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Toeplitz and Translation Operators on the *q*-Fock Spaces*

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Abstract

In this work, we introduce a class of Hilbert spaces $\mathcal{F}_q$ of entire functions on the disk $D\left(o,\frac{1}{\sqrt{1-q}}\right)$, $0<q<1$, with reproducing kernel given by the *q*-exponential function $e_q(z)$; and we prove some properties concerning Toeplitz operators on this space. The definition and properties of the space $\mathcal{F}_q$ extend naturally those of the well-known classical Fock space. Next, we study the multiplication operator Q by z and the *q*-Derivative operator D_q on the Fock space $\mathcal{F}_q$; and we prove that these operators are adjoint-operators and continuous from this space into itself. Lastly, we study a generalized translation operators and a Weyl commutation relations on $\mathcal{F}_q$.

Keywords: *q*-Fock Spaces, *q*-Exponential Function, *q*-Derivative Operator, *q*-Translation Operators, *q*-Toeplitz Operators, *q*-Weyl Commutation Relations

1. Introduction

In 1961, Bargmann [1] introduced a Hilbert space $\mathcal{F}$ of entire functions $f(z)=\sum_{n=0}^{\infty}a_n z^n$ on $\mathbb{C}$ such that

$$\|f\|_{\mathcal{F}}^2 := \sum_{n=0}^{\infty}|a_n|^2 n! < \infty.$$

On this space the author study the differential operator $D = \mathrm{d}/\mathrm{dz}$ and the multiplication operator by z, and proves that these operators are densely defined, closed and adjoint-operators on $\mathcal{F}$ (see [1]). Next, the Hilbert space $\mathcal{F}$ is called Segal-Bargmann space or Fock space and it was the aim of many works [2,3].

In this paper, we consider the *q*-exponential function:

$$e_q(z) := \sum_{n=0}^{\infty}\frac{(1-q)^n}{(q;q)_n}z^n,$$

where

$$(q;q)_n := \prod_{i=0}^{n-1}\left(1-q^{i+1}\right),\ n=1,2,\cdots,\infty.$$

We discuss some properties of a class of Fock spaces associated to the *q*-exponential function and we give some applications.

In the first part of this work, building on the ideas of Bargmann [1], we define the *q*-Fock space $\mathcal{F}_q$ as the space of entire functions $f(z)=\sum_{n=0}^{\infty}a_n z^n$ on the disk $D\left(o,\frac{1}{\sqrt{1-q}}\right)$ of center o and radius $\frac{1}{\sqrt{1-q}}$, and such that

$$\|f\|_{\mathcal{F}_q}^2 := \sum_{n=0}^{\infty}|a_n|^2\frac{(q;q)_n}{(1-q)^n} < \infty.$$

Let f and g be in $\mathcal{F}_q$, such that $f(z)=\sum_{n=0}^{\infty}a_n z^n$ and $g(z)=\sum_{n=0}^{\infty}b_n z^n$, the inner product is given by

$$\langle f,g\rangle_{\mathcal{F}_q} = \sum_{n=0}^{\infty}a_n\overline{b_n}\frac{(q;q)_n}{(1-q)^n}.$$

The *q*-Fock space $\mathcal{F}_q$ has also a reproducing kernel $\mathcal{K}_q$ given by

$$\mathcal{K}_q(w,z)=e_q(\overline{w}z);\quad w,z\in D\left(o,\frac{1}{\sqrt{1-q}}\right).$$

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Then if $f \in \mathcal{F}_q$, we have

$$\left\langle f, \mathcal{K}_q(w,.)\right\rangle_{\mathcal{F}_q} = f(w), \quad w \in D\left(o, \frac{1}{\sqrt{1-q}}\right).$$

Using this property, we prove that the space $\mathcal{F}_q$ is a Hilbert space and we give an Hilbert basis.

Next, we define and study the Toeplitz operators of the q-Fock space $\mathcal{F}_q$.

In the second part of this work, we consider the multiplication operator Q by z and the q-Derivative operator D_q on the Fock space $\mathcal{F}_q$, and we prove that these operators are continuous from $\mathcal{F}_q$ into itself, and satisfy:

$$\left\|D_q f\right\|_{\mathcal{F}_q} \le \frac{1}{\sqrt{1-q}}\|f\|_{\mathcal{F}_q}, \quad \|Qf\|_{\mathcal{F}_q} \le \frac{1}{\sqrt{1-q}}\|f\|_{\mathcal{F}_q}.$$

Then, we prove that these operators are adjoint-operators on $\mathcal{F}_q$:

$$\langle Qf, g\rangle_{\mathcal{F}_q} = \left\langle f, D_q g\right\rangle_{\mathcal{F}_q}; \quad f, g \in \mathcal{F}_q.$$

