Advances in Pure Mathematics, **2011, 1, 286-294**
doi:10.4236/apm.2011.15052 Published Online September 2011 (http://www.SciRP.org/journal/apm)

Squares from $D(-4)$ and $D(20)$ Triples

Zvonko Čerin
Kopernikova 7, Zagreb, Croatia
E-mail: cerin@math.hr
Received May 5, 2011; revised May 20, 2011; accepted June 1, 2011

Abstract

We study the eight infinite sequences of triples of natural numbers $A=(F_{2n+1},4F_{2n+3},F_{2n+7})$, $B=(F_{2n+1},4F_{2n+5},F_{2n+7})$, $C=(F_{2n+1},5F_{2n+1},4F_{2n+3})$, $D=(F_{2n+3},4F_{2n+1},5F_{2n+3})$ and $\mathcal{A}=(L_{2n+1},4L_{2n+3},L_{2n+7})$, $\mathcal{B}=(L_{2n+1},4L_{2n+5},L_{2n+7})$, $\mathcal{C}=(L_{2n+1},5L_{2n+1},4L_{2n+3})$, $\mathcal{D}=(L_{2n+3},4L_{2n+1},5L_{2n+3})$. The sequences A,B,C and D are built from the Fibonacci numbers F_n while the sequences $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ from the Lucas numbers L_n. Each triple in the sequences A,B,C and D has the property $D(-4)$ (*i. e.*, adding -4 to the product of any two different components of them is a square). Similarly, each triple in the sequences $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ has the property $D(20)$. We show some interesting properties of these sequences that give various methods how to get squares from them.

Keywords: Fibonacci Numbers, Lucas Numbers, Square, Symmetric Sum, Alternating Sum, Product, Component

1. Introduction

For integers a, b and c, let us write $a\overset{b}{\sim}c$ provided $a+b=c^2$. For the triples $X=(a,b,c)$, $Y=(d,e,f)$ and $\tilde{X}=(\tilde{a},\tilde{b},\tilde{c})$ the notation $X\overset{Y}{\sim}\tilde{X}$ means that $bc\overset{d}{\sim}\tilde{a}$, $ca\overset{e}{\sim}\tilde{b}$ and $ab\overset{f}{\sim}\tilde{c}$. When $Y=(k,k,k)$, let us write $X\overset{k}{\sim}\tilde{X}$ for $X\overset{Y}{\sim}\tilde{X}$. Hence, X is the $D(k)$ triple (see [1]) if and only if there is a triple $\tilde{X}$ such that $X\overset{k}{\sim}\tilde{X}$.

We now construct the infinite sequences A, B, C and D of the $D(-4)$-triples and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of the $D(20)$ -triples. They are $A=(F_{2n+1},4F_{2n+3},F_{2n+7})$, $B=(F_{2n+1},4F_{2n+5},F_{2n+7})$, $C=(F_{2n+1},5F_{2n+1},4F_{2n+3})$, $D=(F_{2n+3},4F_{2n+1},5F_{2n+3})$ and $\mathcal{A}=(L_{2n+1},4L_{2n+3},L_{2n+7})$, $\mathcal{B}=(L_{2n+1},4L_{2n+5},L_{2n+7})$, $\mathcal{C}=(L_{2n+1},5L_{2n+1},4L_{2n+3})$, $\mathcal{D}=(L_{2n+3},4L_{2n+1},5L_{2n+3})$, where the Fibonacci and Lucas sequences of natural numbers F_n and L_n are defined by the recurrence relations $F_0=0$, $F_1=1$, $F_n=F_{n-1}+F_{n-2}$ for $n\geq 2$ and $L_0=2$, $L_1=1$, $L_n=L_{n-1}+L_{n-2}$ for $n\geq 2$.

The numbers F_k make the integer sequence $\mathcal{A}000045$ from [2] while the numbers L_k make $\mathcal{A}000032$. For an integer k, let us use π_k, ρ_k, $\mathfrak{p}_k$ and $\mathfrak{r}_k$ for F_{2n+k}, L_{2n+k}, F_{4n+k} and L_{4n+k}.

The goal of this article is to explore the properties of the sequences A, B, C, D, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. Each member of these sequences is an Euler $D(-4)$- or $D(20)$-triple (see [3]) so that many of their properties follow from the properties of the general (pencils of) Euler triples (see [4,5]). It is therefore interesting to look for those properties in which at least two of the sequences appear. This paper presents several results of this kind giving many squares from the components, various sums and products of the sequences A, B, C, D, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. Most of our theorems have also versions for the associated sequences $\tilde{A}=(2\pi_5,\pi_4,2\pi_2)$, $\tilde{B}=(2\pi_6,\pi_4,2\pi_3)$, $\tilde{C}=(2\rho_2,2\pi_2,\rho_1)$, $\tilde{D}=(2\rho_2,\rho_3,2\pi_2)$ and $\tilde{\mathcal{A}}=(2\rho_5,\rho_4,2\rho_2)$, $\tilde{\mathcal{B}}=(2\rho_6,\rho_4,2\rho_3)$, $\tilde{\mathcal{C}}=(10\pi_2,2\rho_2,5\pi_1)$, $\tilde{\mathcal{D}}=(10\pi_2,5\pi_3,2\rho_2)$ that satisfy the relations $A\overset{-4}{\sim}\tilde{A}$, $B\overset{-4}{\sim}\tilde{B}$, $C\overset{-4}{\sim}\tilde{C}$, $D\overset{-4}{\sim}\tilde{D}$ and $\mathcal{A}\overset{20}{\sim}\tilde{\mathcal{A}}$, $\mathcal{B}\overset{20}{\sim}\tilde{\mathcal{B}}$, $\mathcal{C}\overset{20}{\sim}\tilde{\mathcal{C}}$, $\mathcal{D}\overset{20}{\sim}\tilde{\mathcal{D}}$.

The overall principle in this paper is that if you can get complete squares by adding a fixed number to the products of different components of some triples of natural numbers then you will be able to get complete squares by adding some other fixed numbers to all kinds of expressions and constructions built from the components of these triples. Our task was to find out these numbers and to identify those expressions and constructions.

All results in this paper are identities among Fibonacci and/or Lucas numbers of varied difficulty. We shall write down the proofs of only a small portion of them to save the space leaving the rest to the dedicated reader. In most cases we prove or only outline the proof of the first among several parts of the theorem. The other parts have similar proofs sometimes with far more complicated details.

Following this introduction, in the section 2, we first show that the selected products of four components among triples from either the sequences A, B, C, D, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ or the sequences $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ become squares by adding some fixed integers.

The Section 3 considers the various products of two symmetric quadratic sums of components and seeks to get squares in the same way (by adding a fixed integer).

The next Section 4 does a similar task for certain products of four symmetric linear sums of components.

In the Section 5 the numerous products of two sums of squares of components are shown as differences of squares.

The long Section 6 contains similar results for products of two symmetric linear sums of components of the three natural products (dot, forward shifted dot and backward shifted dot) of two triples of integers.

Finally, the last section 7 replaces these dot products with the two forms of a standard vector product in the Euclidean 3-space.

2. Squares from Products of Components

The relations $A \overset{-4}{\sim} \tilde{A}$ and $\mathcal{A} \overset{20}{\sim} \tilde{\mathcal{A}}$ imply that the components of A and $\mathcal{A}$ satisfy $A_2A_3 \overset{-4}{\sim} \tilde{A}_1$ and $\mathcal{A}_2\mathcal{A}_3 \overset{20}{\sim} \tilde{\mathcal{A}}_1$. Our first theorem shows that the product $\frac{1}{16}A_2A_3\mathcal{A}_2\mathcal{A}_3$ is in a similar relation with respect to 9. Of course, the other products $A_3A_1\mathcal{A}_3\mathcal{A}_1$, $A_1A_2\mathcal{A}_1\mathcal{A}_2$ as well as $B_2B_3\mathcal{B}_2\mathcal{B}_3$, etc. exhibit a similar property. The missing cases from the list coincide with the one of the previous cases.

Theorem 1. *The following hold for the products of components*:

$$\frac{1}{16}A_2A_3\mathcal{A}_2\mathcal{A}_3 \overset{9}{\sim} \mathfrak{p}_{10},\ \frac{1}{16}B_2B_3\mathcal{B}_2\mathcal{B}_3 \overset{1}{\sim} \mathfrak{p}_{12},\ \frac{1}{400}C_2C_3\mathcal{C}_2\mathcal{C}_3 \overset{1}{\sim} \mathfrak{p}_4,$$

$$A_3A_1\mathcal{A}_3\mathcal{A}_1 \overset{64}{\sim} \mathfrak{p}_8,\quad \frac{1}{16}C_3C_1\mathcal{C}_3\mathcal{C}_1 \overset{1}{\sim} \mathfrak{p}_4,\quad D_3D_1\mathcal{D}_3\mathcal{D}_1 \overset{0}{\sim} 5\mathfrak{p}_6,$$

$$\frac{1}{16}B_1B_2\mathcal{B}_1\mathcal{B}_2 \overset{9}{\sim} \mathfrak{p}_6, \text{ and } C_1C_2\mathcal{C}_1\mathcal{C}_2 \overset{0}{\sim} 5\mathfrak{p}_2 .$$

Proof. Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi}$. Since $F_j = \frac{\varphi^j - \psi^j}{\varphi - \psi}$ and $L_j = \varphi^j + \psi^j$, it follows that $A_2 = \frac{4\left(\varphi^{2n+3} - \psi^{2n+3}\right)}{\varphi - \psi}$, $A_3 = \frac{\varphi^{2n+7} - \psi^{2n+7}}{\varphi - \psi}$ and $\mathcal{A}_2 = 4\left(\varphi^{2n+3} + \psi^{2n+3}\right)$, $\mathcal{A}_3 = \varphi^{2n+7} + \psi^{2n+7}$.

After the substitutions $\psi = -\frac{1}{\varphi}$ and $\Phi = \varphi^n$, the sum of the product $\frac{1}{16}A_2A_3\mathcal{A}_2\mathcal{A}_3$ and 9 becomes $\frac{\varphi^{20}\left(\Phi^8 - \psi^{20}\right)^2}{5\Phi^8}$. However, this is precisely the square of $\mathfrak{p}_{10}$. This shows the first relation.

The version of the previous theorem for the sequences $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ is the following result. Notice that in this theorem there are no repetitions of cases.

