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A Class of Singular Integral Operators Associated to Surfaces of Revolution*

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Abstract

In this paper, the authors establish the L^p-mapping properties of a class of singular integral operators along surfaces of revolution with rough kernels. The size condition on the kernels is optimal and much weaker than that for the classical Calderon-Zygmund singular integral operators.

Keywords: Singular Integrals, Surface of Revolution, Maximal Operator, Rough Kernel

1. Introduction and Main Results

Let R^n, $n \geq 2$, be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in R^n equipped with the normalized Lebesgue measure $\mathrm{d}\sigma = \mathrm{d}\sigma(\bullet)$. Let $\Gamma_\phi = \{y, \phi(|y|);\quad y \in R^n\}$ be the surface of revolution generated by a suitable function ϕ: $[0,\infty) \to R$. For nonzero points $x \in R^n$, we denote $x' = x/|x|$. Let $\Omega \in L^1(S^{n-1})$ be a homogeneous function of degree zero on R^n and satisfy

$$\int_{S^{n-1}} \Omega(y')\mathrm{d}\sigma(y') = 0, \tag{1.1}$$

Suppose that h is a radial measurable function. Define the singular integral operator $T_{\phi,h}$ in R^{n+1} along Γ_ϕ by

$$T_{\phi,h}(f)(x,x_{n+1}) = p.v.\int_{R^n} \frac{h(|y|)\Omega(y')}{|y|^n} \times f(x-y, x_{n+1} - \phi(|y|))\mathrm{d}y \tag{1.2}$$

for all $f \in \mathbb{S}^{n+1}$ (the Schwartz function class on R^{n+1}), where $(x, x_{n+1}) \in R^n \times R = R^{n+1}$.

Operators of the type (1.2) have been studied quite extensively (see [1-13] and therein numerous references). We refer the readers to see Stein-Wainger's report [14, 15] for more background information. In 1996, Kim, Wainger, Wright and Ziesler proved the following result.

Theorem A [10]. *Let $\phi \in C^2([0,\infty))$ be convex, increasing and $\phi(0) = 0$. Suppose that $\Omega \in C^\infty(S^{n-1})$ is a homogeneous function of degree zero on R^n and satisfies (1.1). Then $T_{\phi,1}$ is bounded on $L^p(R^{n+1})$ for $1 < p < \infty$.*

By a minor modification of the proof in Theorem A, one can show that the conclusion of Theorem A remains valid if the condition $\Omega \in C^\infty(S^{n-1})$ is replaced by the condition $\Omega \in L^q(S^{n-1})$ for some $q > 1$ (see [16, pp. 372-373], as well as [6]). Subsequently, this result was improved and extended by many authors (see [1,2,5,6, 11,12] *et al.*). In particular, in 2001, Lu, Pan and Yang gave the general theorem as follows.

Theorem B [11]. *Let ϕ: $[0,\infty) \to R$ be a continuously differentiable on $[0,\infty)$ and satisfy $|\phi(t) - \phi(0)| \leq Ct^\alpha$ for some α and small t, where C is a constant independent of t. Suppose that $\Omega \in H^1(S^{n-1})$ and $h \in L^\infty([0,\infty))$. Then $T_{\phi,h}$ is bounded on $L^p(R^{n+1})$ for $1 < p < \infty$, provided that the maximal operator ν_ϕ defined by*

$$\nu_\phi(f)(r,s) = \sup_{j\in\mathbb{Z}} \frac{1}{2^j}\int_{2^j}^{2^{j+1}} f(r-t, s-\phi(t))\mathrm{d}t$$

is bounded on $L^p(R^2)$ for $p > 1$.

Actually, the condition $h \in L^\infty([0,\infty))$ can be weakened to the case:

$$h \in \Delta_\gamma(R^+) = \{h;\quad \sup_{s>0} s^{-1}\int_0^s |h(t)|^\gamma \mathrm{d}t < \infty\},\ \gamma > 1$$

with $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ (see [9]), and the size condition on Ω in Theorem B is the best one, so far,

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even if $\phi(t)\equiv 0$ (see [8]).