Next, we define and study on the Fock space $\mathcal{F}_q$, the q-translation operators:

$$\tau_z f(w) := e_q\left(zD_q\right) f(w); \quad w, z \in D\left(o, \frac{1}{\sqrt{1-q}}\right),$$

and the generalized multiplication operators:

$$M_z f(w) := e_q(zQ) f(w); \quad w, z \in D\left(o, \frac{1}{\sqrt{1-q}}\right).$$

Using the previous results, we deduce that the operators τ_z and M_z, for $z \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$, are continuous from $\mathcal{F}_q$ into itself, and satisfy:

$$\|\tau_z f\|_{\mathcal{F}_q} \le e_q\left(\frac{|z|}{\sqrt{1-q}}\right)\|f\|_{\mathcal{F}_q},$$

$$\|M_z f\|_{\mathcal{F}_q} \le e_q\left(\frac{|z|}{\sqrt{1-q}}\right)\|f\|_{\mathcal{F}_q}.$$

Lastly, we establish Weyl commutation relations between the translation operators τ_a and the multiplication operators M_b, where $a, b \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$. These relations are realized on the Fock space $\mathcal{F}_q$.

2. The q-Fock Spaces $\mathcal{F}_q$ and the Toeplitz Operators

2.1. Preliminaries

Let a and q be real numbers such that $0 < q < 1$; the q-shifted factorial are defined by

$$(a;q)_0 := 1, \ (a;q)_n := \prod_{i=0}^{n-1}\left(1 - aq^i\right), \ n = 1, 2, \cdots, \infty.$$

Jackson [4] defined the q-analogue of the Gamma function as

$$\Gamma_q(x) := \frac{(q;q)_\infty}{\left(q^x;q\right)_\infty}(1-q)^{1-x}, \quad x \ne 0, -1, -2, \cdots$$

It satisfies the functional equation

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1,$$

where

$$[x]_q := \frac{1-q^x}{1-q};$$

and tends to $\Gamma(x)$ when q tends to 1^-. In particular, for $n = 1, 2, \cdots$, we have

$$\Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n} = [n]_q!.$$

The q-combinatorial coefficients are defined for $n \in \mathbb{N}$ and $k = 0, \cdots, n$, by

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The q-derivative $D_q f$ of a suitable function f (see [5]) is given by

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \ne 0,$$

and $D_q f(0) = f'(0)$ provided $f'(0)$ exists.

If f is differentiable then $D_q f(x)$ tends to $f'(x)$ as $q \to 1^-$.

There are two important q-analogues of the exponential function [5]:

$$E_q(z) := \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{z^n}{[n]_q!},$$

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}.$$

Note that the first series converges for $|z| < \infty$ and the second series converges for $|z| < \frac{1}{1-q}$.

Therefore the function Γ_q has the q-integral representation [6]:

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} r^{x-1} E_q(-qr)\,\mathrm{d}_q r, \quad x > 0, \tag{1}$$

where the q-integral (introduced by Jackson [4]) is defined by

$$\int_0^a f(x)\,\mathrm{d}_q x = (1-q)\sum_{n=0}^{\infty} aq^n f(aq^n).$$

Lemma 1. *The function* $e_q(\lambda.)$, $\lambda \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$, *is the unique analytic solution of the q-problem*:

$$D_q y(z) = \lambda y(z), \quad y(0) = 1. \tag{2}$$

Proof. Searching a solution of (2) in the form $y(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$D_q y(z) = \sum_{n=1}^{\infty} a_n [n]_q z^{n-1}.$$