Theorem 2. *The products of the components of* $\tilde{A}, \cdots, \tilde{\mathcal{D}}$ *satisfy*:

$$\frac{1}{4}\tilde{A}_2\tilde{A}_3\tilde{\mathcal{A}}_2\tilde{\mathcal{A}}_3 \overset{1}{\sim} \mathfrak{p}_6,\quad \frac{1}{4}\tilde{B}_2\tilde{B}_3\tilde{\mathcal{B}}_2\tilde{\mathcal{B}}_3 \overset{1}{\sim} \mathfrak{p}_7,\quad \frac{1}{4}\tilde{C}_2\tilde{C}_3\tilde{\mathcal{C}}_2\tilde{\mathcal{C}}_3 \overset{1}{\sim} \mathfrak{t}_3,$$

$$\frac{1}{4}\tilde{D}_2\tilde{D}_3\tilde{\mathcal{D}}_2\tilde{\mathcal{D}}_3 \overset{1}{\sim} \mathfrak{t}_5,\quad \frac{1}{16}\tilde{A}_3\tilde{A}_1\tilde{\mathcal{A}}_3\tilde{\mathcal{A}}_1 \overset{4}{\sim} \mathfrak{p}_7,\quad \frac{1}{16}\tilde{B}_3\tilde{B}_1\tilde{\mathcal{B}}_3\tilde{\mathcal{B}}_1 \overset{4}{\sim} \mathfrak{p}_9,$$

$$\frac{1}{100}\tilde{C}_3\tilde{C}_1\tilde{\mathcal{C}}_3\tilde{\mathcal{C}}_1 \overset{1}{\sim} \mathfrak{p}_3,\quad \frac{1}{80}\tilde{D}_3\tilde{D}_1\tilde{\mathcal{D}}_3\tilde{\mathcal{D}}_1 \overset{0}{\sim} \mathfrak{p}_4,\quad \frac{1}{4}\tilde{A}_1\tilde{A}_2\tilde{\mathcal{A}}_1\tilde{\mathcal{A}}_2 \overset{1}{\sim} \mathfrak{p}_9,$$

$$\frac{1}{4}\tilde{B}_1\tilde{B}_2\tilde{\mathcal{B}}_1\tilde{\mathcal{B}}_2 \overset{1}{\sim} \mathfrak{p}_{10},\quad \frac{1}{16}\tilde{C}_1\tilde{C}_2\tilde{\mathcal{C}}_1\tilde{\mathcal{C}}_2 \overset{4}{\sim} \mathfrak{t}_4,\quad \frac{1}{100}\tilde{D}_1\tilde{D}_2\tilde{\mathcal{D}}_1\tilde{\mathcal{D}}_2 \overset{1}{\sim} \mathfrak{p}_5.$$

Proof. Since $\tilde{\mathcal{A}}_2 = \varphi^{2n+4} + \psi^{2n+4}$, $\tilde{A}_2 = \frac{\varphi^{2n+4} - \psi^{2n+4}}{\varphi - \psi}$, $\tilde{\mathcal{A}}_3 = 2\left(\varphi^{2n+2} + \psi^{2n+2}\right)$ and $\tilde{A}_3 = \frac{2\left(\varphi^{2n+2} + \psi^{2n+2}\right)}{\varphi - \psi}$, the sum of $\frac{1}{4}\tilde{A}_2\tilde{A}_3\tilde{\mathcal{A}}_2\tilde{\mathcal{A}}_3$ and 1, after the substitutions $\psi = -\frac{1}{\varphi}$ and $\Phi = \varphi^n$, becomes $\frac{\varphi^{12}\left(\Phi^8 - \psi^{12}\right)^2}{5\Phi^8}$. However, the square of $\mathfrak{p}_6$ has the same value. This proves the first relation $\frac{1}{4}\tilde{A}_2\tilde{A}_3\tilde{\mathcal{A}}_2\tilde{\mathcal{A}}_3 \overset{1}{\sim} \mathfrak{p}_6$.

The same kind of relations hold also for the products of components from four among the sequences A, B, C, D, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$.

Theorem 3. *The relations that hold for the products of components*:

$$\frac{1}{16}A_2B_3\mathcal{A}_2\mathcal{B}_3 \overset{9}{\sim} \mathfrak{p}_{10},\quad A_2C_3\mathcal{A}_2\mathcal{C}_3 \overset{0}{\sim} 16\mathfrak{p}_6,\quad A_2D_3\mathcal{A}_2\mathcal{D}_3 \overset{0}{\sim} 20\mathfrak{p}_6,$$

$$\frac{1}{256}B_2C_3\mathcal{B}_2\mathcal{C}_3 \overset{1}{\sim} \mathfrak{p}_8,\ \frac{1}{400}B_2D_3\mathcal{B}_2\mathcal{D}_3 \overset{1}{\sim} \mathfrak{p}_8,\ \frac{1}{625}C_2D_3\mathcal{C}_2\mathcal{D}_3 \overset{1}{\sim} \mathfrak{p}_4,$$

$$A_3B_1\mathcal{A}_3\mathcal{B}_1 \overset{64}{\sim} \mathfrak{p}_8,\quad A_3D_1\mathcal{A}_3\mathcal{D}_1 \overset{9}{\sim} \mathfrak{p}_{10},\quad C_3D_1\mathcal{C}_3\mathcal{D}_1 \overset{0}{\sim} 4\mathfrak{p}_6,$$

$$\frac{1}{16}A_1B_2\mathcal{A}_1\mathcal{B}_2 \overset{9}{\sim} \mathfrak{p}_6,\quad A_1C_2\mathcal{A}_1\mathcal{C}_2 \overset{0}{\sim} 5\mathfrak{p}_2,\quad A_1D_2\mathcal{A}_1\mathcal{D}_2 \overset{0}{\sim} 4\mathfrak{p}_2.$$

Proof. Since $B_3 = A_3$ and $\mathcal{B}_3 = \mathcal{A}_3$, the first relation is the consequence of the first relation in Theorem 1. Similarly, the fifth relation follows from the sixth relation in the Theorem 1.

The other relations in this theorem have proofs similar to the proofs of Theorems 1 and 2.

There is again the version of the previous theorem for the products of components from four among the sequences $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$.

Theorem 4. *The products of components of* $\tilde{A},\cdots,\tilde{\mathcal{D}}$ *satisfy*:

$$\frac{1}{4}\tilde{A}_2\tilde{B}_3\tilde{\mathcal{A}}_2\tilde{\mathcal{B}}_3 \overset{1}{\sim} \mathfrak{p}_7,\quad \tilde{A}_2\tilde{C}_3\tilde{\mathcal{A}}_2\tilde{\mathcal{C}}_3 \overset{16}{\sim} \mathfrak{r}_5,\quad \frac{1}{4}\tilde{A}_2\tilde{D}_3\tilde{\mathcal{A}}_2\tilde{\mathcal{D}}_3 \overset{1}{\sim} \mathfrak{p}_6,$$

$$\tilde{C}_2\tilde{D}_3\tilde{\mathcal{C}}_2\tilde{\mathcal{D}}_3 \overset{0}{\sim} 4\mathfrak{p}_4,\quad \frac{1}{16}\tilde{A}_3\tilde{B}_1\tilde{\mathcal{A}}_3\tilde{\mathcal{B}}_1 \overset{9}{\sim} \mathfrak{p}_8,\quad \frac{1}{16}\tilde{A}_3\tilde{C}_1\tilde{\mathcal{A}}_3\tilde{\mathcal{C}}_1 \overset{4}{\sim} \mathfrak{r}_4,$$

$$\frac{1}{16}\tilde{B}_3\tilde{C}_1\tilde{\mathcal{B}}_3\tilde{\mathcal{C}}_1 \overset{1}{\sim} \mathfrak{r}_5,\quad \frac{1}{100}\tilde{C}_3\tilde{D}_1\tilde{\mathcal{C}}_3\tilde{\mathcal{D}}_1 \overset{1}{\sim} \mathfrak{r}_3,\quad \frac{1}{4}\tilde{A}_1\tilde{B}_2\tilde{\mathcal{A}}_1\tilde{\mathcal{B}}_2 \overset{1}{\sim} \mathfrak{p}_9,$$

$$\frac{1}{16}\tilde{A}_1\tilde{C}_2\tilde{\mathcal{A}}_1\tilde{\mathcal{C}}_2 \overset{4}{\sim} \mathfrak{p}_7,\quad \frac{1}{4}\tilde{A}_1\tilde{D}_2\tilde{\mathcal{A}}_1\tilde{\mathcal{D}}_2 \overset{9}{\sim} \mathfrak{r}_8,\quad \frac{1}{4}\tilde{B}_1\tilde{D}_2\tilde{\mathcal{A}}_1\tilde{\mathcal{D}}_2 \overset{16}{\sim} \mathfrak{r}_9,$$

and $\frac{1}{100}\tilde{C}_1\tilde{D}_2\tilde{\mathcal{C}}\tilde{\mathcal{D}}_2 \overset{1}{\sim} \mathfrak{p}_5$.

Proof. The first, the sixth, the ninth and the tenth relations are the easy consequences of the second, the last, the seventh and the fourth relations in Theorem 2.

In order to prove the second relation, note that the components $\tilde{A}_2$, $\tilde{C}_3$, $\tilde{\mathcal{A}}_2$ and $\tilde{\mathcal{C}}_3$ are $\frac{\varphi^{2n+4}-\psi^{2n+4}}{\varphi-\psi}$, $\varphi^{2n+1}+\psi^{2n+1}$, $\varphi^{2n+4}+\psi^{2n+4}$ and $\frac{5\left(\varphi^{2n+1}-\psi^{2n+1}\right)}{\varphi-\psi}$. It is now clear from the proof of Theorem 1 that the sum of $\tilde{A}_2\tilde{C}_3\tilde{\mathcal{A}}_2\tilde{\mathcal{C}}_3$ and 16 is precisely the square of $\mathfrak{r}_5$. This requires the identities $\pi_1\rho_1 = \mathfrak{p}_2$, $\pi_4\rho_4 = \mathfrak{p}_8$ and $5\mathfrak{p}_2\mathfrak{p}_8 + 16 = \mathfrak{r}_5^2$.

3. Squares from Symmetric Sums

Let $\sigma_1,\sigma_2,\sigma_3 : \mathbb{Z}^3 \to \mathbb{Z}$ be the basic symmetric functions defined for $x = (a,b,c)$ by $x_{\sigma_1} = a+b+c, x_{\sigma_2} = bc+ca+ab, x_{\sigma_3} = abc.$ Let σ_2^*, $\sigma_1^* : \mathbb{Z}^3 \to \mathbb{Z}$ be defined by $x_{\sigma_2^*} = bc-ca+ab$ and $x_{\sigma_1^*} = a-b+c$. Note that $x_{\sigma_1^*}$ is the determinant of the 1×3 matrix $[a,b,c]$ (see [6]).

For the sums σ_2 and σ_2^* of the components the following relations are true.

