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L-Topological Spaces Based on Residuated Lattices

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ABSTRACT

In this paper, we introduce the notion of *L*-topological spaces based on a complete bounded integral residuated lattice and discuss some properties of interior and left (right) closure operators.

Keywords: Residuated Lattice; *L*-Topological Space; Interior Operator; Left (Right) Closure Operator

1. Introduction

Residuation is a fundamental concept of ordered structures and the residuated lattices, obtained by adding a residuated monoid operation to lattices, have been applied in several branches of mathematics, including *L*-groups, ideal lattices of rings and multivalued logic. Commutative residuated lattices have been studied by Krull, Dilworth and Ward. These structures were generalized to the non-commutative situation by Blount and Tsinakis [1].

Definition 1.1. [1-4]. A residuated lattice is an algebra $L=(L,\wedge,\vee,\cdot,\to,\mapsto,0,1)$ of type (2, 2, 2, 2, 2, 0, 0) satisfying the following conditions:

(L1) $(L,\wedge,\vee)$ is a lattice,

(L2) $(L,\cdot,1)$ is a monoid, *i.e.*, $\cdot$ is associative and $x\cdot 1=1\cdot x=x$ for any $x\in L$,

(L3) $x\cdot y\le z$ if and only if $x\le y\to z$ if and only if $y\le x\mapsto z$ for any $x, y, z\in L$.

Generally speaking, 1 is not the top element of *L*. A residuated lattice with a constant 0 is called a pointed residuated lattice or full Lambek algebra (*FL*-algebra, for short). If $x\le 1$ for all $x\in L$, then *L* is called integral residuated lattice. An *FL*-algebra *L* which satisfies the condition $0\le x\le 1$ for all $x\in L$ is called FL_w-algebra or bounded integral residuated lattice (see [2]). Clearly, if *L* is an FL_w-algebra, then $(L,\wedge,\vee,0,1)$ is a bounded lattice.

A bounded integral residuated lattice is called commutative (see [5]) if the operation $\cdot$ is commutative. We adopt the usual convention of representing the monoid operation by juxtaposition, writing *ab* for $a\cdot b$.

The following theorem collects some properties of bounded integral residuated lattices (see [1-4,6]).

Theorem 1.1. *Let L be a bounded integral residuated lattice. Then the following properties hold.*

1) $x\to x=x\mapsto x=1$, $1\to x=1\mapsto x=x$.

2) $x\to(y\mapsto z)=y\mapsto(x\to z)$.

3) $x(x\mapsto y)\le x\wedge y$, $(x\to y)x\le x\wedge y$, $x\le y\to xy$, $y\le x\mapsto xy$.

4) $(x\mapsto y)(y\mapsto z)\le x\mapsto z$, $(y\to z)(x\to y)\le x\to z$.

5) *If* $x\le y$, *then* $xz\le yz, zx\le zy$, $x\to z\ge y\to z$, $x\mapsto z\ge y\mapsto z, z\to x\le z\to y$ *and* $z\mapsto x\le z\mapsto y$.

6) $x\le y$ *if and only if* $x\to y=1$ *if and only if* $x\mapsto y=1$.

7) $xy\mapsto z=y\mapsto(x\mapsto z)$, $xy\to z=x\to(y\to z)$.

8) $(x\vee y)\to z=(x\to z)\wedge(y\to z)$, $(x\vee y)\mapsto z=(x\mapsto z)\wedge(y\mapsto z)$.

9) $x\to(y\wedge z)=(x\to y)(x\to z)$, $x\to(y\wedge z)=(x\to y)(x\to z)$.

