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Interior and Exterior Differential Systems for Lie Algebroids

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Abstract

A theorem of Maurer-Cartan type for Lie algebroids is presented. Suppose that any vector subbundle of a Lie algebroid is called interior differential system (*IDS*) for that Lie algebroid. A theorem of Frobenius type is obtained. Extending the classical notion of exterior diffential system (*EDS*) to Lie algebroids, a theorem of Cartan type is obtained.

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1. Introduction

Using the exterior differential calculus for Lie algebroids (See [1,2]) the structure equations of Maurer-Cartan type are established. Using the Cartan's moving frame method, there exists the following

Theorem (E. Cartan) *If* $N \in |Man_n|$ *is a Riemannian manifold and* $X_\alpha = X_\alpha^i \frac{\partial}{\partial x^i}$, $\alpha \in \overline{1,n}$ *is* n *ortonormal moving frame, then there exists a collection of 1-forms* Ω_β^α, $\alpha, \beta \in \overline{1,n}$ *uniquely defined by the requirements*

$$\Omega_\beta^\alpha = -\Omega_\alpha^\beta$$

and

$$d^F \Theta^\alpha = \Omega_\beta^\alpha \wedge \Theta^\beta, \alpha \in \overline{1,n}$$

where $\{\Theta^\alpha, \alpha \in \overline{1,n}\}$ *is the coframe.* (see [3], p. 151)

We know that an r-dimensional distribution on a manifold N is a mapping D defined on N, which assignees to each point x of N an r-dimensional linear subspace D_x of T_xN. A vector field X belongs to D if we have $X_x \in D_x$ for each $x \in N$. When this happens we write $X \in \Gamma(D)$.

The distribution D on a manifold N is said to be differentiable if for any $x \in N$ there exists r differentiable linearly independent vector fields $X_1, \cdots, X_r \in \Gamma(D)$ in a neighborhood of x. The distribution D is said to be involutive if for all vector fields $X, Y \in \Gamma(D)$ we have $[X, Y] \in \Gamma(D)$.

In the classical theory we have the following

Theorem (Frobenius) *The distribution* D *is involutive if and only if for each* $x \in N$ *there exists a neighborhood* U *and* $n-r$ *linearly independent* 1*-forms* $\Theta^{r+1}, \cdots, \Theta^n$ *on* U *which vanish on* D *and satisfy the condition*

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1,n}} \Omega_\beta^\alpha \wedge \Theta^\beta, \alpha \in \overline{r+1,n}$$

for suitable 1*-forms* Ω_β^α, $\alpha, \beta \in \overline{1,n}$. (see [4], p. 58)

Extending the notion of distribution we obtain the definition of an *IDS* of a Lie algebroid. A characterization of the ivolutivity of an *IDS* in a result of Frobenius type is presented in Theorem 4.7.

This paper studies the intersection between the geometry of Lie algebroids and some aspects of *EDS*. In the classical sense, an *EDS* is a pair (M, I) consisting of a smooth manifold M and a homogeneous, differentially closed ideal I in the algebra of smooth differential forms on M. (see [5,6]) Using the notion of *EDS* of an arbitrary Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_N))$ we obtained a new result of Cartan type in the Theorem 5.1. In the particular case of standard Lie algebroid $((TM,\tau_M,M),[,]_{TM},(Id_{TM},Id_M))$ there are obtained similar results those for distributions.

We know that a submanifold S of N is said to be integral manifold for the distribution D if for every point $x \in N$, D_x coincides with T_xS. The distribution D is said to be integrable if for each point $x \in N$ there exists an integral manifold of D containing x. As a distribution D is involutive if and only if it is integrable, then the study of the integral manifolds of an *IDS* or *EDS* is a new direction by research.

2. Preliminaries

In general, if C is a category, then we denote $|C|$ the class of objects and for any $A, B \in |C|$, we denote $C(A,B)$ the set of morphisms of A source and B target. Let $LieA\lg$, Mod, and B^v be the category of Lie algebras, modules and vector bundles respectively.

We know that if $(E,\pi,M) \in |B^v|$,
$\Gamma(E,\pi,M) = \{u \in Man(M,E) : u \circ \pi = Id_M\}$ and $F(M) = Man(M,\mathrm{R})$, then $(\Gamma(E,\pi,M),+,\cdot)$ is a $F(M)$-module.

