures that $\{a_4, a_5\} \subset V$, then $\sum_{|a_i| \text{odd}} |a_i| \geq |a_2| + |a_3| + |a_4| + |a_5| > 2\,fd(X) - 1$. This is impossible (QED). Put $\alpha_4 = \alpha_1\alpha_2$, $\alpha_5 = \alpha_1\alpha_3$, then $\alpha_2\alpha_3 \neq 0$, because that $\mu = \alpha_1\alpha_2\alpha_3$. This leads us to take $\alpha_6 = \alpha_2\alpha_3$ and to conclude that $\alpha_1^2 = 0$. Hence $(\Lambda V, d) \cong (\Lambda(x, y_1, y_2, y), d)$ with $dx = dy_1 = dy_2 = 0$, $dy = x^2$ and $|x| = 2n$, $|y_1| = |y_2| = 2(n+p) + 1$, *i.e.*, $X \sim \mathbb{S}^{2n} \times \left(\mathbb{S}^{2(n+p)+1}\right)^2$.

Lemma 10 *If* $|\alpha_1| < |\alpha_2| = |\alpha_3| < |\alpha_4| = |\alpha_5| < |\alpha_6|$ *and* $fd(X)$ *is odd, then* $\{a_4, a_5\} \cap V = \varnothing$.

Let us suppose $a_4 \in V$ (for example) and discuss two cases:

- $|a_4|$ is even, then $\alpha_4^2 = 0$ and there exists a generator a_7 of ΛV such that $da_7 = a_4^2$. If $|a_1|$ is odd, then $\sum_{|a_i| \text{odd}} |a_i| \geq |a_1| + |a_2| + |a_3| + |a_7| > 2\,fd(X) - 1$, impossible. Then $|a_1|$ is even and necessary $\dim \mathbb{Q}\{\alpha_1^2, \alpha_2\alpha_3\} \leq 1$, *i.e.*, there exists a generator a_8 of ΛV verifying $da_8 = \lambda_1 a_1^2 + \lambda_2 a_2 a_3$ with $\lambda_1 = 0$ or $\lambda_2 = 0$. Consequently
$$\begin{aligned}\sum_{|a_i| \text{odd}} |a_i| &\geq |a_2| + |a_3| + |a_7| + |a_8| \\ &\geq |a_2| + |a_3| + |a_7| + |a_8| \\ &\geq 2|a_2| + 2|a_4| - 1 + 2|a_1| - 1 \\ &> 2(|a_2| + |a_4|) - 1 = 2\,fd(X) - 1,\end{aligned}$$
what is, once again, an impossible situation.
- $|a_4|$ is odd, because of the Poincar duality, we must have $|a_2|$ be even and $\dim \mathbb{Q}\{\alpha_2\alpha_3, \alpha_3\alpha_4\} = 1$. Let $a_9 \in \Lambda V$ such that $da_9 = \lambda_1 a_2 a_4 - \lambda_2 a_3 a_4$, then
$$\begin{aligned}\sum_{|a_i| \text{even}} |a_i| &\geq |a_2| + |a_3| + |a_9| \geq 2|a_2| + |a_2| + |a_4| - 1 \\ &> |a_2| + |a_4| = fd(X).\end{aligned}$$
This is impossible.

Lemma 11 *If* $|\alpha_1| < |\alpha_2| = |\alpha_3| < |\alpha_4| = |\alpha_5| < |\alpha_6|$ *and* $fd(X)$ *is odd, then* $\alpha_1\alpha_2 \neq 0$ *and* $\alpha_1\alpha_3 \neq 0$.

Proof. (*By contradiction*) Assume, for example, that $\alpha_1\alpha_2 = 0$. By the precedent lemma and the duality of Poincar, we have $\{a_4, a_5\} \subset \Lambda\{a_1, a_2, a_3\}$ and $\alpha_2\alpha_5 \neq 0$, $\alpha_3\alpha_4 \neq 0$. Therefore $\alpha_5 \in \mathbb{Q}\{\alpha_2^2, \alpha_3^2, \alpha_2\alpha_3\}$, but $fd(X)$ is odd, then $\alpha_5 \in \mathbb{Q}\alpha_2\alpha_3$ and $\mu \in \mathbb{Q}\alpha_2\alpha_5 = \mathbb{Q}\alpha_2^2\alpha_3 = 0$. This is a contradiction (*QED*).