On the other hand, for $h\in L^2\left(R^+,r^{-1}dr\right)$, it is known in [17], if $\phi(t)\equiv 0$, that $T_{\phi,h}$ is bounded on $L^p\left(R^{n+1}\right)$ *for* $1<p<\infty$, provided $\Omega\in L\left(\log^+ L\right)^{1/2}\left(S^{n-1}\right)$, which is optimal and much weaker than that for the classical Calderon-Zygmund singular integral operators. It should be noted that the spaces $L\left(\log^+ L\right)^{1/2}\left(S^{n-1}\right)$ and $H^1\left(S^{n-1}\right)$ do not include in each other.

Inspired by Al-Salman's work [17], we shall establish the following main result in this paper.

Theorem 1. *Let* ϕ: $[0,\infty)\to R$ *be a suitable function, which ensures that the integral in* (1.2) *exists in principle-value sense when, say,* $f\in\mathbb{S}^{n+1}$ *(the Schwartz function class on* R^{n+1}*). Suppose that* $h\in L^2\left(R^+,r^{-1}dr\right)$, Ω *is a homogenous function of degree zero on* R^n *and satisfies* (1.1). *If* $\Omega\in L\left(\log^+ L\right)^{1/2}\left(S^{n-1}\right)$, *then*

1) $\left\|T_{\phi,h}(f)\right\|_{L^2\left(R^{n+1}\right)}\le C\|f\|_{L^2\left(R^{n+1}\right)}.$

2) $\left\|T_{\phi,h}(f)\right\|_{L^p\left(R^{n+1}\right)}\le C\|f\|_{L^p\left(R^{n+1}\right)},$

for $1<p<\infty$, *provided that the lower dimensional maximal operator* M_ϕ *defined by*

$$M_\phi(f)(r,s)=\sup_{R>0}R^{-1}\int_{R/2}^{R}\left|f\left(r-t,s-\phi(t)\right)\right|\mathrm{d}t \quad (1.3)$$

is bounded on $L^p\left(R^2\right)$ *for* $1<p<\infty$.

In Theorem 1, if $\phi(t)$ is continuously differentiable on $(0,\infty)$ and satisfy $\left|\phi(t)-\phi(0)\right|\le Ct^\alpha$ for some α and small t, where C is a constant independent of t, in particular, if $\phi(t)\in C^1([0,1))$, then the integral in (1.2) exists in principle-value sense when, say, $f\in\mathbb{S}^{n+1}$ (the Schwartz function class on R^{n+1}) (see [11]).

Remark 1. The $L^p\left(R^2\right)$ boundedness of M_ϕ is known for many $\phi(t)$'s. A few prominent examples are as follows:

1) If $\phi(t)$ is a real-valued polynomial, then M_ϕ is bounded on $L^p\left(R^2\right)$ for $p>1$, see [16, p. 477] or [15].

2) If $\phi(t)=t^\alpha$ with $\alpha\in(0,1]$, then M_ϕ is bounded on $L^p\left(R^2\right)$ for $p>1$, see [13].

3) If $\phi(t)\in C^1([0,1))$ such that $\phi(0)=\phi'(0)=0$ and $\phi(t)'$ is a convex increasing function for $t>0$, M_ϕ is bounded on $L^p\left(R^2\right)$ for $p>1$, see [13], see [7, Corollary 5.3].

4) Let $b(t)=t\phi'(t)-\phi(t)$. If $\phi(t)\in C^2([0,1))$, $\phi(t)$ is convex on $[0,\infty)$ with $\phi(0)=\phi'(0)=0$, and there exists an $\varepsilon>0$ so that for each $t>0$, $b'(t)>\varepsilon b(t)/t$, then M_ϕ is bounded on $L^p\left(R^2\right)$ for $p>1$, see [4, Theorem 1.5]. In particular, if $\phi(t)$ is ether even or odd and there exists a $0<C<\infty$ so that for each $t>0$, $\phi'(Ct)>2\phi(t)$, then M_ϕ is bounded on $L^p\left(R^2\right)$ for $p>1$, see [3] or [4].