We know that a Lie algebroid is a vector bundle $(F,\nu,N) \in |B^v|$ so that there exists

$$(\rho, Id_N) \in B^v((F,\nu,N),(TN,\tau_N,N))$$

and also an operation

$$\begin{array}{ccc}\Gamma(F,\nu,N)\times\Gamma(F,\nu,N) & \xrightarrow{[,]_F} & \Gamma(F,\nu,N)\\ (u,v) & \mapsto & [u,v]_F\end{array}$$

with the following properties:

LA_1. the equality holds good

$$[u, f\cdot v]_F = f[u,v]_F + \Gamma(\rho, Id_N)(u) f \cdot v$$

for all $u, v \in \Gamma(F,\nu,N)$ and $f \in F(N)$,

LA_2. the 4-tuple $(\Gamma(F,\nu,N),+,\cdot,[,]_F)$ is a Lie $F(N)$-algebra,

LA_3. the Mod-morphism $\Gamma(\rho, Id_N)$ is a $LieA\lg$-morphism of $(\Gamma(F,\nu,N),+,\cdot,[,]_F)$ source and $(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$ target.

Let $((F,\nu,N),[,]_F,(\rho,Id_N))$ be a Lie algebroid.

Locally, for any $\alpha,\beta \in \overline{1,p}$, we set $[t_\alpha, t_\beta]_F = L_{\alpha\beta}^\gamma t_\gamma$. We easily obtain that $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$, for any $\alpha,\beta,\gamma \in \overline{1,p}$.

The real local functions $L_{\alpha\beta}^\gamma, \alpha,\beta,\gamma \in \overline{1,p}$ are called the *structure functions*.

We assume that (F,ν,N) is a vector bundle with type fibre the real vector space $(R^p,+,\cdot)$ and structure group a Lie subgroup of $(GL(p,R),\cdot)$. We denote (x^i, z^α) the canonical local coordinates on (F,ν,N), where $i \in \overline{1,n}$ and $\alpha \in \overline{1,p}$.

Consider $(x^i, z^\alpha) \to (x^{i'}, z^{\alpha'})$ a change of coordinates on (F,ν,N). Then the coordinates z^α change to $z^{\alpha'}$ according to the rule:

$$z^{\alpha'} = \Lambda_\alpha^{\alpha'} z^\alpha \tag{2.1}$$

If $z^\alpha t_\alpha \in \Gamma(F,\nu,N)$ is arbitrary, then

$$\left[\Gamma(\rho, Id_N)(z^\alpha t_\alpha) f\right](x) = \left(\rho_\alpha^i z^\alpha \frac{\partial f}{\partial x^i}\right)(x) \tag{2.2}$$

for any $f \in F(N)$ and $x \in N$.

The coefficients ρ_α^i change to $\rho_{\alpha'}^{i'}$ according to the rule:

$$\rho_{\alpha'}^{i'} = \Lambda_{\alpha'}^\alpha \rho_\alpha^i \frac{\partial x^{i'}}{\partial x^i}, \tag{2.3}$$

where $\|\Lambda_{\alpha'}^\alpha\| = \|\Lambda_\alpha^{\alpha'}\|^{-1}$.

The following equalities hold good:

$$\left(\rho_\alpha^i \frac{\partial}{\partial x^i}\right)(f) = \left(\rho_\alpha^i \frac{\partial f}{\partial x^i}\right), \forall f \in F(N) \tag{2.4}$$

and

$$L_{\alpha\beta}^\gamma \cdot \rho_\gamma^k = \rho_\alpha^i \frac{\partial \rho_\beta^k}{\partial x^i} - \rho_\beta^j \frac{\partial \rho_\alpha^k}{\partial x^j}. \tag{2.5}$$

3. Interior Differential Systems

Let $((F,\nu,N),[,]_F,(\rho,Id_N))$ be a Lie algebroid.

Definition 3.1 Any vector subbundle (E,π,M) of the vector bundle (F,ν,N) will be called *interior differential system* (*IDS*) *of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_N))$.

