

Left Eigenvector of a Stochastic Matrix

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Abstract

We determine the left eigenvector of a stochastic matrix M associated to the eigenvalue 1 in the commutative and the noncommutative cases. In the commutative case, we see that the eigenvector associated to the eigenvalue 0 is (N_1, \dots, N_n) , where N_i is the i -th principal minor of $N = M - I_n$, where I_n is the identity matrix of dimension n . In the noncommutative case, this eigenvector is $(P_1^{-1}, \dots, P_n^{-1})$, where P_i is the sum in $\mathbb{Q}\langle\langle a_{ij} \rangle\rangle$ of the corresponding labels of nonempty paths starting from i and not passing through i in the complete directed graph associated to M .

Keywords: Generic Stochastic Noncommutative Matrix, Commutative Matrix, Left Eigenvector Associated To The Eigenvalue 1, Skew Field, Automata

1. Introduction

It is well known that 1 is one of the eigenvalue of a stochastic matrix (*i.e.* the sum of the elements of each row is equal to 1) and its associated right eigenvector is the vector $(1, 1, \dots, 1)^T$. But, what is the left eigenvector associated to 1?

In the commutative case, this eigenvector is the vector (M_1, \dots, M_n) , where M_i is the i -th principal minor of the matrix. This formula is known in probability theory; it amounts to finding the stationary distribution of the finite Markov chain whose transition matrix is the stochastic irreducible matrix.

In the noncommutative case, we must involve inverses of elements of the skew field and as these may be undefined, we take a *generic noncommutative stochastic matrix*: this is the matrix (a_{ij}) of noncommuting variables a_{ij} subject only to the stochastic identities; *i.e.* the sum of each row equals one.

We work in the *free field* generated by these variables (in the sense of Paul Cohn), which we call the *stochastic free field*. Considering the complete digraph on the set $\{1, \dots, n\}$, let M_i be the set of paths from i to i . Let P_i be the paths starting from i and not passing through i again. We identify P_i with the noncommutative power series which is equal to the sum of all the words in P_i and we still denote this series by P_i . Next we show that the elements P_i^{-1} can be evaluated in the

stochastic free field and that the vector

$(P_1^{-1}, \dots, P_n^{-1})$ is fixed by our matrix; moreover, the sum of the P_i^{-1} is equal to 1, hence they form a kind of noncommutative limiting probability.

These results have been proved in [1] but the proof proposed in this paper are completely different.

Indeed for the major part the proof in these two instances, we use elementary operations on the rows and the columns of the matrix.

2. Commutative Case

The commutative case is well known in probability theory. Indeed, we calculate the limit probability of a finite Markov chain by replacing the stochastic matrix M by $M - I$, where I is the identity matrix of the appropriate dimension. For this, we use the Markov chain tree Theorem, where this calculus is expressed in terms of spanning trees. The Markov chain tree theorem gives a formula for the stationary distribution of a finite Markov chain. Equivalently, this formula gives a row vector fixed by a matrix fixing $(1, 1, \dots, 1)^T$. This theorem is attributed to Kirchoff by Persi Diaconis, who gives a probabilistic proof of it (see [2] p. 443 and 444). See also [3,4].

The proof uses only the definition of the determinant which involves the principal minors.

Proposition 1 Let M be a stochastic matrix and

$N = M - I_n = (b_{ij})_{1 \leq i, j \leq n}$; i.e. for all $i = 1, \dots, n$, $\sum_{j=1}^n b_{ij} = 0$. Then (N_1, \dots, N_n) , where N_i is the i -th principal minor of N , is the left eigenvector associated to the eigenvalue 0.

Proof. By definition of N , we have $\det(N) = 0$. We show that

$$(N_1, N_2, \dots, N_n)N = 0$$

We verify this Equation with the first column of N .