Proposition 12 *If* $|\alpha_1| < |\alpha_2| = |\alpha_3| < |\alpha_4| = |\alpha_5| < |\alpha_6|$ *and* $fd(X)$ *is odd, then x have one of the following r.h.t*:

- $\mathbb{S}^{2n+1} \times \mathbb{S}^k \times \mathbb{S}^k$ with $k \geq 2n + 2$,
- $\mathbb{S}^{2n+1} \times Y_\lambda$, where $\lambda \in \mathbb{Q}^*$ and Y_λ have a minimal

model of the form $(\Lambda(a,b,u,v),d)$ with $da = db = 0$, $du = ab$, $dv = b^2 - \lambda a^2$.

Proof. By the two last lemmas, we have $|\alpha_1|$ is odd and $(\Lambda V,d) \cong (\Lambda a_1,0) \otimes (\Lambda V',d)$ with $\{a_2,a_3\} \in V'$. But $\dim H^*(\Lambda V',d) = 4$ (case classified by the first author in his thesis), then $X \sim \mathbb{S}^{n_1} \times Y$ where $Y \sim \mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ and $n_2 \geq n_1 + 1$ or $Y \sim Y_\lambda$ where $(\Lambda V',d) \cong (\Lambda(a,b,u,v),d)$ and $da = db = 0$, $du = ab$, $dv = b^2 - \lambda a^2$.

3.5. Case Where $|\alpha_1| < |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| < |\alpha_6|$

Lemma 13 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| < |\alpha_6|$, *then* $a_3 \in \Lambda(a_1,a_2)$ *or* $a_4 \in \Lambda(a_1,a_2)$.

Proof. Suppose that $\{a_3,a_4\} \in V$, then there exist two generators a_7 and a_8 of ΛV satisfying $da_7 = a_3a_4 - \lambda a_2a_5 + \omega$ and $da_8 = \lambda_1 a_1 a_3 + \lambda_2 a_1 a_4$ with $\omega \in \Lambda^{\geq 3} V$. We distinguish two cases:

- First case: $|\alpha_3|$ is even, then $|\alpha_1|$ is odd. As $fd(X)$ is even and $\chi_c = 0$, then $|a_8|$ is even and consequently $\sum_{|a_i| \text{even}} |a_i| \geq |a_3| + |a_4| + |a_8| > fd(X)$.
- Second case: $|\alpha_3|$ is odd. As $fd(X)$ is even and $\chi_c = 0$, then $|a_8|$ is even and $\sum_{|a_i| \text{even}} |a_i| \geq |a_3| + |a_4| + |a_8| > fd(X)$.

The two cases are both impossible.

Lemma 14 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| < |\alpha_6|$, *then*:

1) $\alpha_1^2 = 0$,
2) $\{a_3,a_4\} \cap \mathbb{Q}\{a_1a_2\} \neq \varnothing$,
3) $\{a_3,a_4\} \cap V \neq \varnothing$,
4) $\alpha_5 \in \mathbb{Q}^*\alpha_1\alpha_4$ and $\alpha_6 \in \mathbb{Q}^*\alpha_2\alpha_4$.

Proof. 1) suppose that $\alpha_1^2 = 0$, then $|\alpha_1|$ is even. Since $fd(X)$ is even and $\chi_c = 0$, then $|\alpha_2|$, $|\alpha_3|$ and $|\alpha_5|$ are both odd. Put $\alpha_6 = \alpha_1^2$, then $\{a_2,a_3,a_4,a_4\} \subset V$ and $\sum_{|a_i| \text{odd}} |a_i| \geq |a_2| + |a_3| + |a_4| + |a_5| > 2\,fd(X) - 1$ (contradiction).

2) We have $\{a_3,a_4\} \cap \mathbb{Q}\{a_1a_2, a_2^2\} \neq \varnothing$. If $|\alpha_2|$ is even, then $|\alpha_3|$ is odd and $\{a_3,a_4\} \cap \mathbb{Q}\{a_1a_2\} \neq \varnothing$. If $|\alpha_2|$ is odd, the result is evident because that $\alpha_2^2 = 0$.