In order to obtain Theorem 1, we let $S_{\Omega,\phi}$ be the operator defined by

$$S_{\Omega,\phi}(f)(x,x_{n+1}) := \left(\int_0^\infty\left|\int_{S^{n-1}}\Omega(y')f\left(x-ry',x_{n+1}-\phi(|y|)\right)\mathrm{d}\sigma(y')\right|^2\frac{\mathrm{d}r}{r}\right)^{1/2}.$$

Clearly, if $h\in L^2\left(R^+,r^{-1}dr\right)$, then

$$\left|T_{\phi,h}(f)(x,x_{n+1})\right|\le\|h\|_{L^2\left(R^+,r^{-1}dr\right)}S_{\Omega,\phi}(f)(x,x_{n+1}).$$

Therefore Theorem 1 can be deduced immediately from the next theorem.

Theorem 2. *Let* ϕ *and* Ω *be as in Theorem* 1. *Then*

1) $\left\|S_{\Omega,\phi}(f)\right\|_{L^2\left(R^{n+1}\right)}\le C\|f\|_{L^2\left(R^{n+1}\right)}.$

2) $\left\|S_{\Omega,\phi}(f)\right\|_{L^p\left(R^{n+1}\right)}\le C\|f\|_{L^p\left(R^{n+1}\right)},$

$2<p<\infty$, *provided that the maximal operator* M_ϕ *in* (1.3) *is bounded on* $L^p\left(R^2\right)$ *for* $1<p<\infty$.

Remark 2. By the similar arguments as in [1], we remark that the condition $\Omega\in L\left(\log^+ L\right)^{1/2}\left(S^{n-1}\right)$ is optimal. Precisely, there exists an Ω lies in $\Omega\in L\left(\log^+ L\right)^{\varepsilon}\left(S^{n-1}\right)$ for all $\varepsilon<1/2$ and satisfies (1.1) such that $S_{\Omega,\phi}$ is unbounded on $L^p\left(R^{n+1}\right)$. And it is worth pointing out that the size condition is much weaker than that for the classical Calderón-Zygmund singular integral operators.

This paper is organize as follows. In Section 2 we will give the proofs of our theorems. An extension of our main results will be given in Section 3. We would like to remark that the main ideas in the proofs of our results are taken from [7,9,17].

Throughout this paper, we always use letter C to denote positive constants that may vary at each occurrence but are independent of the essential variables.

2. Proofs of Main Results

Let us begin with a lemma, which will play a key role in the proofs of our main results.

Lemma 2.1. *Let* $\Omega\in L^1\left(S^{n-1}\right)$ *and satisfy* (1.1). *If the maximal operator* M_ϕ *in* (1.3) *is bounded on* $L^p\left(R^2\right)$ *for, then the following maximal operator* $M_{\Omega,\phi}$ *defined by*

$$M_{\Omega,\phi}(f)(x,x_{n+1}) := \sup_{R>0}\int_{R/2<|y|\le R}\frac{|\Omega(y')|}{|y|^n}\times\left|f\left(x+y,x_{n+1}+\phi(|y|)\right)\right|\mathrm{d}y \quad (2.1)$$

is bounded on $L^p\left(R^{n+1}\right)$ *with bound* $C\|\Omega\|_{L^1\left(S^{n-1}\right)}$, $1<p<\infty$.

Proof. Since

$$M_{\Omega,\phi}(f)(x,x_{n+1})$$
$$=\sup_{R>0}\int_{R/2}^{R}\int_{S^{n-1}}|\Omega(y')||f(x+ry',x_{n+1}+\phi(r))|\mathrm{d}\sigma(y')\frac{\mathrm{d}r}{r}$$
$$\le\int_{S^{n-1}}|\Omega(y')||\sup_{R>0}\int_{R/2}^{R}|f(x+ry',x_{n+1}+\phi(r))|\frac{\mathrm{d}r}{r}\mathrm{d}\sigma(y')$$
$$\le\int_{S^{n-1}}|\Omega(y')|M_{\phi,y'}(f)(x,x_{n+1})\mathrm{d}\sigma(y')$$

where

$$M_{\phi,y'}(f)(x,x_{n+1}):=\sup_{R>0}\frac{1}{R}\int_{R}^{2R}|f(x+ry',x_{n+1}+\phi(r))|\mathrm{d}r.$$

Thus by Minkowski's inequality, we have

$$\|M_{\Omega,\phi}(f)\|_{L^p(R^{n+1})}\le\int_{S^{n-1}}|\Omega(y')|\|M_{\phi,y'}(f)\|_{L^p(R^{n+1})}\mathrm{d}\sigma(y').$$

It remains to show that

$$\|M_{\phi,y'}(f)\|_{L^p(R^{n+1})}\le C\|f\|_{L^p(R^{n+1})}$$

with C independent of y'.