Replacing in (2), we obtain

$$\sum_{n=1}^{\infty} a_n [n]_q z^n = \lambda \sum_{n=1}^{\infty} a_{n-1} x^n.$$

Thus,

$$a_n [n]_q = \lambda a_{n-1}, \quad n = 1, 2, \cdots$$

We deduce that

$$a_n = \frac{\lambda}{[n]_q} a_{n-1}.$$

We get

$$a_n = \frac{\lambda^n}{[n]_q!}.$$

Therefore,

$$y(z) = \sum_{n=0}^{\infty} \frac{(\lambda z)^n}{[n]_q!} = e_q(\lambda z),$$

which completes the proof of the lemma. □

2.2. The *q*-Fock Spaces $\mathcal{F}_q$

We denote by

- $H\left(D\left(o, \frac{1}{\sqrt{1-q}}\right)\right)$ the space of entire functions on $D\left(o, \frac{1}{\sqrt{1-q}}\right)$.
- m_q the measure defined on $D\left(o, \frac{1}{\sqrt{1-q}}\right)$ by
$$dm_q(z) := \frac{1}{2\pi} E_q(-qr)\,d_q r\,d\theta, \quad z = \sqrt{r}\mathrm{e}^{i\theta}.$$
- $L^2\left(D\left(o, \frac{1}{\sqrt{1-q}}\right), m_q\right)$ the space of measurable functions f on $D\left(o, \frac{1}{\sqrt{1-q}}\right)$ satisfying
$$\|f\|^2_{L^2\left(D\left(o, \frac{1}{\sqrt{1-q}}\right), m_q\right)} := \int_{D\left(o, \frac{1}{\sqrt{1-q}}\right)} |f(z)|^2\, dm_q(z) < \infty.$$

Definition 1. We define the prehilbertian space $\mathcal{F}_q$, to be the space of functions in $H\left(D\left(o, \frac{1}{\sqrt{1-q}}\right)\right) \cap L^2\left(D\left(o, \frac{1}{\sqrt{1-q}}\right), m_q\right)$, equipped with the inner product

$$\langle f, g\rangle_{\mathcal{F}_q} = \int_{D\left(o, \frac{1}{\sqrt{1-q}}\right)} f(z)\overline{g(z)}\,dm_q(z),$$

and the norm

$$\|f\|_{\mathcal{F}_q} = \left[\int_{D\left(o, \frac{1}{\sqrt{1-q}}\right)} |f(z)|^2\, dm_q(z)\right]^{1/2}.$$

Remark 1. If $q \to 1^-$, the space $\mathcal{F}_q$ agrees with the Segal-Bargmann's space (see [1]).

Proposition 1. 1) *For all* $f \in \mathcal{F}_q$ *such that* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *we have*

$$\|f\|^2_{\mathcal{F}_q} = \sum_{n=0}^{\infty} |a_n|^2 [n]_q!. \tag{3}$$

2) For all $f, g \in \mathcal{F}_q$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, we have

$$\langle f, g\rangle_{\mathcal{F}_q} = \sum_{n=0}^{\infty} a_n \overline{b_n} [n]_q!. \tag{4}$$

3) For $f, g \in \mathcal{F}_q$, we have

$$\langle f, g\rangle_{\mathcal{F}_q} = f(D_q)\tilde{g}(0), \quad \tilde{g}(z) = \overline{g(\overline{z})}.$$

Proof. Given $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}_q$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{F}_q$.

1) By dominated convergence theorem's, we have

$$\|f\|^2_{\mathcal{F}_q} = \sum_{m,n=0}^{\infty} a_m \overline{a_n} \int_{D\left(o, \frac{1}{\sqrt{1-q}}\right)} z^m \overline{z}^n\, dm_q(z).$$

We put $z=\sqrt{r}e^{i\theta}$, then we deduce

$$\|f\|_{\mathcal{F}_q}^2=\sum_{n=0}^{\infty}|a_n|^2\int_0^{\frac{1}{1-q}}r^nE_q(-qr)\mathrm{d}_qr.$$

But from (1), we have

$$\int_0^{\frac{1}{1-q}}r^nE_q(-qr)\mathrm{d}_qr=\Gamma_q(n+1)=[n]_q!.$$

Thus,

$$\|f\|_{\mathcal{F}_q}^2=\sum_{n=0}^{\infty}|a_n|^2[n]_q!.$$

2) We obtain the result from (1) by polarization.

3) Since

$$D_qz^k=[k]_qz^{k-1},\quad k\geq 1,$$

then

$$D_q^nz^k=\frac{[k]_q!}{[k-n]_q!}z^{k-n},\quad k\geq n, \tag{5}$$

and

$$D_q^ng(z)=\sum_{k=n}^{\infty}\frac{[k]_q!}{[k-n]_q!}b_kz^{k-n}.$$

Thus,

$$b_n=\frac{D_q^ng(0)}{[n]_q!},$$

and

$$g(z)=\sum_{n=0}^{\infty}\frac{D_q^ng(0)}{[n]_q!}z^n. \tag{6}$$

Using (4) and (6), we get

$$\langle f,g\rangle_{\mathcal{F}_q}=\sum_{n=0}^{\infty}a_n\overline{D_q^ng(0)}=\sum_{n=0}^{\infty}a_nD_q^n\tilde{g}(0).$$

Thus

$$\langle f,g\rangle_{\mathcal{F}_q}=f(D_q)\tilde{g}(0),$$

which gives the desired result. □

The following theorem prove that $\mathcal{F}_q$ is a reproducing kernel space.