Remark 3.1 If (E,π,N) is an *IDS* of the Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_N))$ then we obtain a vector subbundle (E^0,π^0,N) of the dual vector bundle $(\overset{*}{F},\overset{*}{\nu},N)$ so that

$$\Gamma(E^0,\pi^0,N) = \left\{\Omega \in \Gamma(\overset{*}{F},\overset{*}{\nu},N) : \Omega(S) = 0, \forall S \in \Gamma(E,\pi,N)\right\}.$$

The vector subbundle (E^0,π^0,N) will be called the *annihilator vector subbundle of the IDS* (E,π,M).

Proposition 3.1 *If* (E,π,N) *is an IDS of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_N))$ *so that* $\Gamma(E,\pi,N) = \langle S_1,\cdots,S_r\rangle$, *then it exists* $\Theta^{r+1},\cdots,\Theta^p \in \Gamma(\overset{*}{F},\overset{*}{\nu},N)$ *linearly independent so that*

$$\Gamma(E^0,\pi^0,N) = \langle \Theta^{r+1},\cdots,\Theta^p\rangle.$$

Definition 3.2 The *IDS* (E,π,N) of the Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_N))$ will be called *involutive* if $[S,T]_F \in \Gamma(E,\pi,N)$, for any $S,T \in \Gamma(E,\pi,N)$.

Proposition 3.2 *If* (E,π,N) *is an IDS of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_N))$ *and* $\{S_1,\cdots,S_r\}$, *is a base of the* $F(N)$ *-submodule* $(\Gamma(E,\pi,N),+,\cdot)$, *then* (E,π,N) *is involutive if and only if*

$$[S_a,S_b]_F \in \Gamma(E,\pi,N), \text{ for any } a,b \in \overline{1,r}.$$

4. Exterior Differential Calculus

Let $\left(\left(F,\nu,N\right),\left[,\right]_F,\left(\rho,Id_N\right)\right)$ be a Lie algebroid. We denote $\Lambda^q\left(F,\nu,N\right)$ the set of differential forms of degree q. If $\Lambda\left(F,\nu,N\right)=\underset{q\geq 0}{\oplus}\Lambda^q\left(F,\nu,N\right)$, then we obtain the exterior differential algebra $\left(\Lambda\left(F,\nu,N\right),+,\cdot,\wedge\right)$.

Definition 4.1 For any $z\in\Gamma\left(F,\nu,N\right)$ the application

$$\Lambda\left(F,\nu,N\right)\xrightarrow{L_z}\Lambda\left(F,\nu,N\right),$$

defined by

$$L_z\left(f\right)=\left[\Gamma\left(\rho,Id_N\right)z\right]\left(f\right),$$

for any $f\in F\left(N\right)$ and

$$L_z\omega\left(z_1,\cdots,z_q\right)=\left[\Gamma\left(\rho,Id_N\right)z\right]\left(\omega\left(z_1,\cdots,z_q\right)\right)-\sum_{i=1}^{q}\omega\left(z_1,\cdots,\left[z,z_i\right]_F,\cdots,z_q\right),$$

for any $\omega\in\Lambda^q\left(F,\nu,N\right)$ and $z_1,\cdots,z_q\in\Gamma\left(F,\nu,N\right)$, is called the *covariant Lie derivative with respect to the section* z.

Theorem 4.1 *If* $z\in\Gamma\left(F,\nu,N\right)$, $\omega\in\Lambda^q\left(F,\nu,N\right)$ *and* $\theta\in\Lambda^r\left(F,\nu,N\right)$, *then*

$$L_z\left(\omega\wedge\theta\right)=L_z\omega\wedge\theta+\omega\wedge L_z\theta. \quad (4.1)$$

Definition 4.2 If $z\in\Gamma\left(F,\nu,N\right)$, then the application

$$\begin{array}{ccc}\Lambda\left(F,\nu,N\right) & \xrightarrow{i_z} & \Lambda\left(F,\nu,N\right)\\ \omega\in\Lambda^q\left(F,\nu,N\right) & \mapsto & i_z\omega\in\Lambda^{q-1}\left(F,\nu,N\right)\end{array}$$

defined by $i_z\left(f\right)=0$, for any $f\in F\left(N\right)$ and

$$i_z\omega\left(z_2,\cdots,z_q\right)=\omega\left(z,z_2,\cdots,z_q\right),$$

for any $z_2,\cdots,z_q\in\Gamma\left(F,\nu,N\right)$, is called the *interior product associated to the section* z.