If bounded integral residuated lattice *L* is complete, then

$$x\to z=\vee\{y\in L|yx\le z\}, x\mapsto z=\vee\{y\in L|xy\le z\}$$

Thus, it follows from some results in [7] that

Theorem 1.2. *Let L be a complete bounded integral residuated lattice and* $a,b,a_j,b_j\in L(j\in J)$. *Then the following properties hold.*

1) $a(\vee_{j\in J}b_j)=\vee_{j\in J}ab_j$ *and* $(\vee_{j\in J}a_j)b=\vee_{j\in J}a_jb$, *i.e., the operation* $\cdot$ *is infinitely* $\vee$*-distributive.*

2) $(\vee_{j\in J}a_j)\to b=\wedge_{j\in J}(a_j\to b)$ *and* $(\vee_{j\in J}a_j)\mapsto b=\wedge_{j\in J}(a_j\mapsto b)$.

3) $a\to(\wedge_{j\in J}b_j)=\wedge_{j\in J}(a\to b_j)$ *and* $a\mapsto(\wedge_{j\in J}b_j)=\wedge_{j\in J}(a\mapsto b_j)$, *i.e., the two residuation operations* $\to$ *and* $\mapsto$ *are all right infinitely* $\wedge$*-distributive* (*see* [8]).

4) $(\wedge_{j\in J}a_j)\to b\ge\vee_{j\in J}(a_j\to b)$ *and* $(\wedge_{j\in J}a_j)\mapsto b\ge\vee_{j\in J}(a_j\to b)$.

5) $a \to \left(\vee_{j\in J} b_j\right) \ge \vee_{j\in J}\left(a \to b_j\right)$ *and* $a \mapsto \left(\vee_{j\in J} b_j\right) \ge \vee_{j\in J}\left(a \mapsto b_j\right)$.

Let us define on *L* two negations, $\neg^L$ and $\neg^R$: $\neg^L x = x \to 0$ and $\neg^R x = x \mapsto 0$.

For any $x, x_j \left(j \in J\right), b \in L$, it follows from Theorems 1.1 and 1.2 that

$$\neg^L\neg^R x \ge x,\quad \neg^R\neg^L x \ge x,\quad x \to \neg^L y = \neg^L\left(xy\right),$$

$$x \mapsto \neg^R y = \neg^R\left(xy\right),\quad x \to \neg^R y = y \mapsto \neg^L x,$$

$$\neg^L\neg^R\neg^L x = \neg^L x,\quad \neg^R\neg^L\neg^R x = \neg^R x,$$

$$x \mapsto y \le \neg^R y \to \neg^R x,\quad x \to y \le \neg^L y \mapsto \neg^L x,$$

$$\neg^L\left(\vee_{j\in J} x_j\right) = \wedge_{j\in J}\neg^L x_j,\quad \neg^R\left(\vee_{j\in J} x_j\right) = \wedge_{j\in J}\neg^R x_j,$$

$$\neg^L\left(\wedge_{j\in J} x_j\right) \ge \wedge_{j\in J}\neg^L x_j,\quad \neg^R\left(\wedge_{j\in J} x_j\right) \ge \wedge_{j\in J}\neg^R x_j.$$

A bounded residuated lattice *L* is called an involutive residuated lattice (see [3]) if $\neg^L\neg^R x = \neg^R\neg^L x = x$ for any $x \in L$. In a complete involutive residuated lattice *L*,

$$x \mapsto y = \neg^R y \to \neg^R x, x \to y = \neg^L y \mapsto \neg^L x,$$

$$\neg^L\left(\wedge_{j\in J} x_j\right) = \wedge_{j\in J}\neg^L x_j, \neg^R\left(\wedge_{j\in J} x_j\right) = \wedge_{j\in J}\neg^R x_j.$$

In the sequel, unless otherwise stated, *L* always represents any given complete bounded integral residuated lattice with maximal element 1 and minimal element 0.