Theorem 1. *The function* $\mathcal{K}_q$ *given for* $w,z\in D\left(o,\frac{1}{\sqrt{1-q}}\right)$, *by*

$$\mathcal{K}_q(w,z)=e_q(\overline{w}z),$$

is a reproducing kernel for the *q*-Fock space $\mathcal{F}_q$, that is:

1) for all $w\in D\left(o,\frac{1}{\sqrt{1-q}}\right)$, the function $z\to\mathcal{K}_q(w,z)$ belongs to $\mathcal{F}_q$.

2) For all $w\in D\left(o,\frac{1}{\sqrt{1-q}}\right)$ and $f\in\mathcal{F}_q$, we have

$$\langle f,\mathcal{K}_q(w,.)\rangle_{\mathcal{F}_q}=f(w).$$

Proof. 1) Since

$$\mathcal{K}_q(w,z)=\sum_{n=0}^{\infty}\frac{\overline{w}^nz^n}{[n]_q!};\quad z,w\in D\left(o,\frac{1}{\sqrt{1-q}}\right), \tag{7}$$

then from (3), we deduce that

$$\|\mathcal{K}_q(w,.)\|_{\mathcal{F}_q}^2=\sum_{n=0}^{\infty}\frac{|w|^{2n}}{[n]_q!}=e_q\left(|w|^2\right)<\infty,$$

which proves 1).

2) If $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{F}_q$, from (4) and (7), we deduce

$$\langle f,\mathcal{K}_q(w,.)\rangle_{\mathcal{F}_q}=\sum_{n=0}^{\infty}a_nw^n=f(w),\quad w\in D\left(o,\frac{1}{\sqrt{1-q}}\right).$$

This completes the proof of the theorem. □

Remark 2. From Theorem 1 (2), for $f\in\mathcal{F}_q$ and $w\in D\left(o,\frac{1}{\sqrt{1-q}}\right)$, we have

$$|f(w)|\leq\|\mathcal{K}_q(w,.)\|_{\mathcal{F}_q}\|f\|_{\mathcal{F}_q}=\left[e_q\left(|w|^2\right)\right]^{1/2}\|f\|_{\mathcal{F}_q}. \tag{8}$$

Proposition 2. *The space* $\mathcal{F}_q$ *equipped with the inner product* $\langle .,.\rangle_{\mathcal{F}_q}$ *is an Hilbert space; and the set* $\{\xi_n\}_{n\in\mathbb{N}}$ *given by*

$$\xi_n(z)=\frac{z^n}{\sqrt{[n]_q!}},\quad z\in D\left(o,\frac{1}{\sqrt{1-q}}\right),$$

forms an Hilbert basis for the space $\mathcal{F}_q$.

Proof. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{F}_q$. We put

$$f=\lim_{n\to\infty}f_n,\quad \textit{in}\quad \mathcal{F}_q.$$

From (8), we have

$$|f_{n+p}(w)-f_n(w)|\leq\left[e_q\left(|w|^2\right)\right]^{1/2}\|f_{n+p}-f_n\|_{\mathcal{F}_q}.$$

This inequality shows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to f. Since the function $w\to\left[e_q\left(|w|^2\right)\right]^{1/2}$ is continuous on $D\left(o,\frac{1}{\sqrt{1-q}}\right)$, then

$\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on all compact set of $D\left(o,\frac{1}{\sqrt{1-q}}\right)$. Consequently, f is an entire function on $D\left(o,\frac{1}{\sqrt{1-q}}\right)$, then f belongs to the space $\mathcal{F}_q$.

On the other hand, from the relation (4), we get

$$\langle \xi_n, \xi_m\rangle_{\mathcal{F}_q} = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker symbol.

This shows that the family $\{\xi_n\}_{n\in\mathbb{N}}$ is an orthonormal set in $\mathcal{F}_q$.

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an element of $\mathcal{F}_q$ such that

$$\langle f, \xi_n\rangle_{\mathcal{F}_q} = 0, \quad \forall\, n\in\mathbb{N}.$$

From the relation (4), we deduce that

$$a_n = 0, \quad \forall\, n\in\mathbb{N}.$$

This completes the proof. □

2.3. Toepliz Operators on $\mathcal{F}_q$

In this paragraph we study the Toeplitz operators on $\mathcal{F}_q$. These operators generalize the classical Toeplitz operators [2].

First we define the orthogonal projection operator P from $L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)$ into $\mathcal{F}_q$, by

$$Pf(w) := \langle f, K_q(w,.)\rangle_{L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)},$$
$$w\in D\left(o,\frac{1}{\sqrt{1-q}}\right),$$

where K_q is the reproducing kernel given by (7).