Theorem 5. *The following is true for the sums* σ_2 *of the components*:

$$A_{\sigma_2}\mathcal{A}_{\sigma_2} \overset{384}{\sim} 9\mathfrak{p}_8 + 8\mathfrak{p}_6,\qquad B_{\sigma_2}\mathcal{B}_{\sigma_2} \overset{384}{\sim} 11\mathfrak{r}_8 + 14\mathfrak{p}_6,$$

$$C_{\sigma_2}\mathcal{C}_{\sigma_2} \overset{576}{\sim} 7\mathfrak{r}_5 + 4\mathfrak{p}_0,\qquad D_{\sigma_2}\mathcal{D}_{\sigma_2} \overset{576}{\sim} 4\mathfrak{p}_8 + 7\mathfrak{p}_3,$$

$$\tilde{A}_{\sigma_2}\tilde{\mathcal{A}}_{\sigma_2} \overset{128}{\sim} 4\mathfrak{p}_9,\qquad \tilde{B}_{\sigma_2}\tilde{\mathcal{B}}_{\sigma_2} \overset{80}{\sim} 8\mathfrak{p}_9,$$

$$\tilde{C}_{\sigma_2}\tilde{\mathcal{C}}_{\sigma_2} \overset{336}{\sim} 8\mathfrak{r}_4,\qquad \tilde{D}_{\sigma_2}\tilde{\mathcal{D}}_{\sigma_2} \overset{320}{\sim} 20\mathfrak{p}_5.$$

Proof. Since $\tilde{B}_{\sigma_2} = \frac{4}{5}(2\mathfrak{r}_9+3)$, $\tilde{\mathcal{B}}_{\sigma_2} = 4(2\mathfrak{r}_9-3)$, the sum $\tilde{B}_{\sigma_2}\tilde{\mathcal{B}}_{\sigma_2}+80$ is $\frac{16}{5}\left[(2\mathfrak{r}_9)^2+391\right]$ that we recognize as the square of $8\mathfrak{p}_9$. This proves the sixth relation $\tilde{B}_{\sigma_2}\tilde{\mathcal{B}}_{\sigma_2} \overset{80}{\sim} 8\mathfrak{p}_9$.

The sums $\tilde{B}_{\sigma_2^*}$ and $\tilde{\mathcal{B}}_{\sigma_2^*}$ have constant values –4 and 20. On the other hand, $\tilde{A}_{\sigma_2^*} \overset{0}{\sim} 2\pi_3$, $\tilde{\mathcal{A}}_{\sigma_2^*} \overset{0}{\sim} 2\rho_3$, $\tilde{C}_{\sigma_2^*}\tilde{\mathcal{C}}_{\sigma_2^*} \overset{-64}{\sim} 4\mathfrak{r}_3$ and $\tilde{D}_{\sigma_2^*}\tilde{\mathcal{D}}_{\sigma_2^*} \overset{-64}{\sim} 4\mathfrak{r}_5$.

Theorem 6. *The following is true for the sums* σ_2^* *of the components*:

$$A_{\sigma_2^*}\mathcal{A}_{\sigma_2^*} \overset{-128}{\sim} 7\mathfrak{p}_8 + 8\mathfrak{p}_6,\quad B_{\sigma_2^*}\mathcal{B}_{\sigma_2^*} \overset{-128}{\sim} 8\mathfrak{p}_{10} + 7\mathfrak{p}_8,$$

$$C_{\sigma_2^*}\mathcal{C}_{\sigma_2^*} \overset{256}{\sim} 8\mathfrak{p}_5 + 13\mathfrak{p}_2,\qquad D_{\sigma_2^*}\mathcal{D}_{\sigma_2^*} \overset{576}{\sim} 4\mathfrak{p}_6 + 3\mathfrak{p}_0.$$

Proof. Since $C_{\sigma_2^*} = \pi_1(23\pi_2 + 7\rho_0)$ and $\mathcal{C}_{\sigma_2^*} = \rho_1(23\pi_4 + 12\pi_0)$, the sum $C_{\sigma_2^*}\mathcal{C}_{\sigma_2^*} + 256$ is the square of $8\mathfrak{p}_5 + 13\mathfrak{p}_2$. This proves the third relation.

Some similar relations make up the following two results.

Theorem 7. *The following is true for the sums* σ_2^* *of the components*:

$$\tilde{A}_{\sigma_2^*}\tilde{\mathcal{A}}_{\sigma_2^*} + \tilde{B}_{\sigma_2^*}\tilde{\mathcal{B}}_{\sigma_2^*} \overset{80}{\sim} 4\mathfrak{p}_6,\quad \tilde{B}_{\sigma_2^*}\tilde{\mathcal{B}}_{\sigma_2^*} + \tilde{C}_{\sigma_2^*}\tilde{\mathcal{C}}_{\sigma_2^*} \overset{16}{\sim} 4\mathfrak{r}_3,$$

and $\tilde{B}_{\sigma_2^*}\tilde{\mathcal{B}}_{\sigma_2^*} + \tilde{D}_{\sigma_2^*}\tilde{\mathcal{D}}_{\sigma_2^*} \overset{16}{\sim} 4\mathfrak{r}_5.$

Proof. Since $\tilde{A}_{\sigma_2^*} = (2\pi_3)^2$, $\tilde{\mathcal{A}}_{\sigma_2^*} = (2\rho_3)^2$, $\tilde{B}_{\sigma_2^*} = -4$ and $\tilde{\mathcal{B}}_{\sigma_2^*} = 20$, it follows that

$$\tilde{A}_{\sigma_2^*}\tilde{\mathcal{A}}_{\sigma_2^*}+\tilde{B}_{\sigma_2^*}\tilde{\mathcal{B}}_{\sigma_2^*}+80=\left(4\pi_3\rho_3\right)^2=\left(4\mathfrak{p}_6\right)^2.$$

Theorem 8. *The following is true for the sums* σ_2 *of the components*:

$$\tilde{A}_{\sigma_2}\tilde{\mathcal{A}}_{\sigma_2}+\tilde{B}_{\sigma_2}\tilde{\mathcal{B}}_{\sigma_2}\overset{144}{\sim}4\mathfrak{r}_9.$$

Proof. Since $\tilde{A}_{\sigma_2}=\frac{4}{5}\left(\mathfrak{r}_9-6\right)$, $\tilde{\mathcal{A}}_{\sigma_2}=4\left(\mathfrak{r}_9+6\right)$, $\tilde{B}_{\sigma_2}=\frac{4}{5}\left(2\mathfrak{r}_9+3\right)$ and $\tilde{\mathcal{B}}_{\sigma_2}=4\left(2\mathfrak{r}_9-3\right)$, it follows that

$$\begin{aligned}&\tilde{A}_{\sigma_2}\tilde{\mathcal{A}}_{\sigma_2}+\tilde{B}_{\sigma_2}\tilde{\mathcal{B}}_{\sigma_2}+144\\&=\frac{16}{5}\left(\mathfrak{r}_9^2-36\right)+\frac{16}{5}\left(4\mathfrak{r}_9^2-9\right)+144=\left(4\mathfrak{r}_9\right)^2.\end{aligned}$$

4. Products of Sums as Differences of Squares

The products of the sums σ_1 and σ_1^* of the components of the four triples among A, B, C, D, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ show the same kind of relations. This is also true for the associated triples $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$. Notice that in the next four theorems the added third number is always a square so that the product on the left hand side in each relation is a difference of squares.

Theorem 9. *The following relations hold for the sums* σ_1:

$$\frac{1}{16}A_{\sigma_1}B_{\sigma_1}\mathcal{A}_{\sigma_1}\mathcal{B}_{\sigma_1}\overset{64}{\sim}19\mathfrak{p}_8,\quad \frac{1}{16}A_{\sigma_1}C_{\sigma_1}\mathcal{A}_{\sigma_1}\mathcal{C}_{\sigma_1}\overset{400}{\sim}3\mathfrak{r}_8+2\mathfrak{p}_0,$$

$$\frac{1}{16}A_{\sigma_1}D_{\sigma_1}\mathcal{A}_{\sigma_1}\mathcal{D}_{\sigma_1}\overset{225}{\sim}4\mathfrak{p}_9+11\mathfrak{p}_4,\quad \frac{1}{16}B_{\sigma_1}C_{\sigma_1}\mathcal{B}_{\sigma_1}\mathcal{C}_{\sigma_1}\overset{1296}{\sim}\mathfrak{r}_{11}+8\mathfrak{p}_4,$$

$$\frac{1}{16}B_{\sigma_1}D_{\sigma_1}\mathcal{B}_{\sigma_1}\mathcal{D}_{\sigma_1}\overset{841}{\sim}3\mathfrak{p}_{11}+\mathfrak{p}_0,\quad \frac{1}{16}C_{\sigma_1}D_{\sigma_1}\mathcal{C}_{\sigma_1}\mathcal{D}_{\sigma_1}\overset{25}{\sim}31\mathfrak{p}_4,$$

$$\frac{1}{25}\tilde{A}_{\sigma_1}\tilde{B}_{\sigma_1}\tilde{\mathcal{A}}_{\sigma_1}\tilde{\mathcal{B}}_{\sigma_1}\overset{16}{\sim}\mathfrak{p}_{10}+3\mathfrak{p}_9,\quad \frac{1}{25}\tilde{A}_{\sigma_1}\tilde{C}_{\sigma_1}\tilde{\mathcal{A}}_{\sigma_1}\tilde{\mathcal{C}}_{\sigma_1}\overset{0}{\sim}3\mathfrak{p}_8,$$

$$\frac{1}{25}\tilde{A}_{\sigma_1}\tilde{D}_{\sigma_1}\tilde{\mathcal{A}}_{\sigma_1}\tilde{\mathcal{D}}_{\sigma_1}\overset{9}{\sim}2\mathfrak{p}_9+\mathfrak{p}_7,\quad \frac{1}{9}\tilde{B}_{\sigma_1}\tilde{C}_{\sigma_1}\tilde{\mathcal{B}}_{\sigma_1}\tilde{\mathcal{C}}_{\sigma_1}\overset{16}{\sim}9\mathfrak{p}_7+5\mathfrak{p}_6,$$

$$\tilde{B}_{\sigma_1}\tilde{D}_{\sigma_1}\tilde{\mathcal{B}}_{\sigma_1}\tilde{\mathcal{D}}_{\sigma_1}\overset{49}{\sim}11\mathfrak{p}_{10},\qquad \frac{1}{9}\tilde{C}_{\sigma_1}\tilde{D}_{\sigma_1}\tilde{\mathcal{C}}_{\sigma_1}\tilde{\mathcal{D}}_{\sigma_1}\overset{9}{\sim}2\mathfrak{p}_9+\mathfrak{p}_7.$$

Proof. The sums of the components A_{σ_1}, $\mathcal{A}_{\sigma_1}$, B_{σ_1} and $\mathcal{B}_{\sigma_1}$ are equal $2\left(\rho_5-\pi_0\right)$, $2\left(5\pi_5-\rho_0\right)$, $\pi_9-\pi_0$ and $\rho_9-\rho_0$. Hence, the sum of 64 and $\frac{1}{16}A_{\sigma_1}B_{\sigma_1}\mathcal{A}_{\sigma_1}\mathcal{B}_{\sigma_1}$ is the square of $19\mathfrak{p}_8$. This proves the above first relation.