$$\begin{aligned}
 & N_1 b_{11} + \dots + N_n b_{n1} \\
 &= \begin{vmatrix} b_{22} & b_{23} & \dots & b_{2n} \\ b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n2} & b_{n3} & \dots & b_{nn} \end{vmatrix} b_{11} + \dots + \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1,n-1} \\ b_{21} & b_{22} & \dots & b_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,1} & b_{n-1,2} & \dots & b_{n-1,n-1} \end{vmatrix} b_{n1} \\
 &= \begin{vmatrix} b_{22} & b_{23} & \dots & b_{2n} \\ b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n2} & b_{n3} & \dots & b_{nn} \end{vmatrix} b_{11} + \dots + \begin{vmatrix} -\sum_{j=2}^n b_{1j} & b_{12} & \dots & b_{1,n-1} \\ -\sum_{j=2}^n b_{2j} & b_{22} & \dots & b_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_{j=2}^n b_{n-1,j} & b_{n-1,2} & \dots & b_{n-1,n-1} \end{vmatrix} b_{n1} \\
 &= \begin{vmatrix} b_{22} & b_{23} & \dots & b_{2n} \\ b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n2} & b_{n3} & \dots & b_{nn} \end{vmatrix} b_{11} + \dots - \underbrace{\begin{vmatrix} b_{12} & b_{12} & \dots & b_{1,n-1} \\ b_{22} & b_{22} & \dots & b_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,2} & b_{n-1,2} & \dots & b_{n-1,n-1} \end{vmatrix}}_{=0} b_{n1} - \dots \\
 &- \underbrace{\begin{vmatrix} b_{1,n-1} & b_{12} & \dots & b_{1,n-1} \\ b_{2,n-1} & b_{22} & \dots & b_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,n-1} & b_{n-1,2} & \dots & b_{n-1,n-1} \end{vmatrix}}_{=0} b_{n1} - \begin{vmatrix} b_{1n} & b_{12} & \dots & b_{1,n-1} \\ b_{2n} & b_{22} & \dots & b_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,n} & b_{n-1,2} & \dots & b_{n-1,n-1} \end{vmatrix} b_{n1} \\
 &= \begin{vmatrix} b_{22} & b_{23} & \dots & b_{2n} \\ b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n2} & b_{n3} & \dots & b_{nn} \end{vmatrix} b_{11} - \dots - \begin{vmatrix} b_{1n} & b_{12} & \dots & b_{1,n-1} \\ b_{2n} & b_{22} & \dots & b_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,n} & b_{n-1,2} & \dots & b_{n-1,n-1} \end{vmatrix} b_{n1} \\
 &= \begin{vmatrix} b_{22} & b_{23} & \dots & b_{2n} \\ b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n2} & b_{n3} & \dots & b_{nn} \end{vmatrix} b_{11} - \dots - (-1)^{n-2} \begin{vmatrix} b_{12} & b_{13} & \dots & b_{1n} \\ b_{22} & b_{23} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n} \end{vmatrix} b_{n1} \\
 &= \begin{vmatrix} b_{22} & b_{23} & \dots & b_{2n} \\ b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n2} & b_{n3} & \dots & b_{nn} \end{vmatrix} b_{11} - \dots + (-1)^{n+1} \begin{vmatrix} b_{12} & b_{13} & \dots & b_{1n} \\ b_{22} & b_{23} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n} \end{vmatrix} b_{n1} \\
 &= \det(N) = 0
 \end{aligned}$$

Remark 2 If M is reducible, then this proposition is asserting that the zero vector is an eigenvector. If M is

irreducible and stochastic, then its adjoint will be of the form $(1, \dots, 1)u^T$, where u^T is a null vector. The pro-

position follows from the cofactor formula for the adjoint.

3. Noncommutative Case

In [1], the authors have prove in two manners that are actually similar, Theorem 9. In the first, results from variable-length prefix codes are required; and the second deals with general variable-length codes, not necessarily prefix. Moreover, in Appendix 2 of [1], we see how the theory of quasideterminants may be used to obtain these results on noncommutative matrices.

The major part of the new proof of this Theorem involves only the elementary operations on the rows and on the columns of the stochastic matrix. Therefore, we need the following results.

3.1. Languages and Series

Let A be a finite alphabet and A^* be the free monoid generated by A . A *language* is a subset of a free monoid A^* . A language is *rational* if it is obtained from finite languages by the operations (called rational) *union*, *product* (concatenation) and *star*. The product of two languages $L_1 L_2$ is $\{w_1 w_2 \mid w_1 \in L_1, w_2 \in L_2\}$, and the star of L is $L^* = \{w_1 \cdots w_n \mid w_i \in L, n \geq 0\} = \bigcup_{n \geq 0} L^n$. Rational languages may be obtained by using only *unambiguous rational operations*; these are: disjoint union, unambiguous product (meaning that if $w \in L_1 L_2$, then w has a unique factorization $w = w_1 w_2$, $w_i \in L_i$) and the star L^* restricted to languages which are free submonoids of A^* .