Definition 2. Let ϕ be a measurable function on $D\left(o,\frac{1}{\sqrt{1-q}}\right)$. The Toeplitz operator T_ϕ is the operator given by

$$T_\phi f := P(\phi f),$$

for every

$$f\in D(T_\phi) := \left\{ f\in\mathcal{F}_q:\ \varphi f\in L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)\right\}.$$

Remark 3. Let $\phi\in L^\infty\left(D\left(o,\frac{1}{\sqrt{1-q}}\right)\right)$.

1) The operator T_ϕ is bounded and $\|T_\phi\|\le\|\phi\|_\infty$.

2) By derivation under the integral sign and using (2), we have $T_{\bar z} = D_q$.

Theorem 2. *If* $\phi\in L^\infty\left(D\left(o,\frac{1}{\sqrt{1-q}}\right)\right)$ *has compact support, then* T_ϕ *is a compact operator.*

Proof. For $\phi\in L^\infty\left(D\left(o,\frac{1}{\sqrt{1-q}}\right)\right)$, we have

$$\langle T_\phi\xi_n, \xi_k\rangle_{L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)} = \int_{D\left(o,\frac{1}{\sqrt{1-q}}\right)} T_\phi\xi_n(w)\overline{\xi_k(w)}\,\mathrm{d}m_q(w).$$

Since

$$T_\phi\xi_n(w) = \int_{D\left(o,\frac{1}{\sqrt{1-q}}\right)} \phi(z)\xi_n(z)K_q(\bar w,\bar z)\,\mathrm{d}m_q(z).$$

Applying Fubini's theorem and Theorem 1, we obtain

$$\langle T_\phi\xi_n, \xi_k\rangle_{L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)} = \langle \phi\xi_n, \xi_k\rangle_{L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)}.$$

Thus,

$$\sum_{n,k=0}^{\infty}\left|\langle T_\phi\xi_n, \xi_k\rangle_{L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)}\right|^2 = \sum_{n,k=0}^{\infty}\left|\langle \phi\xi_n, \xi_k\rangle_{L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)}\right|^2.$$

Since $\phi\in L^\infty\left(D\left(o,\frac{1}{\sqrt{1-q}}\right)\right)$ with compact support, there are positive constants a and K so that $|\phi(z)|\le K, a.e.$ and $\phi(z)=0$, for all $|z|>a$. Then for $k,n\in\mathbb{N}$, we get

$$\langle \phi\xi_n, \xi_k\rangle_{L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)} = \frac{1}{\sqrt{[n]_q![k]_q!}}\int_{|z|\le a}\phi(z)z^n\bar z^k\,\mathrm{d}m_q(z).$$

Thus,

$$\left|\left\langle \phi\xi_n,\xi_k\right\rangle_{L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)}\right|$$

$$\leq \frac{K}{\sqrt{[n]_q![k]_q!}}\int_{|z|\leq a}|z|^{n+k}\,\mathrm{d}m_q(z)$$

$$\leq \frac{K}{\sqrt{[n]_q![k]_q!}}\int_0^{a^2} r^{(n+k)/2}E_q(-qr)\mathrm{d}_q r$$

$$\leq \frac{Ka^{n+k}}{\sqrt{[n]_q![k]_q!}}\int_0^{\frac{1}{1-q}}E_q(-qr)\,\mathrm{d}_q r.$$

But from (1), we have

$$\int_0^{\frac{1}{1-q}}E_q(-qr)\,\mathrm{d}_q r=\Gamma_q(1)=1.$$

Hence

$$\left|\left\langle \phi\xi_n,\xi_k\right\rangle_{L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)}\right|\leq \frac{K}{\sqrt{[n]_q![k]_q!}}a^{n+k}.$$

Thus, we obtain

$$\sum_{n,k=0}^{\infty}\left|\left\langle T_\phi\xi_n,\xi_k\right\rangle_{L^2\left(D\left(o,\frac{1}{\sqrt{1-q}}\right),m_q\right)}\right|^2\leq K^2\left[e_q\left(a^2\right)\right]^2<\infty.$$

Then, T_ϕ is an Hilbert-Schmidt operator [7], and consequently it is compact. □

3. The Multiplication and Translation Operators on $\mathcal{F}_q$

3.1. The Derivative and Multiplication Operators on $\mathcal{F}_q$

On $\mathcal{F}_q$, we consider the multiplication operator Q given by

$$Qf(z):=zf(z).$$

By straightforward calculation we obtain.