Theorem 10. *The following relations hold for the sums* σ_1^*:

$$A_{\sigma_1^*}B_{\sigma_1^*}\mathcal{A}_{\sigma_1^*}\mathcal{B}_{\sigma_1^*}\overset{0}{\sim}4\mathfrak{p}_8,\quad \frac{1}{64}A_{\sigma_1^*}C_{\sigma_1^*}\mathcal{A}_{\sigma_1^*}\mathcal{C}_{\sigma_1^*}\overset{1}{\sim}\mathfrak{p}_6,$$

$$\frac{1}{16}A_{\sigma_1^*}D_{\sigma_1^*}\mathcal{A}_{\sigma_1^*}\mathcal{D}_{\sigma_1^*}\overset{1}{\sim}\mathfrak{r}_7,\quad \frac{1}{64}C_{\sigma_1^*}D_{\sigma_1^*}\mathcal{C}_{\sigma_1^*}\mathcal{D}_{\sigma_1^*}\overset{1}{\sim}\mathfrak{r}_5,$$

$$\frac{1}{9}\tilde{A}_{\sigma_1^*}\tilde{B}_{\sigma_1^*}\tilde{\mathcal{A}}_{\sigma_1^*}\tilde{\mathcal{B}}_{\sigma_1^*}\overset{16}{\sim}\mathfrak{p}_8+2\mathfrak{r}_8,\quad \frac{1}{9}\tilde{A}_{\sigma_1^*}\tilde{C}_{\sigma_1^*}\tilde{\mathcal{A}}_{\sigma_1^*}\tilde{\mathcal{C}}_{\sigma_1^*}\overset{16}{\sim}\mathfrak{p}_8+2\mathfrak{p}_5,$$

$$\frac{1}{9}\tilde{A}_{\sigma_1^*}\tilde{D}_{\sigma_1^*}\tilde{\mathcal{A}}_{\sigma_1^*}\tilde{\mathcal{D}}_{\sigma_1^*}\overset{25}{\sim}\mathfrak{p}_7+2\mathfrak{p}_5,\quad \tilde{B}_{\sigma_1^*}\tilde{C}_{\sigma_1^*}\tilde{\mathcal{B}}_{\sigma_1^*}\tilde{\mathcal{C}}_{\sigma_1^*}\overset{256}{\sim}3\mathfrak{p}_{10}+\mathfrak{r}_3,$$

$$\tilde{B}_{\sigma_1^*}\tilde{D}_{\sigma_1^*}\tilde{\mathcal{B}}_{\sigma_1^*}\tilde{\mathcal{D}}_{\sigma_1^*}\overset{529}{\sim}2\mathfrak{p}_{10}+5\mathfrak{p}_4,\quad \tilde{C}_{\sigma_1^*}\tilde{D}_{\sigma_1^*}\tilde{\mathcal{C}}_{\sigma_1^*}\tilde{\mathcal{D}}_{\sigma_1^*}\overset{9}{\sim}11\mathfrak{p}_4.$$

Proof. The sums of the components $A_{\sigma_1^*}$, $\mathcal{A}_{\sigma_1^*}$, $B_{\sigma_1^*}$ and $\mathcal{B}_{\sigma_1^*}$ are equal $2\pi_4$, $2\rho_4$, $-2\pi_4$ and $-2\rho_4$. Hence, the product $A_{\sigma_1^*}B_{\sigma_1^*}\mathcal{A}_{\sigma_1^*}\mathcal{B}_{\sigma_1^*}$ is the square of $4\mathfrak{p}_8$ since $\pi_4\rho_4=\mathfrak{p}_8$. This proves the above first relation.

In the next result we combine the sums σ_1 and σ_1^* in each product.

Theorem 11. *The following relations hold for the sums* σ_1 *and* σ_1^*:

$$\frac{1}{16}A_{\sigma_1}B_{\sigma_1^*}\mathcal{A}_{\sigma_1}\mathcal{B}_{\sigma_1^*}\overset{16}{\sim}\mathfrak{p}_{10}+\mathfrak{r}_6,\quad \frac{1}{16}A_{\sigma_1^*}B_{\sigma_1}\mathcal{A}_{\sigma_1^*}\mathcal{B}_{\sigma_1}\overset{16}{\sim}\mathfrak{p}_{11}+2\mathfrak{p}_7,$$

$$\frac{1}{64}A_{\sigma_1}C_{\sigma_1^*}\mathcal{A}_{\sigma_1}\mathcal{C}_{\sigma_1^*}\overset{25}{\sim}4\mathfrak{r}_4+\mathfrak{p}_0,\quad \frac{1}{16}A_{\sigma_1^*}C_{\sigma_1}\mathcal{A}_{\sigma_1^*}\mathcal{C}_{\sigma_1}\overset{64}{\sim}2\mathfrak{r}_6+\mathfrak{p}_5,$$

$$\frac{1}{16}A_{\sigma_1}D_{\sigma_1^*}\mathcal{A}_{\sigma_1}\mathcal{D}_{\sigma_1^*}\overset{81}{\sim}3\mathfrak{p}_9-\mathfrak{p}_2,\quad \frac{1}{16}A_{\sigma_1^*}D_{\sigma_1}\mathcal{A}_{\sigma_1^*}\mathcal{D}_{\sigma_1}\overset{49}{\sim}3\mathfrak{p}_7+2\mathfrak{p}_5,$$

$$\frac{1}{64}B_{\sigma_1}C_{\sigma_1^*}\mathcal{B}_{\sigma_1}\mathcal{C}_{\sigma_1^*}\overset{49}{\sim}4\mathfrak{r}_5-\mathfrak{p}_0,\quad \frac{1}{16}B_{\sigma_1}D_{\sigma_1^*}\mathcal{B}_{\sigma_1}\mathcal{D}_{\sigma_1^*}\overset{121}{\sim}\mathfrak{p}_{12}+3\mathfrak{p}_5,$$

$$\frac{1}{16}C_{\sigma_1}D_{\sigma_1^*}\mathcal{C}_{\sigma_1}\mathcal{D}_{\sigma_1^*}\overset{169}{\sim}5\mathfrak{r}_5+\mathfrak{r}_0,\quad \frac{1}{64}C_{\sigma_1^*}D_{\sigma_1}\mathcal{C}_{\sigma_1^*}\mathcal{D}_{\sigma_1}\overset{25}{\sim}\mathfrak{p}_8-2\mathfrak{p}_2.$$

Proof. With the above information about the sums of the components A_{σ_1}, $\mathcal{A}_{\sigma_1}$, $B_{\sigma_1^*}$ and $\mathcal{B}_{\sigma_1^*}$, the sum $\frac{1}{16}A_{\sigma_1}B_{\sigma_1^*}\mathcal{A}_{\sigma_1}\mathcal{B}_{\sigma_1^*}+16$ is

$$\begin{aligned}\mathfrak{p}_8\left(\rho_5-\pi_0\right)\left(5\pi_5-\rho_0\right)+16&=\mathfrak{p}_8\left(5\mathfrak{p}_{10}-2\mathfrak{r}_5+\mathfrak{p}_0\right)+16\\&=\left(\mathfrak{p}_{10}+\mathfrak{r}_6\right)^2.\end{aligned}$$

This proves the first relation.

Theorem 12. *The following relations hold for the sums* σ_1 *and* σ_1^*:

$$\frac{1}{25}\tilde{A}_{\sigma_1}\tilde{B}_{\sigma_1^*}\tilde{\mathcal{A}}_{\sigma_1}\tilde{\mathcal{B}}_{\sigma_1^*}\overset{16}{\sim}\mathfrak{p}_8+2\mathfrak{r}_8,\quad \frac{1}{9}\tilde{A}_{\sigma_1^*}\tilde{B}_{\sigma_1}\tilde{\mathcal{A}}_{\sigma_1^*}\tilde{\mathcal{B}}_{\sigma_1}\overset{16}{\sim}\mathfrak{p}_{10}+3\mathfrak{p}_9,$$

$$\frac{1}{25}\tilde{A}_{\sigma_1}\tilde{C}_{\sigma_1^*}\tilde{\mathcal{A}}_{\sigma_1}\tilde{\mathcal{C}}_{\sigma_1^*}\overset{16}{\sim}2\mathfrak{p}_6+3\mathfrak{p}_5,\qquad \tilde{A}_{\sigma_1^*}\tilde{C}_{\sigma_1}\tilde{\mathcal{A}}_{\sigma_1^*}\tilde{\mathcal{C}}_{\sigma_1}\overset{0}{\sim}9\mathfrak{p}_8,$$

$$\frac{1}{25}\tilde{A}_{\sigma_1}\tilde{D}_{\sigma_1^*}\tilde{\mathcal{A}}_{\sigma_1}\tilde{\mathcal{D}}_{\sigma_1^*}\overset{25}{\sim}\mathfrak{p}_7+2\mathfrak{p}_5,\quad \frac{1}{9}\tilde{A}_{\sigma_1^*}\tilde{D}_{\sigma_1}\tilde{\mathcal{A}}_{\sigma_1^*}\tilde{\mathcal{D}}_{\sigma_1}\overset{9}{\sim}2\mathfrak{p}_9+\mathfrak{p}_7,$$

$$\tilde{B}_{\sigma_1}\tilde{C}_{\sigma_1^*}\tilde{\mathcal{B}}_{\sigma_1}\tilde{\mathcal{C}}_{\sigma_1^*} \overset{576}{\sim} 11\mathfrak{p}_8,\quad \frac{1}{9}\tilde{B}_{\sigma_1^*}\tilde{C}_{\sigma_1}\tilde{\mathcal{B}}_{\sigma_1^*}\tilde{\mathcal{C}}_{\sigma_1} \overset{16}{\sim} 3\mathfrak{p}_8+4\mathfrak{p}_7,$$

$$\tilde{B}_{\sigma_1}\tilde{D}_{\sigma_1^*}\tilde{\mathcal{B}}_{\sigma_1}\tilde{\mathcal{D}}_{\sigma_1^*} \overset{1089}{\sim} 2\mathfrak{p}_7+5\mathfrak{t}_7,\quad \tilde{B}_{\sigma_1^*}\tilde{D}_{\sigma_1}\tilde{\mathcal{B}}_{\sigma_1^*}\tilde{\mathcal{D}}_{\sigma_1} \overset{1}{\sim} 3\mathfrak{p}_{12}+\mathfrak{t}_5,$$

$$\frac{1}{9}\tilde{C}_{\sigma_1}\tilde{D}_{\sigma_1^*}\tilde{\mathcal{C}}_{\sigma_1}\tilde{\mathcal{D}}_{\sigma_1^*} \overset{25}{\sim} \mathfrak{p}_9-\mathfrak{t}_5,\quad \tilde{C}_{\sigma_1^*}\tilde{D}_{\sigma_1}\tilde{\mathcal{C}}_{\sigma_1^*}\tilde{\mathcal{D}}_{\sigma_1} \overset{121}{\sim} \mathfrak{p}_{10}+8\mathfrak{p}_6.$$