Theorem 4.2 *If* $z\in\Gamma\left(F,\nu,N\right)$, *then for any* $\omega\in\Lambda^q\left(F,\nu,N\right)$ *and* $\theta\in\Lambda^r\left(F,\nu,N\right)$ *we obtain the equality*

$$i_z\left(\omega\wedge\theta\right)=i_z\omega\wedge\theta+\left(-1\right)^q\omega\wedge i_z\theta. \quad (4.2)$$

Theorem 4.3 *For any* $z,v\in\Gamma\left(F,\nu,N\right)$ *we obtain*

$$L_v\circ i_z-i_z\circ L_z=i_{\left[v,z\right]_F}. \quad (4.3)$$

Theorem 4.4 *The application*

$$\begin{array}{ccc}\Lambda^q\left(F,\nu,N\right) & \xrightarrow{d^F} & \Lambda^{q+1}\left(F,\nu,N\right)\\ \omega & \mapsto & d^F\omega\end{array}$$

defined by

$$d^F f\left(z\right)=L_z\left(f\right),$$

for any $z\in\Gamma\left(F,\nu,N\right)$ and

$$d^F\omega\left(z_0,z_1,\cdots,z_q\right)=\sum_{i=1}^{q}\left(-1\right)^i\left[\Gamma\left(\rho,Id_N\right)z_i\right]\left(\omega\left(z_0,z_1,\cdots,\hat{z}_i,\cdots,z_q\right)\right)+\sum_{i<j}\left(-1\right)^{i+j}\omega\left(\left[z_i,z_j\right]_F,z_0,z_1,\cdots,\hat{z}_i,\cdots,\hat{z}_j,\cdots,z_q\right)$$

for any $z_0,z_1,\cdots,z_q\in\Gamma\left(F,\nu,N\right)$, *is unique having the following property*:

$$L_z=d^F\circ i_z-i_z\circ d^F,\forall z\in\Gamma\left(F,\nu,N\right). \quad (4.4)$$

This application is called the *exterior differentiation operator of the exterior differential algebra of the Lie algebroid* $\left(\left(F,\nu,N\right),\left[,\right]_F,\left(\rho,Id_N\right)\right)$.

Theorem 4.5 *The exterior differentiation operator* d^F *given by the previous theorem has the following properties:*

1) *For any* $\omega\in\Lambda^q\left(F,\nu,N\right)$ *and* $\theta\in\Lambda^r\left(F,\nu,N\right)$ *we obtain*

$$d^F\left(\omega\wedge\theta\right)=d^F\omega\wedge\theta+\left(-1\right)^q\omega\wedge d^F\theta. \quad (4.5)$$

2) *For any* $z\in\Gamma\left(F,\nu,N\right)$ *we obtain*

$$L_z\circ d^F=d^F\circ L_z. \quad (4.6)$$

3) $d^F\circ d^F=0$.

Theorem 4.6 (of Maurer-Cartan type) *If* $\left(\left(F,\nu,N\right),\left[,\right]_F,\left(\rho,Id_N\right)\right)$ *is a Lie algebroid and* d^F *is the exterior differentiation operator of the exterior differential* $F\left(N\right)$ *-algebra* $\left(\Lambda\left(F,\nu,N\right),+,\cdot,\wedge\right)$, *then we obtain the structure equations of Maurer-Cartan type*

$$d^F t^\alpha=-\frac{1}{2}L_{\beta\gamma}^{\alpha}t^\beta\wedge t^\gamma,\alpha\in\overline{1,p} \quad (\mathrm{MC}_1)$$

and

$$d^F x^i=\rho_\alpha^i t^\alpha,i\in\overline{1,n}. \quad (\mathrm{MC}_2)$$

where $\left\{t^\alpha,\alpha\in\overline{1,p}\right\}$ *is the coframe of the vector bundle* $\left(F,\nu,N\right)$.

These equations will be called *the structure equations of Maurer-Cartan type associated to the Lie algebroid* $\left(\left(F,\nu,N\right),\left[,\right]_F,\left(\rho,Id_N\right)\right)$.