3) It is an immediate consequence of 2). Hence we can take $\alpha_6 = \alpha_1^2$ and $\{a_2,a_3,a_4,a_5\} \subset V^{\text{odd}}$.

4) Since $\alpha_3^2 = \alpha_1^2\alpha_2^2 = 0$, then there exists $\lambda \in \mathbb{Q}^*$ such that $\alpha_3\alpha_4 = \lambda\mu$. So $\alpha_1\alpha_4 \neq 0$, $\alpha_2\alpha_4 \neq 0$ and $\alpha_5 \in \mathbb{Q}^*\alpha_1\alpha_4$, $\alpha_6 \in \mathbb{Q}^*\alpha_2\alpha_4$.

Proposition 15 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| = |\alpha_4| < |\alpha_5| < |\alpha_6|$, *then X have the h.r.t of* $\mathbb{S}^{2n} \times \mathbb{S}^{2k+1} \times \mathbb{S}^{2(n+k)+1}$ *or that of* $\mathbb{S}^{2n+1} \times \mathbb{S}^{2(n+k)+1} \otimes_{d_2} \mathbb{S}^{2k}$.

Proof. Put $\alpha_5 = \alpha_1\alpha_4$, $\alpha_6 = \alpha_2\alpha_4$ and $\alpha_2^2 = \lambda_1\alpha_2\alpha_4$, $\alpha_4^2 = \lambda_2\alpha_1\alpha_2\alpha_4$, then the minimal model of *X* have one of the following forms:

- $(\Lambda V,d) \cong (\Lambda(x,y_1,y_2,y_3),d)$ with $dx_1 = dy_1 = dy_2 = 0$, $dy = x_1^2$ and $|x_1| = 2n$, $|y_1| = 2k+1$, $|y_2| = 2(k+n)+1$, *i.e.*, $X \sim \mathbb{S}^{2n} \times \mathbb{S}^{2k+1} \times \mathbb{S}^{2(n+k)+1}$.
- $(\Lambda V,d) \cong (\Lambda(x,y_1,y_2,y_3),d)$ with $dx_1 = dy_1 = dy_2 = 0$, $dy_3 = x^2 - \lambda y_1 y_2$ and $|x_1| = 2k$, $|y_1| = 2n+1$, $|y_2| = 2(k+n)+1$, *i.e.*, $X \sim \mathbb{S}^{2n+1} \times \mathbb{S}^{2(n+k)+1} \otimes_{d_2} \mathbb{S}^{2k}$.

3.6. Case Where $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6|$

Lemma 16 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6|$, *then* $a_5 \in \Lambda(a_1,a_2,a_3,a_4)$.

Proof. Let $a_5 \in V$ and discuss many cases:

1) $a_5 \in V^{\text{even}}$, then there exist two generators *x* and *y* of ΛV such that $dx = a_2a_5 - a_3a_4$ and $dy = a_5^2$

a) $|a_2|$ is odd, then

$$\sum_{|a_i| \text{even}} |a_i| \geq |a_1| + |a_2| + |a_6| + |a_9| > fd(X).$$

b) $|a_2|$ is even. As $fd(X)$ is even and $\chi_c = 0$, then $|a_1|$ is odd, $a_1 \in V^{\text{even}}$ and

$$\sum_{|a_i| \text{odd}} |a_i| \geq |x| + |y| > 2\,fd(X) - 1.$$

2) $a_5 \in V^{\text{odd}}$.

a) $|a_2|$ is odd, then necessary $a_2 \in V^{\text{odd}}$ and $\alpha_2\alpha_5 = \alpha_3\alpha_4 = \mu$. Hence there exists a generator *x* of ΛV such that $dx = a_2a_5 - a_3a_4$.

i) $a_1 \in V^{\text{even}}$, then $|a_3|$ and $|a_2|$ are both odd since $\chi_c = 0$. Since $\alpha_3\alpha_4 \neq 0$ then $a_j \in V^{\text{odd}}$ for $j = 3$ or $j = 4$ with $\alpha_j\alpha_5 = 0$. Hence there exists a generator *y* of ΛV such that $dy = a_j a_5$ and so

$$\sum_{|a_i| \text{odd}} |a_i| \geq |x| + |y| > 2\,fd(X) - 1.$$

ii) $a_1 \in V^{\text{odd}}$, then $x \in V^{\text{odd}}$ and

$$\sum_{|a_i| \text{odd}} |a_i| \geq |a_1| + |a_2| + |a_3| + |x| > 2\,fd(X) - 1.$$

b) $|a_2|$ is even.