Proof. The sums of the components $\tilde{A}_{\sigma_1}$, $\tilde{\mathcal{A}}_{\sigma_1}$, $\tilde{B}_{\sigma_1^*}$ and $\tilde{\mathcal{B}}_{\sigma_1^*}$ are equal $5\pi_4$, $5\rho_4$, $\frac{1}{2}(\pi_9-\pi_0)$ and $\frac{1}{2}(\rho_9-\rho_0)$. Hence, the sum of the product $\frac{1}{25}\tilde{A}_{\sigma_1}\tilde{B}_{\sigma_1^*}\tilde{\mathcal{A}}_{\sigma_1}\tilde{\mathcal{B}}_{\sigma_1}$ and 16 is

$$\begin{aligned}&\frac{1}{4}\mathfrak{p}_8(\rho_9-\rho_0)(\pi_9-\pi_0)+16\\ &=\frac{1}{4}\mathfrak{p}_8(\mathfrak{p}_{18}-2\mathfrak{p}_9+\mathfrak{p}_0)+16=(\mathfrak{p}_8+2\mathfrak{t}_8)^2.\end{aligned}$$

This proves the above first relation.

5. Squares from the Sums of Squares

For a natural number $k>1$, let the sums $\nu_k,\nu_k^*:\mathbb{Z}^3\to\mathbb{Z}$ of powers be defined for $x=(a,b,c)$ by $x_{\nu_k}=a^k+b^k+c^k$ and $x_{\nu_k^*}=a^k-b^k+c^k$.

We proceed with the version of the Theorem 9 for the sums ν_2 of the squares of components.

Theorem 13. *The following relations are true for the sums* ν_2:

$$\frac{1}{4}A_{\nu_2}\mathcal{A}_{\nu_2} \overset{-224}{\sim} 4\mathfrak{p}_{10}+3\mathfrak{t}_5,\quad \frac{1}{4}B_{\nu_2}\mathcal{B}_{\nu_2} \overset{-224}{\sim} 7\mathfrak{p}_{11}+2\mathfrak{p}_4,$$

$$\frac{1}{4}C_{\nu_2}\mathcal{C}_{\nu_2} \overset{-416}{\sim} 7\mathfrak{t}_5+4\mathfrak{p}_0,\quad \frac{1}{4}D_{\nu_2}\mathcal{D}_{\nu_2} \overset{-416}{\sim} 7\mathfrak{t}_3+4\mathfrak{p}_8,$$

$$\tilde{A}_{\nu_2}\tilde{\mathcal{A}}_{\nu_2} \overset{-288}{\sim} 4\mathfrak{p}_{10}+3\mathfrak{t}_5,\quad \tilde{B}_{\nu_2}\tilde{\mathcal{B}}_{\nu_2} \overset{-288}{\sim} 7\mathfrak{p}_{11}+2\mathfrak{p}_4,$$

$$\tilde{C}_{\nu_2}\tilde{\mathcal{C}}_{\nu_2} \overset{-672}{\sim} 7\mathfrak{t}_5+4\mathfrak{p}_0,\quad \tilde{D}_{\nu_2}\tilde{\mathcal{D}}_{\nu_2} \overset{-672}{\sim} 7\mathfrak{t}_3+4\mathfrak{p}_8.$$

Proof. Since A_{ν_2} and $\mathcal{A}_{\nu_2}$ are $\frac{2}{5}(27\mathfrak{p}_8-2\mathfrak{p}_0+18)$ and $2(27\mathfrak{p}_8-2\mathfrak{p}_0-18)$, the difference of $A_{\nu_2}\mathcal{A}_{\nu_2}$ and 896 is equal $\frac{\alpha\beta_+^2+\beta_-^2}{5212840\Phi^8}$, where α and $\beta\pm$ are $320767+143451\sqrt{5}$ and $38\Phi^4\pm567\mp253\sqrt{5}$. But, one can easily check that this is the square of $8\mathfrak{p}_{10}+6\mathfrak{t}_5$. This concludes the proof of the first relation.

The next is the version of the Theorem 13 for the alternating sums ν_2^* of the squares of components.

Theorem 14. *The following relations are true for the sums* ν_2^*:

$$A_{\nu_2^*}\mathcal{A}_{\nu_2^*} \overset{384}{\sim} 11\mathfrak{p}_6+9\mathfrak{t}_6,\quad \frac{1}{4}B_{\nu_2^*}\mathcal{B}_{\nu_2^*} \overset{96}{\sim} 2\mathfrak{p}_6+5\mathfrak{t}_8,$$

$$\frac{1}{64}C_{\nu_2^*}\mathcal{C}_{\nu_2^*} \overset{24}{\sim} \mathfrak{t}_5+2\mathfrak{p}_1,\quad \frac{1}{4}D_{\nu_2^*}\mathcal{D}_{\nu_2^*} \overset{416}{\sim} 2\mathfrak{t}_8+\mathfrak{t}_0,$$

$$\tilde{A}_{\nu_2^*}\tilde{\mathcal{A}}_{\nu_2^*} \overset{-224}{\sim} 7\mathfrak{t}_7+\mathfrak{p}_6,\quad \tilde{B}_{\nu_2^*}\tilde{\mathcal{B}}_{\nu_2^*} \overset{-224}{\sim} 7\mathfrak{t}_9+\mathfrak{p}_{10},$$

$$\tilde{C}_{\nu_2^*}\tilde{\mathcal{C}}_{\nu_2^*} \overset{-128}{\sim} 8\mathfrak{t}_3+\mathfrak{p}_8,\quad \tilde{D}_{\nu_2^*}\tilde{\mathcal{D}}_{\nu_2^*} \overset{160}{\sim} 4\mathfrak{p}_6+3\mathfrak{p}_0.$$

Proof. Notice that the alternating sums of squares of components $A_{\nu_2^*}$ and $\mathcal{A}_{\nu_2^*}$ are $\frac{2}{5}(25\mathfrak{t}_5+2\mathfrak{t}_0-14)$ and $2(25\mathfrak{t}_5+2\mathfrak{t}_0+14)$. Hence, the sum of $A_{\nu_2^*}\mathcal{A}_{\nu_2^*}$ and 384 is equal

$$\frac{\left(77983+34875\sqrt{5}\right)\left(142\Phi^8+77983-34875\sqrt{5}\right)^2}{50410\Phi^8}.$$

However, one can easily check that this is the square of $11\mathfrak{p}_6+9\mathfrak{t}_6$. This proves the first relation.

Multiplied by five these products of the sums ν_2^* of components show the same behavior.

Theorem 15. *For the sums* ν_2^*, *the following relations hold*:

$$\frac{5}{4}A_{\nu_2^*}\mathcal{A}_{\nu_2^*} \overset{196}{\sim} 3\mathfrak{p}_{11}+4\mathfrak{p}_4,\quad \frac{5}{4}B_{\nu_2^*}\mathcal{B}_{\nu_2^*} \overset{196}{\sim} 3\mathfrak{p}_{11}+14\mathfrak{p}_8,$$

$$\frac{5}{64}C_{\nu_2^*}\mathcal{C}_{\nu_2^*} \overset{4}{\sim} \mathfrak{p}_8+6\mathfrak{p}_2,\quad \frac{1}{20}D_{\nu_2^*}\mathcal{D}_{\nu_2^*} \overset{4}{\sim} 2\mathfrak{p}_8+\mathfrak{p}_0,$$

$$5\tilde{A}_{\nu_2^*}\tilde{\mathcal{A}}_{\nu_2^*} \overset{4}{\sim} 5\mathfrak{p}_{10}+11\mathfrak{t}_6,\quad 5\tilde{B}_{\nu_2^*}\tilde{\mathcal{B}}_{\nu_2^*} \overset{4}{\sim} 4\mathfrak{t}_{12}+5\mathfrak{p}_5,$$

$$5\tilde{C}_{\nu_2^*}\tilde{\mathcal{C}}_{\nu_2^*} \overset{1444}{\sim} 2\mathfrak{t}_9-25\mathfrak{p}_2,\quad 5\tilde{D}_{\nu_2^*}\tilde{\mathcal{D}}_{\nu_2^*} \overset{1764}{\sim} 4\mathfrak{t}_6+3\mathfrak{t}_0.$$

Proof. With the above values of $A_{\nu_2^*}$ and $\mathcal{A}_{\nu_2^*}$, the sum of $5A_{\nu_2^*}\mathcal{A}_{\nu_2^*}$ and 784 is equal

$$\frac{\left(77983+34875\sqrt{5}\right)\left(142\Phi^8-77983+34875\sqrt{5}\right)^2}{10082\Phi^8}.$$

But, this is the square of $6\mathfrak{p}_{11}+8\mathfrak{p}_4$. This outlines the proof the first relation.

6. Squares from the Products $\odot$, $\triangleright$ and $\triangleleft$

Let us introduce three binary operations $\odot$, $\triangleright$ and $\triangleleft$ on the set $\mathbb{Z}^3$ of triples of integers by the rules

$(a,b,c)\odot(u,v,w)=(au,bv,cw)$,

$(a,b,c) \triangleright (u,v,w) = (av,bw,cu)$, and

$$(a,b,c) \triangleleft (u,v,w) = (aw,bu,cv).$$

This section contains four theorems which show that the operations $\odot$, $\triangleright$ and $\triangleleft$ are also the source of squares from components of the sixteen sequences A, $\cdots \tilde{\mathcal{D}}$.