Let $\mathbf{1}=(1,0,\cdots,0)\in S^{n-1}$. For each fixed $y'\in S^{n-1}$, choose a rotation ρ such that $\rho y'=\mathbf{1}$. Denote the inverse of ρ by ρ^{-1} and define the function f_ρ by $f_\rho(x,x_{n+1})=f(\rho x,x_{n+1})$. Then

$$f(x+ry',x_{n+1}+\phi(r))=f_{\rho^{-1}}(\rho x+r\mathbf{1},x_{n+1}+\phi(r)).$$

This together with the L^p-bondedness of M_ϕ, and change of variables, show that

$$\|M_{\phi,y'}(f)\|_{L^p(R^{n+1})}\le C\|f\|_{L^p(R^{n+1})},\ 1<p<\infty,$$

where C is independent of. Lemma 2.1 is proved.

Next we introduce some notations. Assume that $\Omega\in L(\log^+ L)^{1/2}(S^{n-1})$ and satisfies (1.1). For any $l\in\mathbb{N}$, let $E_l=\{y'\in S^{n-1}:\ 2^l\le|\Omega(y')|<2^{l+1}\}$ Also, we let $E_0=\{y'\in S^{n-1}:\ |\Omega(y')|<2\}$ Set $D=\{l\in\mathbb{N}:\ \sigma(E_l)>2^{-4l}\}$ and for $l\ge1$,

$$\Omega_l(y'):=\Omega(y')\chi_{E_l}(y')-\sigma(S^{n-1})^{-1}\int_{S^{n-1}}\Omega(y')\chi_{E_l}(y')\mathrm{d}\sigma(y'),$$

and $\Omega_0(y'):=\Omega(y')-\sum_{l\in D}\Omega_l(y')$. Then we have the following:

$$\int_{S^{n-1}}\Omega_l(y')\mathrm{d}\sigma(y')=0,\ l\ge0, \tag{2.2}$$

$$\|\Omega_l\|_1\le2\|\Omega\chi_{E_l}\|_1:=2A_l,\ l\in D, \tag{2.3}$$

$$\|\Omega_0\|_1\le C\|\Omega_0\|_2\le C<\infty, \tag{2.4}$$

$$\Omega(y')=\sum_{l\in D\cup\{0\}}\Omega_l(y'), \tag{2.5}$$

$$\sum_{l\in D\cup\{0\}}(l+1)^{1/2}A_l\le C\|\Omega\|_{L(\log^+L)^{1/2}(S^{n-1})}, \tag{2.6}$$

where $A_l=\|\Omega\chi_{E_l}\|_1$ for $l\in D$ and $A_0=1$.

Now we give the proofs of our theorems as follows.

Proof of Theorem 2. For each $l\in D\cup\{0\}$, we let

$$S_{\Omega_l,\phi}(f)(x,x_{n+1}):=\left(\int_0^\infty\left|\int_{S^{n-1}}\Omega_l(y')f(x-ry',x_{n+1}-\phi(|y|))\mathrm{d}\sigma(y')\right|^2\frac{\mathrm{d}r}{r}\right)^{1/2}.$$

By (2.5) and Minkowski's inequality, we have

$$S_{\Omega,\phi}(f)(x,x_{n+1})\le\sum_{l\in D\cup\{0\}}S_{\Omega_l,\phi}(f)(x,x_{n+1}). \tag{2.7}$$

For any $l\in D\cup\{0\}$, let's argue as in the proof of Theorem 2.1 in [1], choose a collection of C^∞ function $\{\psi_{l,j}\}_{j\in\mathbb{Z}}$ on $(0,\infty)$ with the following properties:

$$\operatorname{supp}(\psi_{l,j})\subseteq\left[2^{-(j+1)(l+1)},2^{-(j-1)(l+1)}\right], \tag{2.8}$$

$$0\le\psi_{l,j}(t)\le1,\ \text{and}\ \sum_{j\in\mathbb{Z}}[\psi_{l,j}(t)]^2=1, \tag{2.9}$$

$$\left|\frac{\mathrm{d}^\alpha\psi_{l,j}(t)}{\mathrm{d}t^\alpha}\right|\le\frac{C_\alpha}{t^\alpha}, \tag{2.10}$$

where $t>0$, $\alpha\in\mathbb{N}$, and C_α is a constant independent of *l*.