i) $a_2 \in V^{\text{even}}$, then necessary $x \in V^{\text{even}}$ and

$$\sum_{|a_i| \text{even}} |a_i| \geq |a_2| + |x| > fd(X).$$

ii) $a_2 = a_1^2$, then $\alpha_1^2\alpha_5 \neq 0$ (because of the Poincar duality) and $\alpha_1\alpha_5 \neq 0$. Put $a_6 = a_1a_5$, then $a_j \in V^{\text{odd}}$ for $j = 3$ or $j = 4$ ($\chi_c = 0$) and $\alpha_j\alpha_5 = 0$ (Poincar duality). Let *x* be a generator of ΛV such that $dx = a_j a_5$, then

$$\sum_{|a_i| \text{odd}} |a_i| \geq |a_j| + |a_5| + |x|$$
$$\geq 2(|a_j| + |a_5|) - 1 > 2\,fd(X) - 1.$$

Lemma 17 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6|$, *then* $a_6 \in \Lambda(a_1,a_2,a_3,a_4)$.

Proof. Let $a_6 \in V$ and discuss many cases:

1) $a_6 \in V^{\text{even}}$. Then there exist generators a_7, a_8 of

ΛV such that $da_7 = a_1a_6 - a_3a_4$ and $da_8 = a_6^2$.

a) $|a_1|$ is even, then

$$\sum_{|a_i|\text{odd}} |a_i| \geq |a_7| + |a_8| \geq fd(X) - 1 + 2|a_6| - 1$$
$$\geq 2\,fd(X) - 1 + |a_6| - 1 > 2\,fd(X) - 1.$$

b) $|a_1|$ is odd, then

$$\sum_{|a_i|\text{even}} |a_i| \geq |a_6| + |a_7| > fd(X).$$

2) $a_6 \in V^{\text{odd}}$.

a) $a_2 \in V^{\text{even}}$, then

$$\sum_{|a_i|\text{even}} |a_i| \geq |a_2| + |a_9| > fd(X).$$

b) $a_2 \in V^{\text{odd}}$, then

$$\sum_{|a_i|\text{odd}} |a_i| \geq |a_1| + |a_2| + |a_6| + |a_9| > 2\,fd(X) - 1.$$

Lemma 18 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6|$, *then* $2 \leq \left|\{a_1, a_2, a_3, a_4\} \cap V\right| \leq 3$.

Proof. Put $N = \left|\{a_1, a_2, a_3, a_4\} \cap V\right|$.

1) If $N = 1$, then $a_i \in \Lambda(a_i)$ for all $i = 1, \cdots, 6$, this implies the contradiction $\chi_c \neq 0$.

2) If $N = 4$. We have $\alpha_3\alpha_4 = \alpha_1\alpha_6 = \mu$.

a) $|a_3|$ and $|a_4|$ are both even, then $fd(X)$ is even and $\alpha_4^2 = 0$. Let a_7 and a_8 be some generators of ΛV with $da_7 = a_1a_6 - a_3a_4$ and $da_8 = a_4^2$, therefore

$$\sum_{|a_i|\text{odd}} |a_i| \geq |a_7| + |a_8| > 2\,fd(X) - 1.$$

b) $|a_3|$ and $|a_4|$ are both odd, then $fd(X)$ is odd and $\chi_c = 2\left((-1)^{|a_1|} + (-1)^{|a_2|}\right) = 0$, so $|\alpha_1|$ (for example) is odd and

$$\sum_{|a_i|\text{odd}} |a_i| \geq |a_1| + |a_3| + |a_4| + |a_7| > 2\,fd(X) - 1.$$

c) $|a_3|$ is even and $|a_4|$ is odd (for example), then

$$\sum_{|a_i|\text{even}} |a_i| \geq |a_3| + |a_7| > fd(X).$$

Lemma 19 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6|$ *and if* $\left|\{a_1, a_2, a_3, a_4\} \cap V\right| = 2$, *then* a_1 *and* a_j *have different parities where* $a_j = \{a_2, a_3, a_4\} \cap V$.