A *formal series* is an element of the \mathbb{Q} -algebra of noncommutative series $\mathbb{Q}\langle\langle A \rangle\rangle$, where A is a set of noncommuting variables. A *rational series* is an element of the smallest subalgebra of $\mathbb{Q}\langle\langle A \rangle\rangle$, which contains the \mathbb{Q} -algebra of noncommutative polynomials $\mathbb{Q}\langle A \rangle$, and which is closed under the operation

$$S \mapsto S^* = \sum_{n=0}^{\infty} S^n = (1 - S)^{-1}$$

which is defined if S has zero constant term. We denote by $\mathbb{Q}\langle\langle A \rangle\rangle^{rat}$ the \mathbb{Q} -algebra of rational series.

Let L be a rational language. Since L may be obtained by unambiguous rational expressions, it follows

that its *characteristic series* $\sum_{w \in L} w \in \mathbb{Q}\langle\langle A \rangle\rangle$ is rational. We shall identify a language and its characteristic series. This is exposed in [5] or [6].

3.2. Free Field

Let \mathbb{K} be a field. If the multiplicative group of \mathbb{K} is commutative, then \mathbb{K} is a commutative field. Else, \mathbb{K} is called *skew field*. The ring of rational formal series in noncommutative variables $\mathbb{Q}^{rat}\langle\langle A \rangle\rangle$ is not a skew field. However, skew fields containing $\mathbb{Q}\langle A \rangle$ do exist. One of them is the *free field* (in the sense of Cohn), denoted \mathcal{F} . The free field is the noncommutative analogue of the field of fractions of the ring of commutative polynomials. There are several constructions of the free field: Amitsur [7], Bergman [8], Malcolmson [9], Cohn [10] and [11].

A square matrix of order n is *full* if it can't expressed as a product of matrices $n \times n-1$ by $n-1 \times n$.

Theorem 3 ([11], Thm 4.5.8.) *Each full matrix M with coefficients in $\mathbb{K}\langle X \rangle$ is invertible in the free field.*

A square matrix of order n is called *hollow* if it contains a $p \times q$ block of zeros such that $p + q \geq n + 1$.

Example 4 The matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is hollow since

there is a 2×2 block of 0 and $4 > 3$.

Proposition 5 ([11], Prop. 4.5.4) *A hollow matrix is not full.*

Proposition 6 Take $\lambda \in \mathcal{D}^{1 \times n}$, $M \in \mathcal{D}^{n \times n}$ and $\gamma \in \mathcal{D}^{n \times 1}$, where \mathcal{D} is a skew field. We suppose that

M is invertible. Then $\begin{bmatrix} M & \gamma \\ \lambda & 0 \end{bmatrix}$ is invertible if and only if $\lambda M^{-1} \gamma \neq 0$.

Proof.

(\Leftarrow) Suppose that $c = \lambda M^{-1} \gamma \neq 0$, then the inverse of $\begin{bmatrix} M & \gamma \\ \lambda & 0 \end{bmatrix}$ is

$$\begin{bmatrix} M^{-1} - M^{-1} \gamma c^{-1} \lambda M^{-1} & M^{-1} \gamma c^{-1} \\ c^{-1} \lambda M^{-1} & -c^{-1} \end{bmatrix}$$

Indeed, this matrix is the right inverse of M since

$$\begin{aligned} & \begin{bmatrix} M & \gamma \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} M^{-1} - M^{-1} \gamma c^{-1} \lambda M^{-1} & M^{-1} \gamma c^{-1} \\ c^{-1} \lambda M^{-1} & -c^{-1} \end{bmatrix} \\ &= \begin{bmatrix} M(M^{-1} - M^{-1} \gamma c^{-1} \lambda M^{-1}) + \gamma c^{-1} \lambda M^{-1} & M(M^{-1} \gamma c^{-1}) - \gamma c^{-1} \\ \lambda(M^{-1} - M^{-1} \gamma c^{-1} \lambda M^{-1}) & \lambda M^{-1} \gamma c^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 - \gamma c^{-1} \lambda M^{-1} + \gamma c^{-1} \lambda M^{-1} & \gamma c^{-1} - \gamma c^{-1} \\ \lambda M^{-1} - \underbrace{\lambda M^{-1} \gamma c^{-1} \lambda M^{-1}}_{=c} & \underbrace{\gamma M^{-1} \gamma c^{-1}}_{=c} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