For each $j\in\mathbb{Z}$ and $l\in D\cup\{0\}$, denote by $S_{l,j}$ the multiplier

$$\widehat{S_{l,j}}(f)(\xi,\xi_{n+1})=\psi_{l,j}(|\xi|)\hat{f}(\xi,\xi_{n+1}),$$

and $S_{l,j}^2$ by

$$S_{l,j}^2(f)(x,x_{n+1})=S_{l,j}(S_{l,j}(f))(x,x_{n+1}).$$

Then by (2.9) and Minkowski's inequality,

$$S_{\Omega_l,\phi}(f)(x,x_{n+1})$$
$$=\left(\sum_{j\in\mathbb{Z}}\int_{2^{j(l+1)}}^{2^{(j+1)(l+1)}}\left|\sum_{k\in\mathbb{Z}}\int_{S^{n-1}}\Omega_l(y')\times S_{l,j+k}^2(f)(x-ry',x_{n+1}-\phi(r))\mathrm{d}\sigma(y')\right|^2\frac{\mathrm{d}r}{r}\right)^{1/2}$$
$$\le\sum_{k\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}}\int_1^{2^{l+1}}\left|\int_{S^{n-1}}\Omega_l(y')\times S_{l,j+k}^2(f)(x-ry',x_{n+1}-\phi(r))\mathrm{d}\sigma(y')\right|^2\frac{\mathrm{d}r}{r}\right)^{1/2}$$
$$:=\sum_{k\in\mathbb{Z}}I_{l,k}(f)(x,x_{n+1}),$$

where

$$I_{l,k}(f)(x,x_{n+1}) = \left(\sum_{j\in\mathbb{Z}}\int_1^{2^{l+1}}\left|\int_{S^{n-1}}\Omega_l(y') \times S_{l,j+k}^2(f)(x-ry',x_{n+1}-\phi(r))\mathrm{d}\sigma(y')\right|^2\frac{\mathrm{d}r}{r}\right)^{1/2}.$$

This together with (2.7) implies

$$\|S_{\Omega,\phi}(f)\|_{L^p(R^{n+1})} \le \sum_{l\in D\cup\{0\}}\sum_{k\in\mathbb{Z}}\|I_{l,k}(f)\|_{L^p(R^{n+1})}. \quad (2.11)$$

Now we estimate $\|I_{l,k}(f)\|_{L^p(R^{n+1})}$ in the following cases:

Case 1. For $p=2$, we claim that there exists $\delta>0$ such that for $l\in D\cup\{0\}$,

$$\|I_{l,k}(f)\|_{L^2(R^{n+1})} \le C2^{-\delta|k|}(l+1)^{1/2}A_l\|f\|_{L^2(R^{n+1})}, \quad (2.12)$$

where C is independent of l and k.

Indeed, by Plancheral's theorem and Fubini's theorem,

$$\|I_{l,k}(f)\|_2^2 \le \sum_{j\in\mathbb{Z}}\int_{\Delta_{l,j+k}}\left|\hat{f}(\xi,\xi_{n+1})\right|^2\times J_{l,j}(\xi)\mathrm{d}\xi\mathrm{d}\xi_{n+1}, \quad (2.13)$$

where

$$\Delta_{l,j+k} := \left\{\xi\in R^n:\ 2^{-(j+k+l)(l+1)} \le |\xi| \le 2^{-(j+k-1)(l+1)}\right\}\times R,$$

$$J_{l,j}(\xi) = \int_1^{2^{l+1}}\left|\int_{S^{n-1}}\Omega_l(y')\mathrm{e}^{-i2^{j(l+1)}ry'\bullet\xi}\mathrm{d}\sigma(y')\right|^2\frac{\mathrm{d}r}{r}$$

In order to prove (2.12), we first estimate $J_{l,j}(\xi)$. Obviously, we have

$$J_{l,j}(\xi) \le (l+1)\|\Omega_l\|_1^2 \le C(l+1)A_l^2. \quad (2.14)$$

By (2.2), a straightforward computing shows that

$$J_{l,j}(\xi) \le (l+1)\|\Omega_l\|_1^2\left|2^{(j+1)(l+1)}\xi\right|^2 \le C(l+1)A_l^2\left|2^{(j+1)(l+1)}\xi\right|^2, \quad (2.15)$$

