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A Closed Form Solution to a Special Normal Form of Riccati Equation

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Abstract

We present the general solution to the Riccati differential equation, $\frac{\mathrm{d}w}{\mathrm{d}z} = z^n + w^2$, for arbitrary *real* number *n*.

Keywords: Riccati Differential Equation, Bessel Functions

1. Introduction

We consider a special normal form of Riccati equation, $w' = z^n + w^2$, where $w' = \mathrm{d}w/\mathrm{d}z$. We prove, for any *real* number *n*, the equation has a closed analytic solution. We show, the general solution for $n \neq -2$ is a product of $\sqrt{z^n}$ and a combination of Bessel functions of various *n*—dependent orders and arguments. We discuss the variable transformations leading to this general result. Introducing another set of variable transformations conducive to a closed analytic solution, we by-pass the singular behavior of the Bessel functions for *n* = –2. We show, the general solution in the limit of $n \to -2$ is identical to the solution of *n* = –2.

2. Procedure

Riccati equation is given by,

$$u' = a(z) + b(z)u + c(z)u^2, \tag{1}$$

where $a(z)$, $b(z)$ and $c(z)$ are analytic functions of *z*. It is known, for $c(z) \neq 0$,

$$u = \frac{1}{c(z)}w - \frac{b(z)}{2c(z)} - \frac{c'(z)}{2c^2(z)}$$

transforms (1) into the normal form

$$w' = A(z) + w^2, \tag{2}$$

where

$$A(z) = ac - \frac{b^2}{4} + \frac{b'}{2} - \frac{3}{4}\left(\frac{c'}{c}\right)^2 - \frac{b}{2}\frac{c'}{c} + \frac{1}{2}\frac{c''}{c}.$$

It is the objective of this paper to solve (2) for a special case where $A(z) = z^n$. We prove the solution for any *real* value of *n* is analytic.

We begin with Euler variable transformation, [1, p. 112], namely $-w(z) = \mathrm{d}/\mathrm{d}z \ln y(z)$. This linearizes (2)

$$y'' + z^n y = 0, \tag{3}$$

Multiplying both sides of (3) by z^2 we compare the result, $z^2 y'' + z^{n+2} y = 0$ vs. $z^2 y'' + \left(\alpha^2 \beta^2 z^{2\beta} + 1/4 - v^2 \beta^2\right) y = 0$ of [**1**, p. 206] and deduce the following identities

$$\beta = (n+2)/2, v = 1/(2\beta) = 1/(n+2),$$
$$\text{and } \alpha = 1/\beta = 2/(n+2)$$

Furthermore, according to the last reference, the solution of the given equation is, $y = \sqrt{z} f\left(\alpha z^{\beta}\right)$ where $f(\xi)$ is a solution of the Bessel equation of order *v*. Therefore, we conclude the solution to, $z^2 y'' + z^{n+2} y = 0$ is

$$y = \sqrt{z}\left[c_1 J_v(\xi) + c_2 Y_v(\xi)\right], \tag{4}$$

where J_v and Y_v are the Bessel functions of the first and the second kind of order v, with c_1 and c_2 being two arbitrary constants. Substituting (4) in Euler transformation and applying the chain differentiation, $\mathrm{d}/\mathrm{d}z = z^{\beta-1}\mathrm{d}/\mathrm{d}\xi$, we deduce

$$-w = \frac{c_1 J_v(\xi) + c_2 Y_v(\xi) + 2z^{\beta}\left[c_1 J_v'(\xi) + c_2 Y_v'(\xi)\right]}{2\sqrt{z}\left[c_1 J_v(\xi) + c_2 Y_v(\xi)\right]}, \tag{5}$$

with the prime notation indicating the derivative of the Bessel functions with respect to variable ξ. Furthermore, applying two sets of recurrence relationships [2, p. 361],