The family of all *L*-fuzzy set in *X* will be denoted by L^X. For any family $\mu, \mu_j \in L^X \left(j \in J\right)$ of *L*-fuzzy sets, we will write $\neg^L\mu, \neg^R\mu, \vee_{j\in J}\mu_j$ and $\wedge_{j\in J}\mu_j$ to denote the *L*-fuzzy sets in *X* given by

$$\left(\neg^L\mu\right)\left(x\right) = \neg^L\left(\mu\left(\left(x\right)\right), \left(\neg^R\mu\right)\left(x\right) = \neg^R\left(\mu\left(\left(x\right)\right),\right.\right.$$

$$\left(\vee_{j\in J}\mu_j\right)\left(x\right) = \vee_{j\in J}\mu_j\left(x\right), \left(\wedge_{j\in J}\mu_j\right)\left(x\right) = \wedge_{j\in J}\mu_j\left(x\right).$$

Besides this, we define $1_X, 0_X \in L^X$ as follows: $1_X(x) = 1 \forall x \in X$ and $0_X\left(x\right) = 0 \forall x \in X$.

2. *L*-Topological Spaces

A completely distributive lattice *L* is called a *F*-lattice, if *L* has an order-reversing involution $': L \to L$. When *L* is a *F*-lattice, Liu and Luo [9] studied the concept of *L*-topology. Below, we consider the notion of *L*-topological space based on a complete bounded integral residuated lattice.

Definition 2.1. Let $\tau \subseteq L^X$. If τ satisfies the following three conditions:

(LFT1) $0_X, 1_X \in \tau$,

(LFT2) $\mu, \nu \in \tau \Rightarrow \mu \wedge \nu \in \tau$,

(LFT3) $\mu_j \in \tau \Rightarrow \vee_{j\in J}\mu_j \in \tau$,

then τ is called an *L*-topology on *X* and $\left(L^X, \tau\right)$ *L*-topological space.

When $L = [0,1]$, called an *L*-topological space $\left(L^X, \tau\right)$ an *F*-topological space.

Every element in τ is called an open subset in L^X.

Let $\tau'_L = \left\{\neg^L\mu \middle| \mu \in \tau\right\}$ and $\tau'_R = \left\{\neg^R\mu \middle| \mu \in \tau\right\}$. The elements of τ'_L and τ'_R are called, respectively, left closed subsets and right closed subsets in L^X.

Definition 2.2. Let τ be an *L*-topology on *X* and μ *L*-fuzzy subset of *X*. The interior, left closure and right closure of μ w.r.t τ are, respectively, defined by

$$\operatorname{int}\left(\mu\right) = \vee\left\{\eta \in \tau \middle| \eta \le \mu\right\},$$

$$cl_L\left(\mu\right) = \wedge\left\{\xi \in \tau'_L \middle| \mu \le \xi\right\},$$

$$cl_R\left(\mu\right) = \wedge\left\{\zeta \in \tau'_R \middle| \mu \le \zeta\right\}.$$

int, cl_L and cl_R are, respectively, called interior, left closure and right closure operators.

For the sake of convenience, we denote $\operatorname{int}\left(\mu\right)$, $cl_L\left(\mu\right)$, and $cl_R\left(\mu\right)$ by μ^o, μ_L^- and μ_R^-, respectively.

In view of Definitions 2.1 and 2.2, for any $\mu \in L^X$,

$$\mu^o = \vee\left\{\eta \in \tau \middle| \eta \le \mu\right\} \in \tau,$$

$$\mu_L^- = \wedge\left\{\neg^L\xi \middle| \mu \le \neg^L\xi, \xi \in \tau\right\} = \neg^L\left(\vee\left\{\xi \middle| \mu \le \neg^L\xi, \xi \in \tau\right\}\right) = \neg^L\mu_1,$$

$$\mu_R^- = \wedge\left\{\neg^R\zeta \middle| \mu \le \neg^R\zeta, \zeta \in \tau\right\} = \neg^R\left(\vee\left\{\zeta \middle| \mu \le \neg^R\zeta, \zeta \in \tau\right\}\right) = \neg^R\mu_2,$$

where

$$\mu_1 = \vee\left\{\xi \middle| \mu \le \neg^L\xi, \xi \in \tau\right\} \in \tau,$$

$$\mu_2 = \vee\left\{\zeta \middle| \mu \le \neg^R\zeta, \zeta \in \tau\right\} \in \tau,$$

i.e., μ^o is just the largest open subset contained in μ, μ_L^- and μ_L^- are, respectively, the smallest left closed and right closed subsets containing μ.