Using interpolation between (2.14) and (2.15), we get

$$J_{l,j}(\xi) \le C(l+1)A_l^2\left|2^{(j+1)(l+1)}\xi\right|^{2/(l+1)}. \quad (2.16)$$

On the other hand, we have

$$\begin{aligned}J_{l,j}(\xi) &= \int_1^{2^{l+1}}\iint_{S^{n-1}\times S^{n-1}}\Omega_l(y')\overline{\Omega_l(u')}\times\mathrm{e}^{-i2^{j(l+1)}r(y'-u)\bullet\xi}\mathrm{d}\sigma(y')\mathrm{d}\sigma(u')\frac{\mathrm{d}r}{r}\\ &= \iint_{S^{n-1}\times S^{n-1}}\Omega_l(y')\overline{\Omega_l(u')}\times\int_1^{2^{l+1}}\mathrm{e}^{-i2^{j(l+1)}r(y'-u)\bullet\xi}\frac{\mathrm{d}r}{r}\mathrm{d}\sigma(y')\mathrm{d}\sigma(u')\end{aligned}.$$

And by integration by parts,

$$\left|\int_1^{2^{l+1}}\mathrm{e}^{-i2^{j(l+1)}r(y'-u)\bullet\xi}\frac{\mathrm{d}r}{r}\right| \le C(l+1)\left|2^{j(l+1)}(y'-u')\bullet\xi\right|^{-1},$$

with the easy fact

$$\left|\int_1^{2^{l+1}}\mathrm{e}^{-i2^{j(l+1)}r(y'-u)\bullet\xi}\frac{\mathrm{d}r}{r}\right| \le l+1,$$

we obtain

$$\left|\int_1^{2^{l+1}}\mathrm{e}^{-i2^{j(l+1)}r(y'-u)\bullet\xi}\frac{\mathrm{d}r}{r}\right| \le C(l+1)\left|2^{j(l+1)}(y'-u')\bullet\xi\right|^{-1/4}.$$

Thus, by Holder's inequality

$$\begin{aligned}J_{l,j}(\xi) &\le C(l+1)\left|2^{j(l+1)}\xi\right|^{-1/4}\|\Omega_l\|_2^2\left\{\iint_{S^{n-1}\times S^{n-1}}\times\left|(y'-u')\bullet\xi'\right|^{-1/2}d\sigma(y')d\sigma(u')\right\}^{1/2}\\ &\le C(l+1)\left|2^{j(l+1)}\xi\right|^{-1/4}\|\Omega_l\|_2^2.\end{aligned}$$

Note that $\|\Omega_0\|_2 \le C = CA_0$, and for $l\in D$, $A_l \le C2^l\sigma(E_l) \le C2^{-3l}$, we have

$$\|\Omega_l\|_2 \le C2^{l+1}\sigma(E_l)^{1/2} \le C2^{2(l+1)}A_l.$$

Consequently,

$$J_{l,j}(\xi) \le C(l+1)A_l^2 2^{4(l+1)}\left|2^{j(l+1)}\xi\right|^{-1/4}.$$

This together with (2.14) and an interpolation implies

$$J_{l,j}(\xi) \le C(l+1)A_l^2\left|2^{j(l+1)}\xi\right|^{-1/4(l+1)}. \quad (2.17)$$

Then by the fact that $\Delta_{l,j+k}\cap\Delta_{l,j'+k}=\varnothing$ whenever $j'\notin\{j-1,j,j+1\}$, (2.12) follows from (2.13), (2.16) and (2.17).

Case 2. For $p>2$, we shall show that there exists $\theta>0$ such that for $l\in D\cup\{0\}$,

$$\|I_{l,k}(f)\|_{L^p(R^{n+1})} \le C2^{-\theta|k|}(l+1)^{1/2}A_l\|f\|_{L^p(R^{n+1})}. \quad (2.18)$$

where C is independent of l and k.