This matrix is also the left inverse of M since

$$\begin{aligned} & \begin{bmatrix} M^{-1} - M^{-1}\gamma c^{-1}\lambda M^{-1} & M^{-1}\gamma c^{-1} \\ c^{-1}\lambda M^{-1} & -c^{-1} \end{bmatrix} \begin{bmatrix} M & \gamma \\ \lambda & 0 \end{bmatrix} \\ &= \begin{bmatrix} (M^{-1} - M^{-1}\gamma c^{-1}\lambda M^{-1})M + M^{-1}\gamma c^{-1}\lambda & (M^{-1} - M^{-1}\gamma c^{-1}\lambda M^{-1})\gamma \\ (c^{-1}\lambda M^{-1})M - c^{-1}\lambda & (c^{-1}\lambda M^{-1})\gamma \end{bmatrix} \\ &= \begin{bmatrix} 1 - M^{-1}\gamma c^{-1}\lambda + M^{-1}\gamma c^{-1}\lambda & M^{-1}\gamma - M^{-1}\gamma c^{-1}\lambda M^{-1}\gamma \\ c^{-1}\lambda - c^{-1}\lambda & c^{-1}\lambda M^{-1}\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

(\Rightarrow) Suppose that $\begin{bmatrix} M & \gamma \\ \lambda & 0 \end{bmatrix}$ is invertible. Let

$\begin{bmatrix} M' & \gamma' \\ \lambda' & \alpha \end{bmatrix}$ be its inverse. Then we have

$$\begin{bmatrix} M & \gamma \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} \gamma' \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Hence $M\gamma' + \gamma\alpha = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0$ and $\lambda\gamma' = 1$. It follows

that

$$\begin{aligned} \gamma' &= -M^{-1}\gamma\alpha \\ \Rightarrow \lambda\gamma' &= -\lambda M^{-1}\gamma\alpha = 1 \\ \Rightarrow \lambda M^{-1}\gamma &\neq 0 \end{aligned}$$

From this proposition, we deduce the following result.

Corollary 7 Let λ , M and γ be matrices with coefficients in $\mathbb{K}\langle A \rangle$ where M is full. If $\begin{bmatrix} M & \gamma \\ \lambda & 0 \end{bmatrix}$ is not full (in particular, if $\begin{bmatrix} M & \gamma \\ \lambda & 0 \end{bmatrix}$ is equivalent by elementary operations on the lines and on the columns to a hollow matrix) then $\lambda M^{-1}\gamma = 0$ in the free field.

3.3. Generic Stochastic Matrices

Let $M = (a_{ij})_{1 \leq i, j \leq n}$ be a generic noncommutative matrix; i.e. the a_{ij} are noncommutative variables. We denote \mathcal{F} the corresponding free field. We associate to M the matrix S : it's exactly the same matrix of which the coefficients satisfy the stochastic identities

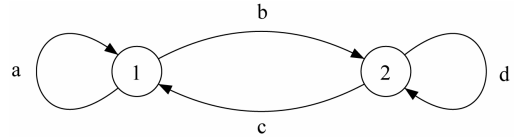
$$\forall i = 1, \dots, n, \sum_{j=1}^n a_{ij} = 1 \quad (1)$$

In other words, the sum of each line of S is equal to 1; hence S is a stochastic matrix. We call S a *generic noncommutative stochastic matrix*. The algebra over \mathbb{Q} generated by its coefficients is a free associative algebra, since it is isomorph to the algebra $\mathbb{Q}\langle a_{ij}, i \neq j \rangle$. Indeed, we can eliminate the a_{ii} with the stochastic relations (1). We denote this algebra by $\mathbb{Q}\langle a_{ij} / (1) \rangle$. Hence, there is a corresponding free field called *stochastic free field* denoted \mathcal{S} .