For any $\mu \in L^X$,

$$\neg^L\left(\mu^o\right) = \neg^L\left(\vee\left\{\eta \in \tau \middle| \eta \le \mu\right\}\right) = \wedge\left\{\neg^L\eta \middle| \eta \le \mu, \eta \in \tau\right\} \ge \wedge\left\{\neg^L\eta \middle| \neg^L\mu \le \neg^L\xi, \eta \in \tau\right\} = \left(\neg^L\mu\right)_L^-.$$

Similarly, $\neg^R\left(\mu^o\right) \ge \left(\neg^R\mu\right)_R^-$.

Theorem 2.1. *If L is an involutive residuated lattice and* $\mu \in L^X$, *then*

1) $\neg^L\left(\mu^o\right) = \left(\neg^L\mu\right)_L^-$ *and* $\neg^R\left(\mu^o\right) = \left(\neg^R\mu\right)_R^-$;

2) $\mu^o = \neg^L\left(\neg^R\mu\right)_R^- = \neg^R\left(\neg^L\mu\right)_L^-$;

3) $\left(\neg^L\mu\right)^o = \neg^L\mu_R^-, \left(\neg^R\mu\right)^o = \neg^R\mu_L^-$,

$\mu_L^- = \neg^L\left(\neg^R\mu\right)^o$ *and* $\mu_R^- = \neg^R\left(\neg^L\mu\right)^o$.

Proof. When L is an involutive residuated lattice, $\neg^R\left(\neg^L\mu\right) = \neg^L\left(\neg^R\mu\right) = \mu \forall \mu \in L^X$.

1) If $\eta \in L^X$ and $\neg^L\mu \le \neg^L\eta$, then

$$\mu = \neg^R\left(\neg^L\mu\right) \ge \neg^R\left(\neg^L\eta\right) = \eta.$$

Thus, $\neg^L\left(\mu^o\right) = \left(\neg^L\mu\right)_L^-$. Similarly, $\neg^R\left(\mu^o\right) = \left(\neg^R\mu\right)_R^-$.

2) It follows from 1) that

$$\mu^o = \neg^R\neg^L\left(\mu^o\right) = \neg^R\left(\neg^L\mu\right)_L^-,$$

$$\mu^o = \neg^L\neg^R\left(\mu^o\right) = \neg^L\left(\neg^R\mu\right)_R^-.$$

3) By 2), we see that

$$\left(\neg^L\mu\right)^o = \neg^L\left(\neg^R\neg^L\mu\right)_R^- = \neg^L\left(\mu_R^-\right),$$

$$\left(\neg^R\mu\right)^o = \neg^R\left(\neg^L\neg^R\mu\right)_L^- = \neg^R\left(\mu_L^-\right),$$

$$\neg^L\left(\neg^R\mu\right)^o = \neg^L\left(\neg^R\left(\neg^L\neg^R\mu\right)_L^-\right) = \mu_L^-,$$

$$\neg^R\left(\neg^L\mu\right)^o = \neg^R\left(\neg^L\left(\neg^R\neg^L\mu\right)_R^-\right) = \mu_R^-.$$

Theorem 2.2. *Let* $\mu, \nu \in L^X$. *Then the following properties hold*:

1) $\left(1_X\right)^o = 1_X, \left(0_X\right)_L^- = \left(0_X\right)_R^- = 0_X$.

2) $\mu^o \le \mu, \mu \le \mu_L^-, \mu \le \mu_R^-$.

3) *If* $\mu \le \nu$, *then* $\mu^o \le \nu^o$, $\mu_L^- \le \nu_L^-$, $\mu_R^- \le \nu_R^-$.

4) $\left(\mu^o\right)^o = \mu^o$, $\left(\mu_L^-\right)_L^- = \mu_L^-$ *and* $\left(\mu_R^-\right)_R^- = \mu_R^-$.