Proof. Let $\alpha\in\overline{1,p}$ be arbitrary. Since

$$d^F t^\alpha\left(t_\beta,t_\gamma\right)=-L_{\beta\gamma}^{\alpha},\ \forall\beta,\gamma\in\overline{1,p}$$

it results that

$$d^F t^\alpha=-\sum_{\beta<\gamma}L_{\beta\gamma}^{\alpha}t^\beta\wedge t^\gamma. \quad (1)$$

Since $L_{\alpha\beta}^{\gamma}=-L_{\beta\alpha}^{\gamma}$ and $t^\beta\wedge t^\gamma=-t^\gamma\wedge t^\beta$, for any $\beta,\gamma\in\overline{1,p}$, it results that

$$\sum_{\beta<\gamma}L_{\beta\gamma}^{\alpha}t^\beta\wedge t^\gamma=\frac{1}{2}L_{\beta\gamma}^{\alpha}t^\beta\wedge t^\gamma,\alpha\in\overline{1,p}. \quad (2)$$

Using the equalities (1) and (2) it results the structure equation (MC_1).

Let $i \in \overline{1,n}$ be arbitrary. Since

$$d^F x^i(t_\alpha) = \rho_\alpha^i, \forall \alpha \in \overline{1,p}$$

it results the structure equation (MC_2). *q.e.d.*

Remark 4.1 In the particular case of the standard Lie algebroid $\left((TN,\tau_N,N),[,]_{TN},(Id_{TN},Id_N)\right)$ we obtain

$$d^{TN} x^i = dx^i, i \in \overline{1,n}, \qquad (MC_2)'$$

where $\{dx^i, i \in \overline{1,n}\}$ is the coframe of the vector bundle (TN,τ_N,N).

As $d^{TN} \circ d^{TN} = 0.$ and $L^i_{jk} = 0$, for all $i,j,k \in \overline{1,n}$ we obtain

$$d^F(dx^i) = 0 = -\frac{1}{2} L^i_{jk} dx^j \wedge dx^k, i \in \overline{1,n}. \qquad (MC_1)'$$

These equations are the *structure equations of Maurer-Cartan type associated to the standard Lie algebroid* $\left((TN,\tau_N,N),[,]_{TN},(Id_{TN},Id_N)\right)$.

Theorem 4.7 (of Frobenius type) *Let* (E,π,N) *be an IDS of the Lie algebroid* $\left((F,\nu,N),[,]_F,(\rho,Id_N)\right)$. *If* $\{\Theta^{r+1},\cdots,\Theta^p\}$ *is a base of* $F(N)-$*submodule* $\left(\Gamma(E^0,\pi^0,N),+,\cdot\right)$, *then the IDS* (E,π,N) *is involutive if and only if it exists* $\Omega^\alpha_\beta \in \Lambda^1(F,\nu,N)$, $\alpha,\beta \in \overline{r+1,p}$ *so that*

$$d^F\Theta^\alpha = \sum_{\beta \in \overline{r+1,p}} \Omega^\alpha_\beta \wedge \Theta^\beta, \alpha \in \overline{r+1,p}.$$

Proof. Let $\{S_1,\cdots,S_r\}$ is a base of the $F(N)$-submodule $\left(\Gamma(E,\pi,N),+,\cdot\right)$.

Let $\{S_{r+1},\cdots,S_p\} \subseteq \Gamma(F,\nu,N)$ so that $\{S_1,\cdots,S_r,S_{r+1},\cdots,S_p\}$ is a base of the $F(N)$-module $\left(\Gamma(F,\nu,N),+,\cdot\right)$.

Let $\{\Theta^1,\cdots,\Theta^r\} \subseteq \Gamma\left(\overset{*}{F},\overset{*}{\nu},N\right)$ so that $\{\Theta^1,\cdots,\Theta^r,\Theta^{r+1},\cdots,\Theta^p\}$ is a base of the $F(N)$-module $\left(\Gamma\left(\overset{*}{F},\overset{*}{\nu},N\right),+,\cdot\right)$.