Indeed, choose $g\in L^{(p/2)'}(R^{n+1})$ such that $\|g\|_{L^{(p/2)}(R^{n+1})}=1$ and

$$\|I_{l,k}(f)\|_{L^p(R^{n+1})}^2 = \int_{R^{n+1}}\sum_{j\in\mathbb{Z}}\int_1^{2^{l+1}}\left|\int_{S^{n-1}}\Omega_l(y')\times S_{l,j+k}^2(f)\left(x-2^{j(l+1)}ry',x_{n+1}-\phi\left(2^{j(l+1)}r\right)\right)\mathrm{d}\sigma(y')\right|^2\times\frac{\mathrm{d}r}{r}g(x,x_{n+1})\mathrm{d}x\mathrm{d}x_{n+1}.$$

By Holder's inequality, we get

$$\left\|I_{l,k}(f)\right\|_p^2 \le \int_{R^{n+1}} \sup_{j\in\mathbb{Z}} \int_1^{2^{l+1}} \int_{S^{n-1}} \left|\Omega_l(y')\right| \times\left|g\left(x+2^{j(l+1)}ry', x_{n+1}+\phi\left(2^{j(l+1)}r\right)\right)\right| \mathrm{d}\sigma(y')\frac{\mathrm{d}r}{r} \times\sum_{j\in\mathbb{Z}}\left|S_{l,j+k}^2(f)(x,x_{n+1})\right|^2 \mathrm{d}x\mathrm{d}x_{n+1}\left\|\Omega_l\right\|_1.$$

Note that

$$\sup_{j\in\mathbb{Z}} \int_1^{2^{l+1}} \int_{S^{n-1}} \left|\Omega_l(y')\right| \times\left|g\left(x+2^{j(l+1)}ry', x_{n+1}+\phi\left(2^{j(l+1)}r\right)\right)\right| \mathrm{d}\sigma(y')\frac{\mathrm{d}r}{r} \le (l+1)M_{\Omega_l,\phi}(g)(x,x_{n+1}).$$

Applying Holder's inequality again, it follows from Lemma 2.1 and the Littlewood-Paley theory (see [18, Chapter 4]) that

$$\left\|I_{l,k}(f)\right\|_p^2 \le C(l+1)\left\|\Omega_l\right\|_1^2\left\|g\right\|_{(p/2)'} \times\left\|\left(\sum_{j\in\mathbb{Z}}\left|S_{l,j+k}^2(f)\right|^2\right)^{1/2}\right\|_p \le C(l+1)\left\|\Omega_l\right\|_1^2\left\|\left(\sum_{j\in\mathbb{Z}}\left|S_{l,j+k}^2(f)\right|^2\right)^{1/2}\right\|_p \le C(l+1)\left\|\Omega_l\right\|_1^2\left\|f\right\|_p^2 \le C(l+1)A_l^2\left\|f\right\|_p^2.$$

This together with (2.12) and an interpolation implies (2.18).

Therefore, by (2.11), (2.12) and (2.6), we get

$$\left\|S_{\Omega,\phi}(f)\right\|_{L^2(R^{n+1})} \le C\sum_{l\in D\cup\{0\}}\sum_{k\in\mathbb{Z}}(l+1)^{1/2} \times A_l 2^{-\delta|k|}\left\|f\right\|_{L^2(R^{n+1})} \le C\sum_{l\in D\cup\{0\}}(l+1)^{1/2}A_l\left\|f\right\|_{L^2(R^{n+1})} \le C\left\|\Omega\right\|_{L(\log^+ L)^{1/2}(S^{n-1})}\left\|f\right\|_{L^2(R^{n+1})}.$$

This prove 1) of Theorem 2.

If M_ϕ is bounded on $L^p(R^2)$ for $1<p<\infty$, then by (2.11), (2.18) and (2.6), we obtain that for $p>2$,

$$\left\|S_{\Omega,\phi}(f)\right\|_{L^p(R^{n+1})} \le C\left\|\Omega\right\|_{L(\log^+ L)^{1/2}(S^{n-1})}\left\|f\right\|_{L^p(R^{n+1})},$$

which completes the proof of Theorem 2.

Proof of Theorem 1. By the fact

$$\left|T_{\phi,h}(f)(x,x_{n+1})\right| \le \left\|h\right\|_{L^2(R^+,r^{-1}dr)} S_{\Omega,\phi}(f)(x,x_{n+1}),$$

and Theorem 2, it follows that $T_{\phi,h}$ is bounded on $L^p(R^{n+1})$ for $2\le p<\infty$. On the other hand, by duality we can obtain that $T_{\phi,h}$ is bounded on $L^p(R^{n+1})$ for $1<p<2$. Theorem 1 is proved.