5) $\left(\mu \wedge \nu\right)^o = \mu^o \wedge \nu^o$.

6) *If* $\neg^L\left(x \wedge y\right) = \neg^L x \vee \neg^L y \forall\ x, y \in L$, *then* $\left(\mu \vee \nu\right)_L^- = \mu_L^- \vee \nu_L^-$.

7) *If* $\neg^R\left(x \wedge y\right) = \neg^R x \vee \neg^R y \forall x, y \in L$, *then* $\left(\mu \vee \nu\right)_R^- = \mu_R^- \vee \nu_R^-$.

Proof. By Definition 2.2, it is easy to see that 1)-3) hold.

4) By 2) and 3), we have that $\left(\mu^o\right)^o \le \mu^o$. On the other hand, $\mu^o \in \tau$ and $\mu^o \le \mu^o$. Thus, it follows from Definition 2.1 that $\mu^o \le \left(\mu^o\right)^o$ and so $\left(\mu^o\right)^o = \mu^o$. We can prove in an analogous way that $\left(\mu_L^-\right)_L^- = \mu_L^-$ and $\left(\mu_R^-\right)_R^- = \mu_R^-$.

5) Clearly, $\left(\mu \wedge \nu\right)^o \le \mu^o \wedge \nu^o$. Noting that $\mu^o \wedge \nu^o \in \tau$, we see that

$$\mu^o \wedge \nu^o = \left(\mu^o \wedge \nu^o\right)^o \le \mu^o \wedge \nu^o.$$

Thus, $\left(\mu \wedge \nu\right)^o = \mu^o \wedge \nu^o$.

6) There exist $\mu_1, \nu_1 \in \tau$ such that $\mu_L^- = \neg^L\mu_1$, $\nu_L^- = \neg^L\nu_1$.

If $\neg^L\left(x \wedge y\right) = \neg^L x \vee \neg^L y \forall x, y \in L$, then $\mu \vee \nu \le \mu_L^- \vee \nu_L^- = \neg^L\mu_1 \vee \neg^L\nu_1 = \neg^L\left(\mu_1 \wedge \nu_1\right) \in \tau_L^-$. Thus, $\left(\mu \vee \nu\right)_L^- \le \mu_L^- \vee \nu_L^-$. Clearly, $\left(\mu \vee \nu\right)_L^- \ge \mu_L^- \vee \nu_L^-$.

Therefore, $\left(\mu \vee \nu\right)_L^- = \mu_L^- \vee \nu_L^-$.

7) Similar to (6).

Theorem 2.3. *Let* $f : L^X \to L^X$ *be a mapping. Then the following two statements hold.*

1) *If the operator f on* L^X *satisfying the follwing conditions*:

(C1) $f\left(1_X\right) = 1_X$,

(C2) $f\left(\mu\right) \le \mu \forall \mu \in L^X$,

(C3) $f\left(\mu \wedge \nu\right) = f\left(\mu\right) \wedge f\left(\nu\right) \forall \mu, \nu \in L^X$,

then $\tau = \left\{\xi \middle| f\left(\xi\right) = \xi, \xi \in L^X\right\}$ *is an L-topology on X. Moreover, if the operator f also fulfills*

(C4) $f\left(f\left(\mu\right)\right) = f\left(\mu\right) \forall \mu \in L^X$,

then with the L-topology τ, $f\left(\mu\right) = \mu^o$ *for every* $\mu \in L^X$, *i.e., f is the interior operator w.r.t* τ.