Lemma 2. *$\left[D_q,Q\right]=D_qQ-QD_q=\Lambda_q$, where Λ_q is the q-shift operator given by*

$$\Lambda_q f(z):=f(qz).$$

This lemma is the q-analogous commutation rule of [1]. When $q\to 1^-$, then $\left[D_q,Q\right]$ tends to the identity operator I.

We now study the continuous property of the operators Λ_q, D_q and Q on $\mathcal{F}_q$.

Theorem 3. *If $f\in\mathcal{F}_q$ then $\Lambda_q f$, $D_q f$ and Qf belong to $\mathcal{F}_q$, and we have*

1) $\left\|\Lambda_q f\right\|_{\mathcal{F}_q}\leq\|f\|_{\mathcal{F}_q}$,

2) $\left\|D_q f\right\|_{\mathcal{F}_q}\leq\frac{1}{\sqrt{1-q}}\|f\|_{\mathcal{F}_q}$,

3) $\left\|Qf\right\|_{\mathcal{F}_q}\leq\frac{1}{\sqrt{1-q}}\|f\|_{\mathcal{F}_q}$.

Proof. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{F}_q$.

1) We have

$$\Lambda_q f(z)=f(qz)=\sum_{n=0}^{\infty}a_nq^nz^n,$$

and from (3), we obtain

$$\left\|\Lambda_q f\right\|_{\mathcal{F}_q}^2=\sum_{n=0}^{\infty}|a_n|^2q^{2n}[n]_q!\leq\sum_{n=0}^{\infty}|a_n|^2[n]_q!=\|f\|_{\mathcal{F}_q}^2.$$

2) We have

$$D_qf(z)=\sum_{n=1}^{\infty}a_n[n]_qz^{n-1}=\sum_{n=0}^{\infty}a_{n+1}[n+1]_qz^n. \quad (9)$$

Then from (9), we get

$$\left\|D_q f\right\|_{\mathcal{F}_q}^2=\sum_{n=0}^{\infty}|a_{n+1}|^2\left([n+1]_q\right)^2[n]_q!.$$

Since

$$[n+1]_q!=[n+1]_q[n]_q!, \quad (10)$$

we obtain

$$\left\|D_q f\right\|_{\mathcal{F}_q}^2=\sum_{n=0}^{\infty}|a_{n+1}|^2[n+1]_q[n+1]_q!,$$

and consequently,

$$\left\|D_q f\right\|_{\mathcal{F}_q}^2=\sum_{n=0}^{\infty}|a_n|^2[n]_q[n]_q!. \quad (11)$$

Using the fact that $[n]_q\leq\frac{1}{1-q}$, we obtain

$$\left\|D_q f\right\|_{\mathcal{F}_q}\leq\frac{1}{\sqrt{1-q}}\left[\sum_{n=0}^{\infty}|a_n|^2[n]_q!\right]^{1/2}=\frac{1}{\sqrt{1-q}}\|f\|_{\mathcal{F}_q}.$$

3) On the other hand, since

$$Qf(z)=\sum_{n=1}^{\infty}a_{n-1}z^n, \quad (12)$$

then

$$\|Qf\|_{\mathcal{F}_q}^2=\sum_{n=1}^{\infty}|a_{n-1}|^2[n]_q!=\sum_{n=0}^{\infty}|a_n|^2[n+1]_q!.$$

By (10), we deduce

$$\|Qf\|_{\mathcal{F}_q}^2=\sum_{n=0}^{\infty}|a_n|^2[n+1]_q[n]_q!. \quad (13)$$

Using the fact that $[n+1]_q\leq\frac{1}{1-q}$, we obtain

$$\|Qf\|_{\mathcal{F}_q} \le \frac{1}{\sqrt{1-q}}\|f\|_{\mathcal{F}_q}.$$

□

We deduce also the following norm equality.

Theorem 4. 1) *If* $f \in \mathcal{F}_q$ *then*

$$\|Qf\|^2_{\mathcal{F}_q} = \left\|D_q f\right\|^2_{\mathcal{F}_q} + \left\|\Lambda_{\sqrt{q}} f\right\|^2_{\mathcal{F}_q}.$$

2) The operator $Q:\mathcal{F}_q \to \mathcal{F}_q$ is injective on $\mathcal{F}_q$.

Proof. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}_q$.