3. Further Results

In this section, we will extend the definition of $T_{\phi,h}$ to higher dimensional cases. Let $\Phi(t)=(\phi_1(t),\cdots,\phi_m(t))$ be a curve on R^m, $m\ge 2$, where each $\phi_i(t)$ is a real-valued continuous function. For $x, y\in R^n$, and $x^*\in R^m$, we define

$$T_{\Phi,h}(f)(x,x^*) = p.v.\int_{R^n} h(|y|)|y|^{-n}\Omega(y') \times f\left(x-y, x^*-\Phi(|y|)\right)\mathrm{d}y. \tag{4.1}$$

Also, we define the operator $S_{\Omega,\Phi}$ by

$$S_{\Omega,\Phi}(f)(x,x^*) = \left(\int_0^\infty\left|\int_{S^{n-1}}\Omega(y') \times f\left(x-ry', x^*-\Phi(r)\right)\mathrm{d}\sigma(y')\right|^2\frac{\mathrm{d}r}{r}\right)^{1/2}, \tag{4.2}$$

and the lower dimensional maximal function M_Φ by

$$M_\Phi(f)(x_1,x_*) = \sup_{r>0} r^{-1}\int_{r/2}^r\left|f\left(x_1-t, x_*-\Phi(t)\right)\right|\mathrm{d}t. \tag{4.3}$$

In [10], Fan and Zheng extended the result of Theorem B to the operator $T_{\Phi,h}$ for $h\in\Delta_\gamma(R^+)$ for $\gamma > 1$. Here, we will obtain the following results.

Theorem 3. *Suppose that* $h\in L^2(R^+, r^{-1}dr)$ *and* $\Omega\in L(\log^+L)^{1/2}(S^{n-1})$. *Then*

1) $\left\|T_{\Phi,h}(f)\right\|_{L^2(R^{n+m})} \le C\left\|f\right\|_{L^2(R^{n+m})}.$

2) $\left\|T_{\Phi,h}(f)\right\|_{L^p(R^{n+m})} \le C\left\|f\right\|_{L^p(R^{n+m})}$, $1<p<\infty$,

provided that the maximal operator M_Φ *defined in* (4.3) *is bounded on* $L^p(R^{m+1})$ *for all* $p>1$.

Theorem 4. *Let* Φ *and* Ω *be as in Theorem* 3. *Then*

1) $\left\|S_{\Omega,\Phi}(f)\right\|_{L^2(R^{n+m})} \le C\left\|f\right\|_{L^2(R^{n+m})}.$

2) $\left\|S_{\Omega,\Phi}(f)\right\|_{L^p(R^{n+m})} \le C\left\|f\right\|_{L^p(R^{n+m})}$, $2<p<\infty$,

provided that the maximal operator M_Φ *defined in* (4.3) *is bounded on* $L^p(R^{m+1})$ *for all* $p>1$.

Clearly, if $m = 1$ then Theorem 3 and 4 are reduced to Theorem 1 and 2. For $m\ge 2$, by the same arguments as in the proof of Lemma 2.1, we can obtain the following lemma.

Lemma 4.1. *Let* $\Omega\in L^1(S^{n-1})$ *and satisfy* (1.1). *If the maximal operator* M_Φ *in* (4.3) *is bounded on* $L^p(R^{m+1})$ *for* $p > 1$, *then the following maximal operator* $M_{\Omega,\Phi}$ *defined by*

$$M_{\Omega,\Phi}(f)(x,x_*) := \sup_{r>0}\int_{r/2<|y|\le r}\frac{\left|\Omega(y')\right|}{|y|^n}\times\left|f\left(x+y, x_*+\Phi(|y|)\right)\right|dy \tag{4.4}$$

is bounded on $L^p(R^{n+m})$ *with bound* $C\left\|\Omega\right\|_{L^1(S^{n-1})}$,

$1 < p < \infty$.

Then Theorem 3 and 4 follow from this lemma with the arguments and the estimates similar to those in the proofs of our theorems in Section 2. The details are omitted.

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