Proof. Suppose that a_1 and a_j have the same parity.

1) If $|a_1|$ and $|a_j|$ are both even, then all $|a_i|$ are even for $i = 1, \cdots, 6$ and $\chi_c \neq 0$.

2) If $|a_1|$ and $|a_j|$ are odd, then necessary $a_j = a_2$ and $\{\alpha_3, \alpha_4\} \subset \mathbb{Q}\alpha_1\alpha_2$ (because that $\{a_3, a_4\} \subset \Lambda(a_1, a_2)$), but this is impossible.

Proposition 20 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6|$ *and if* $\left|\{a_1, a_2, a_3, a_4\} \cap V\right| = 2$, *then* $X \sim \mathbb{S}^n_{(3)} \times \mathbb{S}^{2m+1}$.

Proof. Put $\{a_1, a_2, a_3, a_4\} \cap V = \{a_1, a_j\}$. Because the duality of Poincar, the fact that $|a_1|$, $|a_j|$ have different parities and the fact that $a_i \in \Lambda(a_1, a_j)$ for all $i = 1, \cdots, 6$. Let $a_\ell \in \{a_1, a_j\}$ such that $|a_\ell|$ is odd and $a_k \in \{a_1, a_j\} - a_\ell$, it is evident that $|a_k|$ is even. As $H^*(\Lambda V, d) = \mathbb{Q}\{\alpha_\ell^m \alpha_k^n, n \in \mathbb{N}, m = 0, 1\}$, then $\alpha_k\alpha_{7-k} = \alpha_\ell\alpha_{7-\ell} = \mu$. This allows us to take $\{a_{7-\ell}, a_{7-k}\} = \{a_k^p, a_\ell a_k^{p-1}\}$ with $p = 3$, because if not $\dim H^*(\Lambda V, d) \neq 8$. Hence $\alpha_k^4 = 0$ and $\alpha_\ell\alpha_k^n \neq 0$. Conclude that $(\Lambda V, d) \cong \Lambda(a_\ell, 0) \otimes (\Lambda W, d)$, that $\dim V^{\text{even}} = 1$ and that $\dim H^*(\Lambda W, d) = 4$. In [?], $(\Lambda W, d)$ is the minimal model of $\mathbb{S}^n_{(3)}$, then $X \sim \mathbb{S}^n_{(3)} \times \mathbb{S}^{2m+1}$ where $n = |a_k|$ and $2m + 1 = |a_\ell|$.

Lemma 21 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6|$ *and if* $\left|\{a_1, a_2, a_3, a_4\} \cap V\right| = 3$, *then only one among* a_3 *or* a_4 *is in V.*

Proof. Assume that $\{a_3, a_4\} \in V$, then $|a_1|$ and $|a_2|$ are both even, because that necessary $a_2 \in \mathbb{Q}a_1^2$. Therefore $\chi_c = \left(1 + (-1)^{fd(X)}\right)\left(3 + (-1)^{|a_3|}\right) = 0$, *i.e.*, $fd(X)$ is odd, and there exists a generator a_7 of ΛV, such that $da_7 = a_3a_4 - a_1a_6$ with

$$\sum_{|a_i|\text{even}} |a_i| \geq |a_3| + |a_7| > fd(X).$$

Lemma 22 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6|$ *and if* $\left|\{a_1, a_2, a_3, a_4\} \cap V\right| = 3$, *then* $a_j = a_1a_2$ *where* $a_j = \{a_3, a_4\} \cap V$.

Proof. Suppose $\alpha_1\alpha_2 = 0$, we know, from the duality of Poincar, that $\alpha_1\alpha_6 = \alpha_2\alpha_5 = \mu$ and by the Lemmas 16 and 17 that $\{a_5, a_6\} \subset \Lambda(a_1, a_2, a_j)$. We deduce that $a_5 \in \Lambda(a_2, a_j)$, $a_6 \in \Lambda(a_1, a_6)$ and that $a_k \in \mathbb{Q}\{a_1^2, a_2^2, a_1a_j, a_2a_j\}$ where $k = \{3, 4\} - j$.