2) *If the operator f on* L^X *satisfying the follwing conditions*:

(C1′) $f\left(0_X\right) = 0_X$,

(C2′) $\mu \le f\left(\mu\right) \forall \mu \in L^X$,

(C3′) $f\left(\mu \vee \nu\right) = f\left(\mu\right) \vee f\left(\nu\right) \forall \mu, \nu \in L^X$, *then*

a) *when* $\neg^L\left(x \wedge y\right) = \neg^L x \vee \neg^L y \forall x, y \in L$,

$$\tau_1 = \left\{\eta \middle| f\left(\neg^L\eta\right) = \neg^L\eta, \eta \in L^X\right\}$$

is an L-topology on X, moreover, if the operator f also fulfills

(C4) $f\left(f\left(\mu\right)\right) = f\left(\mu\right) \forall \mu \in L^X$, *and* $\neg^L : L^X \to L^X$ *is a bijection, then with the L-topology* τ_1, $f\left(\mu\right) = \mu_L^- \forall \mu \in L^X$, *i.e., f is the left closure operator w.r.t* τ_1;

b) *when* $\neg^R\left(x \wedge y\right) = \neg^R x \vee \neg^R y \forall x, y \in L$,

$$\tau_2 = \left\{\xi \middle| f\left(\neg^R\xi\right) = \neg^R\xi, \xi \in L^X\right\}$$

is also an L-topology on X, moreover if (C4) *holds and* $\neg^R : L^X \to L^X$ *is a bijection, then with the L-topology* τ_2, $f\left(\mu\right) = \mu_R^- \forall \mu \in L^X$, *i.e., f is the right closure operator w.r.t* τ_2.

Proof. 1) Refer to the proof of Theorem 2.2.21 in [9].

2) Clearly, $0_X, 1_X \in \tau_1$. If $\eta_1, \eta_2 \in \tau_1$, then

$$\begin{aligned} f\left(\neg^L\left(\eta_1 \wedge \eta_2\right)\right) &= f\left(\neg^L\eta_1 \vee \neg^L\eta_2\right) \\ &= f\left(\neg^L\eta_1\right) \vee f\left(\neg^L\eta_2\right) = \neg^L\eta_1 \vee \neg^L\eta_2 \\ &= \neg^L\left(\eta_1 \wedge \eta_2\right), \end{aligned}$$

i.e., $\eta_1 \wedge \eta_2 \in \tau_1$. If $\eta_j \in \tau_1\left(j \in J\right)$, then

$$f\left(\neg^L\left(\vee_{j\in J}\eta_j\right)\right)=f\left(\wedge_{j\in J}\neg^L\eta_j\right)\le\wedge_{j\in J}f\left(\neg^L\eta_j\right)$$
$$=\wedge_{j\in J}\neg^L\eta_j=\neg^L\left(\vee_{j\in J}\eta_j\right).$$

Combing with (C2′), we have that

$$f\left(\neg^L\left(\vee_{j\in J}\eta_j\right)\right)=\neg^L\left(\vee_{j\in J}\eta_j\right).$$

Thus, $\vee_{j\in J}\eta_j\in\tau_1$ and so τ_1 is an *L*-topology on *X*. For any $\mu\in L^X$,

$$f\left(\mu_L^-\right)=f\left(\wedge\left\{\neg^L\xi\middle|\mu\le\neg^L\xi,\xi\in\tau_1\right\}\right)$$
$$\le\left(\wedge\left\{f\left(\neg^L\xi\right)\middle|\mu\le\neg^L\xi,\xi\in\tau_1\right\}\right)$$
$$=\wedge\left\{\neg^L\xi\middle|\mu\le\neg^L\xi,\xi\in\tau_1\right\}=\mu_L^-,$$

i.e., $f(\mu)\le f\left(\mu_L^-\right)\le\mu_L^-$. Moreover, if (C4) holds and $\neg^L:L^X\to L^X$ is a bijection, then

$$f(\mu)\ge\wedge\left\{\eta\in L^X\,\middle|\,f(\eta)=\eta\ge\mu\right\}$$
$$=\wedge\left\{\neg^L\xi\middle|\mu\le\neg^L\xi,\xi\in\tau_1\right\}=\mu_L^-.$$

Therefore, $f(\mu)=\mu_L^-$, *i.e.*, *f* is the left closure operator w.r.t τ_1.

We can prove in an analogous way that τ_2 is an *L*-topology on *X* and the corresponding *f* is the right closure operator w.r.t τ_2.

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