$$\frac{1}{2}\left[\begin{Bmatrix} J(\xi) \\ Y(\xi) \end{Bmatrix}_{v-1} - \begin{Bmatrix} J(\xi) \\ Y(\xi) \end{Bmatrix}_{v+1}\right] = \begin{Bmatrix} J'(\xi) \\ Y'(\xi) \end{Bmatrix}_{v}, \frac{\xi}{2v}\left[\begin{Bmatrix} J(\xi) \\ Y(\xi) \end{Bmatrix}_{v-1} + \begin{Bmatrix} J(\xi) \\ Y(\xi) \end{Bmatrix}_{v+1}\right] = \begin{Bmatrix} J(\xi) \\ Y(\xi) \end{Bmatrix}_{v},$$

simplifies (5)

$$w = -\sqrt{z^n}\,\frac{c_1 J_{-\frac{n+1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right) + c_2 Y_{-\frac{n+1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right)}{c_1 J_{\frac{1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right) + c_2 Y_{\frac{1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right)}, \quad (6)$$

And, (6) simplifies to its final form

$$w = -\sqrt{z^n}\,\frac{C J_{-\frac{n+1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right) + Y_{-\frac{n+1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right)}{C J_{\frac{1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right) + Y_{\frac{1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right)}, \quad (7)$$

With $C = c_1/c_2$.

We observe that solution (2) is a one-parameter family function.

The $n = -2$ is the pole of the order and the argument of the Bessel functions and needs a special consideration. For $n = -2$ Equation (2) takes the form

$$w' = \frac{1}{z^2} + w^2, \quad (8)$$

By inspection, $w_1 = \lambda/z^2$ solves (8). Substituting w_1 in (8) gives $\lambda^2 + \lambda + 1 = 0$. We select $\lambda = 1/2\left(-1 + \sqrt{3}i\right)$ to further the analysis. The standard variable transformation [1, p. 50]

$$w = w_1 + \frac{1}{v(z)}, \quad (9)$$

gives the general solution. Substituting (9) in (8) yields,

$$v' + 2\frac{\lambda_1}{z} v + 1 = 0, \quad (10)$$

The solution of (10) is, $v = c/z^{2\lambda_1} - z/(2\lambda_1 + 1)$, with C being a constant. The solution to (8) yields,

$$w = \frac{1}{z}\left[\frac{1}{2}\left(-1 + \sqrt{3}i\right) + \frac{1}{Cz^{-\sqrt{3}i} - \frac{1}{\sqrt{3}i}}\right], \quad (11)$$

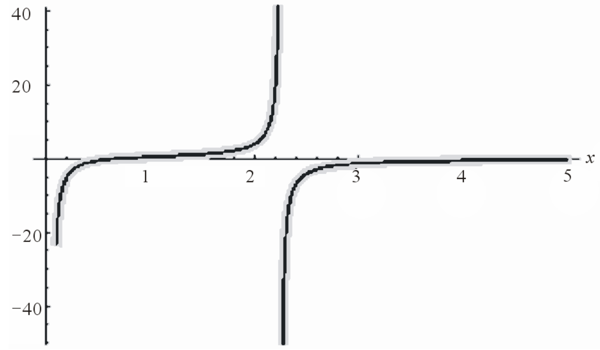

Figure 1. The solid curve is the graph of the solution given in (12) for c_1 = 1. The gray curve is the graph of the solution given in (7) for *n* = –1.99 and *C* = 0.85.

Applying the identities, $z = e^{\ln z}$ and $\tan z = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$ and suitably selecting the value of $C = -\frac{i}{\sqrt{3}} e^{-\sqrt{3}c_1 i}$, (11) becomes

$$w = \frac{1}{2z}\left\{-1 - \sqrt{3}\cot\left[\frac{\sqrt{3}}{2}\left(\ln z + c_1\right)\right]\right\}, \quad (12)$$

We show graphically, see **Figure 1**, that (7) in the limit of $n \to -2$ is identical to (12).

3. A comment and an Acknowledgement

By applying multiple step variable transformations (not reported in this paper) the author devised a method of solving (2) resulting in (7). The author appreciates the in depth discussion with Prof. Rostamian.

4. References

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