1) By (13) and using the fact that $[n+1]_q = [n]_q + q^n$, we obtain

$$\|Qf\|^2_{\mathcal{F}_q} = \sum_{n=0}^{\infty}|a_n|^2\left[[n]_q + q^n\right][n]_q! = \left\|D_q f\right\|^2_{\mathcal{F}_q} + \left\|\Lambda_{\sqrt{q}} f\right\|^2_{\mathcal{F}_q}.$$

2) From (1), we have

$$\|Qf\|^2_{\mathcal{F}_q} \ge \left\|\Lambda_{\sqrt{q}} f\right\|^2_{\mathcal{F}_q}.$$

Therefore $Qf=0$ implies that $f=0$. Then $Q:\mathcal{F}_q \to \mathcal{F}_q$ is injective continuous operator on $\mathcal{F}_q$. □

Proposition 3. *The operators* Q *and* D_q *are adjoint-operators on* $\mathcal{F}_q$; *and for all* $f,g \in \mathcal{F}_q$, *we have*

$$\langle Qf, g\rangle_{\mathcal{F}_q} = \langle f, D_q g\rangle_{\mathcal{F}_q}.$$

Proof. Consider $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum_{n=0}^{\infty} b_n z^n$ in $\mathcal{F}_q$. From (9) and (12),

$$Qf(z) = \sum_{n=1}^{\infty} a_{n-1} z^n, \quad D_q g(z) = \sum_{n=0}^{\infty} b_{n+1}[n+1]_q z^n.$$

Thus from (4), we get

$$\langle Qf, g\rangle_{\mathcal{F}_q} = \sum_{n=1}^{\infty} a_{n-1}\overline{b_n}[n]_q!$$
$$= \sum_{n=0}^{\infty} a_n \overline{b_{n+1}}[n+1]_q! = \langle f, D_q g\rangle_{\mathcal{F}_q},$$

which gives the result. □

3.2. The Translation Operators on $\mathcal{F}_q$

In this section we study a generalized translation operators on $\mathcal{F}_q$. We begin by the following definition.

Definition 3. For $f \in \mathcal{F}_q$ and $w,z \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$, we define the q-translation operators on $\mathcal{F}_q$, by

$$\tau_z f(w) := e_q(zD_q) f(w) = \sum_{n=0}^{\infty} D_q^n f(w)\frac{z^n}{[n]_q!}. \quad (14)$$

For $w,z \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$, the function e_q satisfies the following product formula:

$$\tau_z e_q(w) = e_q(z) e_q(w).$$

Proposition 4. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}_q$ *and* $z,w \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$. *Then*

$$\tau_z f(w) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k}_q w^{n-k} z^k.$$

Proof. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}_q$. From (14), we have

$$\tau_z f(w) = \sum_{n=0}^{\infty} \frac{D_q^n f(w)}{[n]_q!} z^n; \quad w,z \in D\left(o, \frac{1}{\sqrt{1-q}}\right).$$

But from (5), we have

$$D_q^n f(w) = \sum_{k=n}^{\infty} a_k \frac{[k]_q!}{[k-n]_q!} w^{k-n}.$$

Thus we obtain

$$\tau_z f(w) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \frac{[n]_q!}{[k]_q![n-k]_q!} w^{n-k} z^k$$
$$= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k}_q w^{n-k} z^k.$$

□

Definition 4. For $f \in \mathcal{F}_q$ and $w,z \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$, we define:

- The generalized multiplication operators on $\mathcal{F}_q$, by

$$M_z f(w) := e_q(zQ) f(w) = \sum_{n=0}^{\infty} Q^n f(w)\frac{z^n}{[n]_q!}.$$

- The generalized shift operators on $\mathcal{F}_q$, by

$$S_z f(w) := e_q(z\Lambda_q) f(w) = \sum_{n=0}^{\infty} \Lambda_q^n f(w)\frac{z^n}{[n]_q!}.$$

According to Theorem 3 we study the continuous property of the operators τ_z, M_z and S_z on $\mathcal{F}_q$.

Theorem 5. *If* $f \in \mathcal{F}_q$ *and* $z \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$, *then* $\tau_z f$, $M_z f$ *and* $S_z f$ *belong to* $\mathcal{F}_q$, *and we have*

1) $\|\tau_z f\|_{\mathcal{F}_q} \le e_q\left(\frac{|z|}{\sqrt{1-q}}\right)\|f\|_{\mathcal{F}_q}$,

2) $\|M_z f\|_{\mathcal{F}_q} \le e_q\left(\dfrac{|z|}{\sqrt{1-q}}\right)\|f\|_{\mathcal{F}_q}$,

3) $\|S_z f\|_{\mathcal{F}_q} \le e_q(|z|)\|f\|_{\mathcal{F}_q}$.