1) If $a_k \in \mathbb{Q}a_1^2$, then $a_6 = a_1a_j$ and $a_5 \in \mathbb{Q}\{a_2^2, a_2a_j\}$. As $|a_5| \leq |a_6|$, then $a_5 \in \mathbb{Q}a_2^2$, and so $|a_i|$ is even for all $i = 1, \cdots, 6$, but this implies that $\chi_c \neq 0$.

2) If $a_k \in \mathbb{Q}a_2^2$, then necessary $\alpha_1^2 = 0$ and $\alpha_j\alpha_k = \lambda\alpha_2^2\alpha_j = \lambda'\mu$. Hence $fd(X) = 2|a_2| + |a_j|$ and $a_5 \in \mathbb{Q}a_2a_j$, $a_6 \in \mathbb{Q}a_j^2$. Since $|a_j|$ and $|a_2|$ are both even, then $fd(X) = |a_1| + |a_6|$ and $|a_6| = 2|a_j|$. So $|a_i|$ is even for all $i = 1, \cdots, 6$, but this leads to the contradiction $\chi_c \neq 0$.

3) If $a_k \in \mathbb{Q}a_1a_j$, then $\alpha_1^2 = 0$ and $\alpha_1\alpha_j^2 = \lambda\mu$. Suppose that $\alpha_2^2 = 0$ and discuss two cases.

a) $a_2^2 \in \mathbb{Q}a_1a_j$, then $|a_1|$, $|a_2|$ and $|a_3|$ are even and $\chi_c \neq 0$.

b) $a_2^2 \in \mathbb{Q}a_5$, then $fd(X) = 3|a_2|$ is even, but also $fd(X) = |a_1| + |a_6| = |a_1| + 2|a_j|$, then $|a_1|$ is even. Therefore $|a_i|$ is even for all $i = 1, \cdots, 6$ and $\chi_c \neq 0$.

4) If $a_k \in \mathbb{Q}a_1a_j$, then $\alpha_1^2 = \alpha_2^2 = 0$ and $\alpha_j\alpha_k = \lambda\alpha_2\alpha_j^2 = \lambda'\mu$ (Poincar duality). Hence $a_5 \in \mathbb{Q}a_j^2$, $a_6 \in \mathbb{Q}a_j^3$ and $fd(X) = |a_2| + 2|a_j| = |a_1| + 3|a_j|$. So $|a_1| + |a_j| = |a_2| < |a_j|$.

Proposition 23 *If* $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4| < |\alpha_5| < |\alpha_6|$ *and if* $\left|\{a_1, a_2, a_3, a_4\} \cap V\right| = 3$, *then X have one of the following r.h.t*:

- *E*: the total space of the fiber bundle with $\mathbb{S}^{2p+1} \times$

$\mathbb{S}^{2q+1}$ as base space,

- $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \mathbb{S}^{n_3}$ where n_1, n_2 and n_3 are both even.

Proof. We know by the precedent lemmas and by the Poincar duality that $\alpha_k \alpha_j = \alpha_1 \alpha_2 \alpha_j = \mu$. Put $a_5 = a_1 a_k$ and $a_6 = a_2 a_j$, then $\alpha_1^2 = 0$ and $\alpha_2^2 = \lambda \alpha_1 \alpha_j$. We distinguish two cases:

1) $\lambda \neq 0$, then $|a_1|$ and $|a_j|$ are both odd because that $\chi_c \neq 0$. Replace λa_1 by a_1 and put $\lambda = 1$, then the minimal model of X is of the form $(\Lambda(x, y_1, y_2, y), d)$ with $|y_1| = 2p+1 < |x| = 2n < |y_2| = 2q+1$ and $dy = x^2 - y_1 y_2$. Hence $X \sim E$ where $\mathbb{S}^{2n} \to E \to \mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$ is a fibration of the KS-complex

$$(\Lambda(y_1, y_2), 0) \to (\Lambda(y_1, y_2) \otimes \Lambda(x, y), d) \to (\Lambda(x, y), \bar{d}).$$

2) $\lambda = 0$, then X have the minimal model $(\Lambda(x, y_1, y_2, y), d)$ with $dy_1 = dy_2 = 0$ and $dy = x^2$, *i.e.*, $X \sim \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \mathbb{S}^{n_3}$ with n_1, n_2 and n_3 are both even.

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