Theorem 16. *The following relations hold for the sequences* $A, \cdots \mathcal{D}$:

$$(A \odot B)_{\sigma_1} (\mathcal{A} \odot \mathcal{B})_{\sigma_1} \overset{-384}{\sim} 34\mathfrak{p}_8, \quad (A \odot C)_{\sigma_1} (\mathcal{A} \odot \mathcal{C})_{\sigma_1} \overset{-64}{\sim} 8\mathfrak{p}_9 + 9\mathfrak{p}_2,$$

$$\frac{1}{4}(A \odot D)_{\sigma_1} (\mathcal{A} \odot \mathcal{D})_{\sigma_1} \overset{1}{\sim} 5\mathfrak{r}_7 + 6\mathfrak{p}_4, \quad (B \odot C)_{\sigma_1} (\mathcal{B} \odot \mathcal{C})_{\sigma_1} \overset{4736}{\sim} 5\mathfrak{r}_{10} + \mathfrak{p}_0,$$

$$(B \odot D)_{\sigma_1} (\mathcal{B} \odot \mathcal{D})_{\sigma_1} \overset{3716}{\sim} 14\mathfrak{r}_7, \quad (C \odot D)_{\sigma_1} (\mathcal{C} \odot \mathcal{D})_{\sigma_1} \overset{-1599}{\sim} 61\mathfrak{p}_4,$$

$$(A \triangleright B)_{\sigma_1} (\mathcal{A} \triangleright \mathcal{B})_{\sigma_1} \overset{960}{\sim} 13\mathfrak{p}_8, \quad (A \triangleright C)_{\sigma_1} (\mathcal{A} \triangleright \mathcal{C})_{\sigma_1} \overset{-64}{\sim} 14\mathfrak{r}_5,$$

$$(A \triangleright D)_{\sigma_1} (\mathcal{A} \triangleright \mathcal{D})_{\sigma_1} \overset{-439}{\sim} \mathfrak{r}_{11} + 5\mathfrak{r}_3, \quad \frac{1}{4}(B \triangleright C)_{\sigma_1} (\mathcal{B} \triangleright \mathcal{C})_{\sigma_1} \overset{-80}{\sim} 2\mathfrak{p}_{11} + 3\mathfrak{p}_2,$$

$$(B \triangleright D)_{\sigma_1} (\mathcal{B} \triangleright \mathcal{D})_{\sigma_1} \overset{-559}{\sim} \mathfrak{r}_{13} - 6\mathfrak{r}_4, \quad (C \triangleright D)_{\sigma_1} (\mathcal{C} \triangleright \mathcal{D})_{\sigma_1} \overset{561}{\sim} 37\mathfrak{p}_4,$$

$$(A \triangleleft B)_{\sigma_1} (\mathcal{A} \triangleleft \mathcal{B})_{\sigma_1} \overset{-320}{\sim} 29\mathfrak{p}_8, \quad \frac{1}{9}(A \triangleleft C)_{\sigma_1} (\mathcal{A} \triangleleft \mathcal{C})_{\sigma_1} \overset{256}{\sim} 3\mathfrak{p}_7 + \mathfrak{r}_3,$$

$$(A \triangleleft D)_{\sigma_1} (\mathcal{A} \triangleleft \mathcal{D})_{\sigma_1} \overset{1561}{\sim} \mathfrak{r}_{10} + 2\mathfrak{r}_3, \quad (B \triangleleft C)_{\sigma_1} (\mathcal{B} \triangleleft \mathcal{C})_{\sigma_1} \overset{3136}{\sim} 7\mathfrak{p}_8 + 2\mathfrak{p}_2,$$

$$\frac{1}{9}(B \triangleleft D)_{\sigma_1} (\mathcal{B} \triangleleft \mathcal{D})_{\sigma_1} \overset{185}{\sim} 2\mathfrak{p}_9 - \mathfrak{r}_4, \quad \frac{1}{676}(C \triangleleft D)_{\sigma_1} (\mathcal{C} \triangleleft \mathcal{D})_{\sigma_1} \overset{1}{\sim} \mathfrak{p}_4.$$

Proof. Since $(A \odot B)_{\sigma_1} = \frac{1}{5}(47\mathfrak{p}_9 - \mathfrak{p}_0 + 52)$ and $(\mathcal{A} \odot \mathcal{B})_{\sigma_1} = 47\mathfrak{p}_9 - \mathfrak{p}_0 - 52$, it follows that the difference of $(A \odot B)_{\sigma_1} (\mathcal{A} \odot \mathcal{B})_{\sigma_1}$ and 384 is the square of $34\mathfrak{p}_8$. This proves the first relation.

Theorem 17. *The following relations hold for the sequences* $A, \cdots, \mathcal{D}$:

$$(A \odot B)_{\sigma_1^*} (\mathcal{A} \odot \mathcal{B})_{\sigma_1^*} \overset{384}{\sim} 2\mathfrak{p}_8, \qquad (A \odot C)_{\sigma_1^*} (\mathcal{A} \odot \mathcal{C})_{\sigma_1^*} \overset{896}{\sim} 5\mathfrak{p}_8 + 8\mathfrak{r}_4,$$

$$\frac{1}{100}(A \odot D)_{\sigma_1^*} (\mathcal{A} \odot \mathcal{D})_{\sigma_1^*} \overset{9}{\sim} \mathfrak{p}_9 - \mathfrak{r}_5, \quad (B \odot C)_{\sigma_1^*} (\mathcal{B} \odot \mathcal{C})_{\sigma_1^*} \overset{2496}{\sim} 2\mathfrak{p}_6 + 9\mathfrak{p}_5,$$

$$\frac{1}{4}(B \odot D)_{\sigma_1^*} (\mathcal{B} \odot \mathcal{D})_{\sigma_1^*} \overset{321}{\sim} 2\mathfrak{p}_9 + \mathfrak{r}_4, \quad (C \odot D)_{\sigma_1^*} (\mathcal{C} \odot \mathcal{D})_{\sigma_1^*} \overset{1601}{\sim} \mathfrak{p}_{11} + 27\mathfrak{p}_3,$$

$$(A \triangleright B)_{\sigma_1^*} (\mathcal{A} \triangleright \mathcal{B})_{\sigma_1^*} \overset{128}{\sim} 7\mathfrak{p}_8 + 4\mathfrak{p}_5, \quad \frac{1}{4}(A \triangleright C)_{\sigma_1^*} (\mathcal{A} \triangleright \mathcal{C})_{\sigma_1^*} \overset{48}{\sim} 2\mathfrak{r}_6 + 3\mathfrak{p}_5,$$

$$(A \triangleright D)_{\sigma_1^*} (\mathcal{A} \triangleright \mathcal{D})_{\sigma_1^*} \overset{201}{\sim} \mathfrak{r}_8 + 3\mathfrak{r}_6, \qquad \frac{1}{4}(B \triangleright C)_{\sigma_1^*} (\mathcal{B} \triangleright \mathcal{C})_{\sigma_1^*} \overset{240}{\sim} \mathfrak{r}_8 + 6\mathfrak{r}_6,$$

$$(B \triangleright D)_{\sigma_1^*} (\mathcal{B} \triangleright \mathcal{D})_{\sigma_1^*} \overset{1121}{\sim} \mathfrak{r}_{11} + 9\mathfrak{r}_6, \qquad \frac{1}{4}(C \triangleright D)_{\sigma_1^*} (\mathcal{C} \triangleright \mathcal{D})_{\sigma_1^*} \overset{561}{\sim} 13\mathfrak{p}_4,$$

$$\left(A \lhd B\right)_{\sigma_1^*}\left(\mathcal{A} \lhd \mathcal{B}\right)_{\sigma_1^*} \overset{640}{\sim} 10\mathfrak{p}_{10}+7\mathfrak{p}_5, \quad \frac{1}{25}\left(A \lhd C\right)_{\sigma_1^*}\left(\mathcal{A} \lhd \mathcal{C}\right)_{\sigma_1^*} \overset{64}{\sim} \mathfrak{p}_8,$$

$$\left(A \lhd D\right)_{\sigma_1^*}\left(\mathcal{A} \lhd \mathcal{D}\right)_{\sigma_1^*} \overset{1177}{\sim} 2\mathfrak{p}_9-\mathfrak{p}_1, \quad \left(B \lhd C\right)_{\sigma_1^*}\left(\mathcal{B} \lhd \mathcal{C}\right)_{\sigma_1^*} \overset{1024}{\sim} 4\mathfrak{p}_8+\mathfrak{p}_2,$$

$$\left(B \lhd D\right)_{\sigma_1^*}\left(\mathcal{B} \lhd \mathcal{D}\right)_{\sigma_1^*} \overset{1105}{\sim} 5\mathfrak{p}_4, \quad \frac{1}{16}\left(C \lhd D\right)_{\sigma_1^*}\left(\mathcal{C} \lhd \mathcal{D}\right)_{\sigma_1^*} \overset{16}{\sim} 4\mathfrak{p}_4.$$

Proof. Since the sums $\left(A \odot B\right)_{\sigma_1^*}$ and $\left(\mathcal{A} \odot \mathcal{B}\right)_{\sigma_1^*}$ are $\frac{2}{5}\left(\mathfrak{r}_8-22\right)$ and $2\left(\mathfrak{r}_8+22\right)$, it follows that the sum of $\left(A \odot B\right)_{\sigma_1^*}\left(\mathcal{A} \odot \mathcal{B}\right)_{\sigma_1^*}$ and 384 is $\frac{2}{5}\left(\mathfrak{r}_8^2-22^2\right)+384=\frac{4}{5}\left(\mathfrak{r}_8^2-4\right)=4\mathfrak{p}_8^2$, *i. e.*, the square of $2\mathfrak{p}_8$. This concludes the proof of the first relation.