For any $a,b \in \overline{1,r}$ and $\alpha,\beta \in \overline{r+1,p}$ we have the equalities:

$$\Theta^a(S_b) = \delta^a_b, \quad \Theta^\alpha(S_b) = 0,$$
$$\Theta^a(S_\beta) = 0, \quad \Theta^\alpha(S_\beta) = \delta^\alpha_\beta.$$

We remark that the set of the 2-forms

$$\left\{\Theta^a \wedge \Theta^b, \Theta^a \wedge \Theta^\beta, \Theta^\alpha \wedge \Theta^b, \Theta^\alpha \wedge \Theta^\beta; \; a,b \in \overline{1,r} \wedge \alpha,\beta \in \overline{r+1,p}\right\}$$

is a base of the $F(N)$-module $\left(\Lambda^2(F,\nu,N),+,\cdot\right)$.

Therefore, we have

$$d^F\Theta^\alpha = \sum_{b<c} A^\alpha_{bc}\Theta^b \wedge \Theta^c + \sum_{b,\gamma} B^\alpha_{b\gamma}\Theta^b \wedge \Theta^\gamma + \sum_{\beta<\gamma} C^\alpha_{\beta\gamma}\Theta^\beta \wedge \Theta^\gamma, \qquad (1)$$

where, $A^\alpha_{bc}, B^\alpha_{b\gamma}$ and $C^\alpha_{\beta\gamma}$, $a,b,c \in \overline{1,r}$, $\alpha,\beta,\gamma \in \overline{r+1,p}$ are real local functions so that $A^\alpha_{bc} = -A^\alpha_{cb}$ and $C^\alpha_{\beta\gamma} = -C^\alpha_{\gamma\beta}$.

Using the formula

$$d^F\Theta^\alpha(S_b,S_c) = \Gamma(\rho,Id_N)(S_b)\left(\Theta^\alpha(S_c)\right) - \Gamma(\rho,Id_N)(S_c)\left(\Theta^\alpha(S_b)\right) - \Theta^\alpha\left([S_b,S_c]_F\right) \qquad (2)$$

we obtain that

$$A^\alpha_{bc} = -\Theta^\alpha\left([S_b,S_c]_F\right), \qquad (3)$$

for any $b,c \in \overline{1,r}$ and $\alpha \in \overline{r+1,p}$.

We admit that (E,π,N) is an involutive *IDS* of the Lie algebroid $\left((F,\nu,N),[,]_F,(\rho,Id_N)\right)$. As $[S_b,S_c]_F \in \Gamma(E,\pi,N)$, for any $b,c \in \overline{1,r}$, it results that $\Theta^\alpha\left([S_b,S_c]_F\right) = 0$ for any $b,c \in \overline{1,r}$ and $\alpha \in \overline{r+1,p}$.

Therefore, for any $b,c \in \overline{1,r}$ and $\alpha \in \overline{r+1,p}$, we obtain $A^\alpha_{bc} = 0$ and

$$d^F\Theta^\alpha = \sum_{b,\gamma} B^\alpha_{b\gamma}\Theta^b \wedge \Theta^\gamma + \frac{1}{2}C^\alpha_{\beta\gamma}\Theta^\beta \wedge \Theta^\gamma = \left(B^\alpha_{b\gamma}\Theta^b + \frac{1}{2}C^\alpha_{\beta\gamma}\Theta^\beta\right) \wedge \Theta^\gamma$$

As $\Omega^\alpha_\gamma = B^\alpha_{b\gamma}\Theta^b + \frac{1}{2}C^\alpha_{\beta\gamma}\Theta^\beta \in \Lambda^1(F,\nu,N)$, for any $\beta,\gamma \in \overline{r+1,p}$ it results the first implication.

Conversely, we admit that it exists $\Omega^\alpha_\beta \in \Lambda^1(F,\nu,N)$, $\alpha,\beta \in \overline{r+1,p}$ so that

$$d^F\Theta^\alpha = \sum_{\beta \in \overline{r+1,p}} \Omega^\alpha_\beta \wedge \Theta^\beta, \qquad (4)$$

for any $\alpha \in \overline{r+1,p}$.

Using the affirmations (1), (2) and (4) we obtain that $A^\alpha_{bc} = 0$, for any $b,c \in \overline{1,r}$ and $\alpha \in \overline{r+1,p}$.

Using the affirmation (3), we obtain $\Theta^\alpha\left([S_b,S_c]_F\right) = 0$ for any $b,c \in \overline{1,r}$ and $\alpha \in \overline{r+1,p}$.