Proof. From (14) and Theorem 3 (2), we deduce

$$\|\tau_z f\|_{\mathcal{F}_q} \le \sum_{n=0}^{\infty}\|D_q^n f\|_{\mathcal{F}_q}\frac{|z|^n}{[n]_q!} \le \sum_{n=0}^{\infty}\frac{|z|^n}{(1-q)^{n/2}[n]_q!}\|f\|_{\mathcal{F}_q}.$$

Therefore,

$$\|\tau_z f\|_{\mathcal{F}_q} \le e_q\left(\frac{|z|}{\sqrt{1-q}}\right)\|f\|_{\mathcal{F}_q},$$

which gives the first inequality, and as in the same way we prove the second and the third inequalities of this theorem. □

From Proposition 3 we deduce the following results.

Proposition 5. *For all* $f, g \in \mathcal{F}_q$, *we have*

$$\langle M_z f, g\rangle_{\mathcal{F}_q} = \langle f, \tau_{\bar z} g\rangle_{\mathcal{F}_q},$$

$$\langle \tau_z f, g\rangle_{\mathcal{F}_q} = \langle f, M_{\bar z} g\rangle_{\mathcal{F}_q}.$$

We denote by R_z the following operator defined on $\mathcal{F}_q$ by

$$R_z := \tau_{\bar z} M_z - M_{\bar z}\tau_z = e_q(\bar z D_q)e_q(zQ) - e_q(\bar z Q)e_q(zD_q).$$

Then, we prove the following theorem.

Theorem 6. *For all* $f \in \mathcal{F}_q$, *we have*

$$\|M_z f\|_{\mathcal{F}_q}^2 = \|\tau_z f\|_{\mathcal{F}_q}^2 + \langle f, R_z f\rangle_{\mathcal{F}_q}.$$

Proof. From Proposition 5, we get

$$\|M_z f\|_{\mathcal{F}_q}^2 = \langle f, \tau_{\bar z} M_z f\rangle_{\mathcal{F}_q} = \langle f, (M_{\bar z}\tau_z + R_z) f\rangle_{\mathcal{F}_q} = \|\tau_z f\|_{\mathcal{F}_q}^2 + \langle f, R_z f\rangle_{\mathcal{F}_q}.$$

□

3.3. The Weyl Commutation Relations on $\mathcal{F}_q$

Let $a, b \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$. In this paragraph we establish Weyl commutation relations between the translation operators τ_a and the multiplication operators M_b. These relations are realized on the Fock space $\mathcal{F}_q$.

Lemma 3. *For* $a, b \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$, *we have*

1) $[D_q, Q^n] = [n]_q Q^{n-1}\Lambda_q, \quad n = 1, 2, \cdots$.

2) $[D_q, M_b] = bM_b\Lambda_q$.

Proof. 1) From Lemma 2, for $n = 1, 2, \cdots$, we deduce that

$$[D_q, Q^n] = \sum_{k=0}^{n-1}Q^k[D_q, Q]Q^{n-k-1} = \sum_{k=0}^{n-1}Q^k\Lambda_q Q^{n-k-1}.$$

Since

$$\Lambda_q Q = qQ\Lambda_q,$$

we get

$$[D_q, Q^n] = [n]_q Q^{n-1}\Lambda_q.$$

Which proves the first equality.

2) We have

$$[D_q, M_b] = \sum_{n=1}^{\infty}\frac{b^n}{[n]_q!}[D_q, Q^n].$$

Using (1), we obtain

$$[D_q, M_b] = \sum_{n=1}^{\infty}Q^{n-1}\Lambda_q\frac{b^n}{[n-1]_q!} = b\sum_{n=0}^{\infty}Q^n\Lambda_q\frac{b^n}{[n]_q!} = bM_b\Lambda_q.$$

□

Theorem 7. *For* $a, b \in D\left(o, \frac{1}{\sqrt{1-q}}\right)$, *we have*

$$\tau_a M_b = M_b\tau_a S_{ab}.$$

Proof. From Lemma 3 (2), we have

$$D_q M_b = M_b(D_q + b\Lambda_q).$$

Then, for $n = 0, 1, 2, \cdots$, we deduce

$$D_q^n M_b = M_b(D_q + b\Lambda_q)^n.$$

Multiplying by $\dfrac{a^n}{[n]_q!}$ and summing, we get

$$\tau_a M_b = M_b e_q(aD_q + ab\Lambda_q).$$

Since $D_q\Lambda_q = q\Lambda_q D_q$, from [5] we get

$$e_q(aD_q + ab\Lambda_q) = e_q(aD_q)e_q(ab\Lambda_q) = \tau_a S_{ab},$$

which completes the proof of the theorem. □

Remark 4. If $q \to 1^-$, we obtain the classical commutation relations [8]:

$$[D, Q] = I, \quad e^{aD}e^{bQ} = e^{ab}e^{bQ}e^{aD}; \quad a, b \in \mathbb{C}.$$

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