Theorem 18 *The following relations hold for the sequences* $\tilde{A},\cdots,\tilde{\mathcal{D}}$:

$$\left(\tilde{A} \odot \tilde{B}\right)_{\sigma_1}\left(\tilde{\mathcal{A}} \odot \tilde{\mathcal{B}}\right)_{\sigma_1} \overset{256}{\sim} 2\mathfrak{r}_{11}-\mathfrak{p}_2, \quad \frac{1}{16}\left(\tilde{A} \odot \tilde{C}\right)_{\sigma_1}\left(\tilde{\mathcal{A}} \odot \tilde{\mathcal{C}}\right)_{\sigma_1} \overset{-16}{\sim} \mathfrak{p}_{11}-3\mathfrak{r}_6,$$

$$\left(\tilde{A} \odot \tilde{D}\right)_{\sigma_1}\left(\tilde{\mathcal{A}} \odot \tilde{\mathcal{D}}\right)_{\sigma_1} \overset{-127}{\sim} 2\mathfrak{p}_{11}-3\mathfrak{r}_4, \quad \frac{1}{4}\left(\tilde{B} \odot \tilde{C}\right)_{\sigma_1}\left(\tilde{\mathcal{B}} \odot \tilde{\mathcal{C}}\right)_{\sigma_1} \overset{168}{\sim} 2\mathfrak{p}_{10}-\mathfrak{p}_2,$$

$$\left(\tilde{B} \odot \tilde{D}\right)_{\sigma_1}\left(\tilde{\mathcal{B}} \odot \tilde{\mathcal{D}}\right)_{\sigma_1} \overset{721}{\sim} \mathfrak{p}_{13}+2\mathfrak{p}_3, \quad \frac{1}{4}\left(\tilde{C} \odot \tilde{D}\right)_{\sigma_1}\left(\tilde{\mathcal{C}} \odot \tilde{\mathcal{D}}\right)_{\sigma_1} \overset{-160}{\sim} 15\mathfrak{p}_4,$$

$$\frac{1}{4}\left(\tilde{A} \rhd \tilde{B}\right)_{\sigma_1}\left(\tilde{\mathcal{A}} \rhd \tilde{\mathcal{B}}\right)_{\sigma_1} \overset{40}{\sim} \mathfrak{p}_{11}, \quad \left(\tilde{A} \rhd \tilde{C}\right)_{\sigma_1}\left(\tilde{\mathcal{A}} \rhd \tilde{\mathcal{C}}\right)_{\sigma_1} \overset{336}{\sim} \mathfrak{p}_{11}+2\mathfrak{p}_1,$$

$$\frac{1}{4}\left(\tilde{A} \rhd \tilde{D}\right)_{\sigma_1}\left(\tilde{\mathcal{A}} \rhd \tilde{\mathcal{D}}\right)_{\sigma_1} \overset{92}{\sim} \mathfrak{p}_9+7\mathfrak{p}_5, \quad \left(\tilde{B} \rhd \tilde{C}\right)_{\sigma_1}\left(\tilde{\mathcal{B}} \rhd \tilde{\mathcal{C}}\right)_{\sigma_1} \overset{-208}{\sim} 3\mathfrak{r}_8-2\mathfrak{p}_1,$$

$$\frac{1}{4}\left(\tilde{B} \rhd \tilde{D}\right)_{\sigma_1}\left(\tilde{\mathcal{B}} \rhd \tilde{\mathcal{D}}\right)_{\sigma_1} \overset{-91}{\sim} 3\mathfrak{p}_9+\mathfrak{r}_3, \quad \frac{1}{4}\left(\tilde{C} \rhd \tilde{D}\right)_{\sigma_1}\left(\tilde{\mathcal{C}} \rhd \tilde{\mathcal{D}}\right)_{\sigma_1} \overset{100}{\sim} 2\mathfrak{p}_8-\mathfrak{r}_1,$$

$$\frac{1}{4}\left(\tilde{A} \lhd \tilde{B}\right)_{\sigma_1}\left(\tilde{\mathcal{A}} \lhd \tilde{\mathcal{B}}\right)_{\sigma_1} \overset{-20}{\sim} 5\mathfrak{p}_8, \quad \frac{1}{16}\left(\tilde{A} \lhd \tilde{C}\right)_{\sigma_1}\left(\tilde{\mathcal{A}} \lhd \tilde{\mathcal{C}}\right)_{\sigma_1} \overset{9}{\sim} \mathfrak{p}_8,$$

$$\frac{1}{4}\left(\tilde{A} \lhd \tilde{D}\right)_{\sigma_1}\left(\tilde{\mathcal{A}} \lhd \tilde{\mathcal{D}}\right)_{\sigma_1} \overset{12}{\sim} \mathfrak{p}_{10}, \quad \frac{1}{4}\left(\tilde{B} \lhd \tilde{C}\right)_{\sigma_1}\left(\tilde{\mathcal{B}} \lhd \tilde{\mathcal{C}}\right)_{\sigma_1} \overset{120}{\sim} 4\mathfrak{p}_7+\mathfrak{p}_5,$$

$$\frac{1}{36}\left(\tilde{B} \lhd \tilde{D}\right)_{\sigma_1}\left(\tilde{\mathcal{B}} \lhd \tilde{\mathcal{D}}\right)_{\sigma_1} \overset{5}{\sim} 2\mathfrak{p}_7, \quad \frac{1}{4}\left(\tilde{C} \lhd \tilde{D}\right)_{\sigma_1}\left(\tilde{\mathcal{C}} \lhd \tilde{\mathcal{D}}\right)_{\sigma_1} \overset{281}{\sim} \mathfrak{r}_9-5\mathfrak{p}_2.$$

Proof. Since the sums $\left(\tilde{A} \odot \tilde{B}\right)_{\sigma_1}$ and $\left(\tilde{\mathcal{A}} \odot \tilde{\mathcal{B}}\right)_{\sigma_1}$ are $\frac{1}{2}\pi_4\left(59\pi_3+9\pi_0\right)$ and $\frac{1}{2}\rho_4\left(59\rho_3+9\rho_0\right)$, it follows that the sum of $\left(\tilde{A} \odot \tilde{B}\right)_{\sigma_1}\left(\tilde{\mathcal{A}} \odot \tilde{\mathcal{B}}\right)_{\sigma_1}$ and 256 is $\frac{1}{4}\mathfrak{p}_8\left(59\pi_3+9\pi_0\right)\left(59\rho_3+9\rho_0\right)+256$, *i. e.*,

$$\frac{\left(787407+352139\sqrt{5}\right)\left(638\Phi^8+787407-352139\sqrt{5}\right)^2}{4070440\Phi^8}$$

which is the square of $2\mathfrak{r}_{11}-\mathfrak{p}_2$. This is the outline of the proof of the first relation.

Theorem 19 *The following relations hold for the sequences* $\tilde{A},\cdots,\tilde{\mathcal{D}}$:

$$\left(\tilde{A} \odot \tilde{B}\right)_{\sigma_1^*}\left(\tilde{\mathcal{A}} \odot \tilde{\mathcal{B}}\right)_{\sigma_1^*} \overset{256}{\sim} \mathfrak{p}_{14}-2\mathfrak{r}_5, \quad \frac{1}{16}\left(\tilde{A} \odot \tilde{C}\right)_{\sigma_1^*}\left(\tilde{\mathcal{A}} \odot \tilde{\mathcal{C}}\right)_{\sigma_1^*} \overset{1}{\sim} 2\mathfrak{p}_7+\mathfrak{p}_2,$$

$$\left(\tilde{A} \odot \tilde{D}\right)_{\sigma_1^*}\left(\tilde{\mathcal{A}} \odot \tilde{\mathcal{D}}\right)_{\sigma_1^*} \overset{129}{\sim} \mathfrak{p}_{10}+4\mathfrak{r}_5, \quad \frac{1}{4}\left(\tilde{B} \odot \tilde{C}\right)_{\sigma_1^*}\left(\tilde{\mathcal{B}} \odot \tilde{\mathcal{C}}\right)_{\sigma_1^*} \overset{236}{\sim} \mathfrak{p}_{11}+2\mathfrak{p}_3,$$

$$\left(\tilde{B} \odot \tilde{D}\right)_{\sigma_1^*}\left(\tilde{\mathcal{B}} \odot \tilde{\mathcal{D}}\right)_{\sigma_1^*} \overset{945}{\sim} \mathfrak{p}_{12}+7\mathfrak{p}_5, \quad \frac{1}{4}\left(\tilde{C} \odot \tilde{D}\right)_{\sigma_1^*}\left(\tilde{\mathcal{C}} \odot \tilde{\mathcal{D}}\right)_{\sigma_1^*} \overset{4}{\sim} \mathfrak{r}_7-2\mathfrak{r}_2,$$

$$\frac{1}{4}\left(\tilde{A}\rhd\tilde{B}\right)_{\sigma_1^*}\left(\tilde{\mathcal{A}}\rhd\tilde{\mathcal{B}}\right)_{\sigma_1^*}\overset{44}{\sim}3\mathfrak{p}_8,\qquad\left(\tilde{A}\rhd\tilde{C}\right)_{\sigma_1^*}\left(\tilde{\mathcal{A}}\rhd\tilde{\mathcal{C}}\right)_{\sigma_1^*}\overset{208}{\sim}3\mathfrak{p}_8+2\mathfrak{t}_2,$$

$$\frac{1}{4}\left(\tilde{A}\rhd\tilde{D}\right)_{\sigma_1^*}\left(\tilde{\mathcal{A}}\rhd\tilde{\mathcal{D}}\right)_{\sigma_1^*}\overset{72}{\sim}\mathfrak{p}_{10}-2\mathfrak{p}_2,\quad\left(\tilde{B}\rhd\tilde{C}\right)_{\sigma_1^*}\left(\tilde{\mathcal{B}}\rhd\tilde{\mathcal{C}}\right)_{\sigma_1^*}\overset{-144}{\sim}9\mathfrak{p}_7+\mathfrak{p}_0,$$

$$\frac{1}{4}\left(\tilde{B}\rhd\tilde{D}\right)_{\sigma_1^*}\left(\tilde{\mathcal{B}}\rhd\tilde{\mathcal{D}}\right)_{\sigma_1^*}\overset{-59}{\sim}\mathfrak{p}_{11}+\mathfrak{p}_2,\quad\frac{1}{4}\left(\tilde{C}\rhd\tilde{D}\right)_{\sigma_1^*}\left(\tilde{\mathcal{C}}\rhd\tilde{\mathcal{D}}\right)_{\sigma_1^*}\overset{100}{\sim}\mathfrak{p}_9-5\mathfrak{p}_2,$$

$$\frac{1}{4}\left(\tilde{A}\lhd\tilde{B}\right)_{\sigma_1^*}\left(\tilde{\mathcal{A}}\lhd\tilde{\mathcal{B}}\right)_{\sigma_1^*}\overset{8}{\sim}\mathfrak{p}_5,\qquad\frac{1}{16}\left(\tilde{A}\lhd\tilde{C}\right)_{\sigma_1^*}\left(\tilde{\mathcal{A}}\lhd\tilde{\mathcal{C}}\right)_{\sigma_1^*}\overset{28}{\sim}\mathfrak{p}_4,$$

$$\left(\tilde{A}\lhd\tilde{D}\right)_{\sigma_1^*}\left(\tilde{\mathcal{A}}\lhd\tilde{\mathcal{D}}\right)_{\sigma_1^*}\overset{224}{\sim}\mathfrak{p}_9+\mathfrak{t}_3,\qquad\frac{1}{4}\left(\tilde{B}\lhd\tilde{C}\right)_{\sigma_1^*}\left(\tilde{\mathcal{B}}\lhd\tilde{\mathcal{C}}\right)_{\sigma_1^*}\overset{204}{\sim}\mathfrak{p}_8,$$

$$\frac{1}{4}\left(\tilde{B}\lhd\tilde{D}\right)_{\sigma_1^*}\left(\tilde{\mathcal{B}}\lhd\tilde{\mathcal{D}}\right)_{\sigma_1^*}\overset{69}{\sim}2\mathfrak{p}_8,\qquad\frac{1}{25}\left(\tilde{C}\lhd\tilde{D}\right)_{\sigma_1^*}\left(\tilde{\mathcal{C}}\lhd\tilde{\mathcal{D}}\right)_{\sigma_1^*}\overset{1}{\sim}\mathfrak{p}_4.$$

Proof. Since the sums $\left(\tilde{A}\odot\tilde{B}\right)_{\sigma_1^*}$ and $\left(\tilde{\mathcal{A}}\odot\tilde{\mathcal{B}}\right)_{\sigma_1}$ are $\frac{1}{2}\pi_4\left(53\pi_3+7\pi_0\right)$ and $\frac{1}{2}\rho_4\left(53\rho_3+7\rho_0\right)$, it follows that the sum of $\left(\tilde{A}\odot\tilde{B}\right)_{\sigma_1^*}\left(\tilde{\mathcal{A}}\odot\tilde{\mathcal{B}}\right)_{\sigma_1^*}$ and 256 is $\frac{1}{4}\mathfrak{p}_8\left(53\pi_3+7\pi_0\right)\left(53\rho_3+7\rho_0\right)+256$, *i.e.*,

$$\frac{\left(629487+281515\sqrt{5}\right)\left(638\Phi^8+629487-281515\sqrt{5}\right)^2}{4070440\Phi^8}$$

which is the square of $\mathfrak{p}_{14}-2\mathfrak{t}_5$. This is the outline of the proof of the first relation.