Therefore, we have $[S_b,S_c]_F \in \Gamma(E,\pi,N)$, for any $b,c \in \overline{1,r}$. Using the *Proposition 3.2*, we obtain the second implication. *q.e.d.*

5. Exterior Differential Systems

Let $\left((F,\nu,N),[,]_F,(\rho,Id_N)\right)$ be a Lie algebroid.

Definition 5.1 Any ideal $(I,+,\cdot)$ of the exterior differential algebra of the Lie algebroid

$((F,\nu,N),[,]_F,(\rho,Id_N))$ closed under differentiation operator d^F, namely $d^F I \subseteq I$, is called *differential ideal of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_N))$.

Definition 5.2 Let $(I,+,\cdot)$ be a differential ideal of the Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_N))$. If it exists an *IDS* (E,π,N) so that for all $k \in N^*$ and $\omega \in I \cap \Lambda^k(F,\nu,N)$ we have $\omega(u_1,\cdots,u_k)=0$, for any $u_1,\cdots,u_k \in \Gamma(E,\pi,N)$, then we will say that $(I,+,\cdot)$ is an *exterior differential system* (*EDS*) *of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_N))$.

Theorem 5.1 (of Cartan type) *The IDS* (E,π,N) *of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_N))$ *is involutive, if and only if the ideal generated by the* $F(N)$*-submodule* $(\Gamma(E^0,\pi^0,N),+,\cdot)$ *is an EDS of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_N))$.

Proof. Let (E,π,N) be an involutive *IDS* of the Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_N))$. Let $\{\Theta^{r+1},\cdots,\Theta^p\}$ be a base of the $F(N)$-submodule $(\Gamma(E^0,\pi^0,N),+,\cdot)$. We know that

$$I(\Gamma(E^0,\pi^0,N)) = \bigcup_{q\in N}\{\Omega_\alpha \wedge \Theta^\alpha;\{\Omega_{r+1},\cdots,\Omega_p\} \subseteq \Lambda^q(F,\nu,N)\}.$$

Let $q \in N$ and $\{\Omega_{r+1},\cdots,\Omega_p\} \subseteq \Lambda^q(F,\nu,N)$ be arbitrary.

Using the Theorems 4.5 and 4.7 we obtain

$$d^F(\Omega_\alpha \wedge \Theta^\alpha) = d^F\Omega_\alpha \wedge \Theta^\alpha + (-1)^q \Omega_\beta \wedge d^F\Theta^\beta = \left(d^F\Omega_\alpha + (-1)^{q+1}\Omega_\beta \wedge \Omega_\alpha^\beta\right)\wedge\Theta^\alpha.$$

As $d^F\Omega_\alpha + (-1)^{q+1}\Omega_\beta \wedge \Omega_\alpha^\beta \in \Lambda^{q+2}(F,\nu,N)$ it results that

$$d^F(\Omega_\alpha \wedge \Theta^\alpha) \in I(\Gamma(E^0,\pi^0,N)).$$

Therefore, $d^F I(\Gamma(E^0,\pi^0,N)) \subseteq I(\Gamma(E^0,\pi^0,N))$.

Conversely, let (E,π,N) be an *IDS* of the Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_N))$ so that the $F(N)$-submodule $(I(\Gamma(E^0,\pi^0,N)),+,\cdot)$ is an *EDS* of the Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_N))$.

Let $\{\Theta^{r+1},\cdots,\Theta^p\}$ be a base of the $F(N)$-submodule $(\Gamma(E^0,\pi^0,N),+,\cdot)$.

As $d^F I(\Gamma(E^0,\pi^0,N)) \subseteq I(\Gamma(E^0,\pi^0,N))$ it results that it exists $\Omega_\beta^\alpha \in \Lambda^1(F,\nu,N)$, $\alpha,\beta \in \overline{r+1,p}$ so that

$$d^F\Theta^\alpha = \sum_{\beta\in\overline{r+1,p}} \Omega_\beta^\alpha \wedge \Theta^\beta \in I(\Gamma(E^0,\pi^0,N)).$$

Using the Theorem 4.7 there results that (E,π,N) is an involutive *IDS*. *q.e.d.*

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6. References

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