7. Squares from the Products $\downarrow$ and $\uparrow$

This section uses the binary operations $\downarrow$ and $\uparrow$ defined by

$$(a,b,c)\downarrow(d,e,f)=(bf-ce,cd-af,ae-bd),$$
$$(a,b,c)\uparrow(d,e,f)=(bf+ce,cd+af,ae+bd).$$

Note that restricted on the standard Euclidean 3 -space $\mathbb{R}^3$ the product $\downarrow$ is the familiar vector cross-product.

Theorem 20. *The following relations hold for the triples* $A,\cdots,\tilde{\mathcal{D}}$:

$$\frac{1}{16}\left(A\downarrow\tilde{A}\right)_{\sigma_1}\left(\mathcal{A}\downarrow\tilde{\mathcal{A}}\right)_{\sigma_1}\overset{9}{\sim}\mathfrak{t}_6,\qquad\frac{1}{64}\left(B\downarrow\tilde{B}\right)_{\sigma_1}\left(\mathcal{B}\downarrow\tilde{\mathcal{B}}\right)_{\sigma_1}\overset{0}{\sim}\mathfrak{p}_8,$$

$$\frac{1}{64}\left(C\downarrow\tilde{C}\right)_{\sigma_1}\left(\mathcal{C}\downarrow\tilde{\mathcal{C}}\right)_{\sigma_1}\overset{4}{\sim}\mathfrak{t}_2,\qquad\frac{1}{16}\left(D\downarrow\tilde{D}\right)_{\sigma_1}\left(\mathcal{D}\downarrow\tilde{\mathcal{D}}\right)_{\sigma_1}\overset{9}{\sim}\mathfrak{t}_2.$$

Proof. Since the sums $\left(A\downarrow\tilde{A}\right)_{\sigma_1}$ and $\left(\mathcal{A}\downarrow\tilde{\mathcal{A}}\right)_{\sigma_1}$ are $4\pi_2\rho_4$ and $20\pi_4\rho_2$, it follows that the sum of $\frac{1}{16}\left(A\downarrow\tilde{A}\right)_{\sigma_1}\left(\mathcal{A}\downarrow\tilde{\mathcal{A}}\right)_{\sigma_1}$ and 9 is $5\mathfrak{p}_4\mathfrak{p}_8+9$, *i.e.*, the square of $\mathfrak{t}_6$. This concludes the proof of the first relation.

Theorem 21. *The following relations hold for the triples* $A,\cdots,\tilde{\mathcal{D}}$:

$$\frac{1}{16}\left(A\downarrow\tilde{A}\right)_{\sigma_1^*}\left(\mathcal{A}\downarrow\tilde{\mathcal{A}}\right)_{\sigma_1^*}\overset{-71}{\sim}4\mathfrak{p}_8+5\mathfrak{p}_6,\quad\frac{1}{1024}\left(B\downarrow\tilde{B}\right)_{\sigma_1^*}\left(\mathcal{B}\downarrow\tilde{\mathcal{B}}\right)_{\sigma_1^*}\overset{1}{\sim}\mathfrak{p}_9,$$

$$\frac{1}{4}\left(C\downarrow\tilde{C}\right)_{\sigma_1^*}\left(\mathcal{C}\downarrow\tilde{\mathcal{C}}\right)_{\sigma_1^*}\overset{-476}{\sim}\mathfrak{t}_9-\mathfrak{t}_{-2},\quad\frac{1}{16}(D\downarrow\tilde{D})_{\sigma_1^*}\left(\mathcal{D}\downarrow\tilde{\mathcal{D}}\right)_{\sigma_1^*}\overset{-79}{\sim}2\mathfrak{p}_8+5\mathfrak{p}_2.$$

Proof. Since the sums $\left(A\downarrow\tilde{A}\right)_{\sigma_1}$ and $\left(\mathcal{A}\downarrow\tilde{\mathcal{A}}\right)_{\sigma_1}$ are $-\frac{4}{5}\left(4\mathfrak{t}_8+5\mathfrak{t}_6+7\right)$ and $-4\left(4\mathfrak{t}_8+5\mathfrak{t}_6-7\right)$, it follows that the difference of the product $\frac{1}{16}\left(A\downarrow\tilde{A}\right)_{\sigma_1^*}\left(\mathcal{A}\downarrow\tilde{\mathcal{A}}\right)_{\sigma_1^*}$ and 71 is

$\frac{1}{5}\left[(4\mathfrak{r}_8+5\mathfrak{r}_6)^2-404\right]$ which simplifies to

$$\frac{\left(38541+17236\sqrt{5}\right)\left(101\Phi^8-38541+17236\sqrt{5}\right)^2}{51005\Phi^8}$$ *i.e.*, to

the square of $4\mathfrak{p}_8+5\mathfrak{p}_6$. This proves the first relation. Notice that, in our final result, the third added constant value is in all cases the complete square.

Theorem 22. *The following relations hold for the triples* $A,\cdots,\mathcal{D}$:

$$\frac{1}{4}(A\uparrow\mathcal{A})_{\sigma_1}(B\uparrow\mathcal{B})_{\sigma_1}\overset{8^2}{\sim}19\mathfrak{p}_8,\qquad (A\uparrow\mathcal{A})_{\sigma_1}(C\uparrow\mathcal{C})_{\sigma_1}\overset{40^2}{\sim}13\mathfrak{p}_8+9\mathfrak{p}_2,$$

$$(A\uparrow\mathcal{A})_{\sigma_1}(D\uparrow\mathcal{D})_{\sigma_1}\overset{30^2}{\sim}7\mathfrak{r}_8+9\mathfrak{p}_2,\quad \frac{1}{4}(B\uparrow\mathcal{B})_{\sigma_1}(C\uparrow\mathcal{C})_{\sigma_1}\overset{36^2}{\sim}4\mathfrak{p}_{10}+3\mathfrak{p}_2,$$

$$\frac{1}{4}(B\uparrow\mathcal{B})_{\sigma_1}(D\uparrow\mathcal{D})_{\sigma_1}\overset{29^2}{\sim}\mathfrak{r}_{10}+18\mathfrak{p}_6,\qquad \frac{1}{4}(C\uparrow\mathcal{C})_{\sigma_1}(D\uparrow\mathcal{D})_{\sigma_1}\overset{5^2}{\sim}31\mathfrak{p}_4,$$

$$\frac{1}{4}(A\uparrow\mathcal{B})_{\sigma_1}(B\uparrow\mathcal{A})_{\sigma_1}\overset{8^2}{\sim}21\mathfrak{p}_8,\qquad (A\uparrow\mathcal{C})_{\sigma_1}(C\uparrow\mathcal{A})_{\sigma_1}\overset{48^2}{\sim}3\mathfrak{p}_8+20\mathfrak{r}_5,$$

$$\frac{1}{25}(A\uparrow\mathcal{D})_{\sigma_1}(D\uparrow\mathcal{A})_{\sigma_1}\overset{6^2}{\sim}\mathfrak{r}_9-3\mathfrak{p}_3,\quad (B\uparrow\mathcal{C})_{\sigma_1}(C\uparrow\mathcal{B})_{\sigma_1}\overset{72^2}{\sim}11\mathfrak{r}_8-6\mathfrak{p}_1,$$

$$\frac{1}{4}(B\uparrow\mathcal{D})_{\sigma_1}(D\uparrow\mathcal{B})_{\sigma_1}\overset{27^2}{\sim}\mathfrak{r}_{10}+26\mathfrak{p}_6,\qquad (C\uparrow\mathcal{D})_{\sigma_1}(D\uparrow\mathcal{C})_{\sigma_1}\overset{19^2}{\sim}63\mathfrak{p}_4.$$

Proof. Since the sums $(A\uparrow\mathcal{A})_{\sigma_1}$ and $(B\uparrow\mathcal{B})_{\sigma_1}$ are $2(3\mathfrak{p}_{11}-2\mathfrak{r}_4)$ and $2(\mathfrak{p}_{14}+12\mathfrak{p}_8)$, it follows that the sum of the product $\frac{1}{4}(A\uparrow\mathcal{A})_{\sigma_1}(B\uparrow\mathcal{B})_{\sigma_1}$ and 64 is $(3\mathfrak{p}_{11}-2\mathfrak{r}_4)(\mathfrak{p}_{14}+12\mathfrak{p}_8)+64$ which simplifies to the square of $19\mathfrak{p}_8$. This proves the first relation.

8. References

[1] E. Brown, "Sets in Which $xy+k$ Is Always a Square," *Mathematics of Computation*, Vol. 45, No. 172, 1985, pp. 613-620.

[2] N. Sloane, On-Line Encyclopedia of Integer Sequences. http://www.research.att.com/ ~ njas/sequences/.

[3] L. Euler, "Commentationes Arithmeticae I," Opera Omnia, Series I, volume II, B.G. Teubner, Basel, 1915.

[4] Z. Čerin, "On Pencils of Euler Triples I," (in press).

[5] Z. Čerin, "On Pencils of Euler Triples II," (in press).

[6] M. Radić, "A Definition of Determinant of Rectangular Matrix," *Glasnik Matematic*, Vol. 1, No. 21, 1966, pp. 17-22.