3.4. Paths

Consider the set of nonempty paths in the complete directed graph with a set of vertices $\{1, \dots, n\}$ starting from i and not passing through i ; we denote by P_i the sum in $\mathbb{Q}\langle a_{ij} \rangle$ of all the corresponding words. It is classically a rational series, and thus defines an element of the free field \mathcal{F} .

Example 8 $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The graph is



then

$$P_1 = 1 + bd^*, \quad P_2 = 1 + ca^*$$

We shall also consider rational expressions over any skew field D , and say that such an expression is *evaluable in D* if it can be evaluated without inversion of 0. If the elements of D appearing in the rational expression are actually in a subring R of D , we say that the expression is *over R* .

There is a canonical embedding of $\mathbb{Q}\langle A \rangle^{rat}$ into \mathcal{F} , which can be seen as follows: let S be any rational series, we replace in it the operation T^* by $(1 - T)^{-1}$; then one obtains a rational expression in \mathcal{F} , which is evaluable in \mathcal{F} and represents the image of S under the embedding $\mathbb{Q}\langle A \rangle^{rat} \hookrightarrow \mathcal{F}$. Thus, each rational

language and each rational series is naturally an element of the free field. See [12].

By Theorem 1 of [1], the P_i can be evaluated in the stochastic free field.

We can now state the main result of this paper.

Theorem 9 Let $M = (a_{ij})_{1 \leq i, j \leq n}$ be a generic

stochastic noncommutative matrix. Let P_i be defined as above. Then

$$(P_1^{-1}, \dots, P_n^{-1})M = (P_1^{-1}, \dots, P_n^{-1}) \quad (2)$$

In other words, $(P_1^{-1}, \dots, P_n^{-1})$ is the left eigenvector associated to the eigenvalue 1 of M .

Example 10 (Continued.) Let be S be the matrix M with $a+b=1$ and $c+d=1$. We show that

$$(P_1^{-1}, P_2^{-1})M = (P_1^{-1}, P_2^{-1})$$

From this system, we deduce the two following equations:

$$P_1^{-1}a + P_2^{-1}(1-d) = P_1^{-1} \quad (3)$$

$$P_1^{-1}(1-a) + P_2^{-1}d = P_2^{-1} \quad (4)$$

From the Equation (3), we have

$$\begin{aligned} P_1^{-1}a + P_2^{-1}(1-d) &= P_1^{-1} \\ \Leftrightarrow P_2^{-1}(1-d) &= P_1^{-1}(1-a) \\ \Leftrightarrow d^*P_2 &= a^*P_1 \\ \Leftrightarrow d^*(1+ca^*) &= a^*(1+bd^*) \\ \Leftrightarrow d^* + d^*(1-d)a^* &= a^* + a^*(1-a)d^* \\ \Leftrightarrow d^* + a^* &= a^* + d^* \end{aligned} \quad (5)$$

Hence, Equation (3) is satisfied. From the Equation (4), we have

$$\begin{aligned} P_1^{-1}(1-a) + P_2^{-1}d &= P_2^{-1} \\ \Leftrightarrow P_1^{-1}(1-a) &= P_2^{-1}(1-d) \\ \Leftrightarrow a^*P_1 &= d^*P_2 \end{aligned}$$

Since we again obtain Equation (5), it follows that Equation (4) is also satisfied.

Proof of Theorem 9.

Let $M = (a_{ij})_{1 \leq i, j \leq n}$ be the matrix of the automaton \mathcal{A} and P_i be the language of the labels of the paths starting from i and not passing through i . We have

$$P_i = \sum_{j=1}^n L_{ij}, \quad i = 1, \dots, n,$$

where

$$L_{ij} = \begin{cases} 1, & \text{if } i = j \\ \sum_{k=1}^n L_{ik}a_{kj} = a_{ij} + \sum_{k=1, k \neq j}^n L_{ik}a_{kj}, & \text{if } i \neq j \end{cases}$$

Let (S_i) be the system composed of the n Equations $P_i, L_{i1}, \dots, L_{i,i-1}, L_{i,i+1}, \dots, L_{in}$. Multiplying to the left by P_i^{-1} each equation of S_i , we obtain the system

$$(T_i): \begin{cases} 1 = P_i^{-1} + \sum_{j=1, j \neq i}^n P_i^{-1}L_{ij}, & i = 1, \dots, n \\ P_i^{-1}L_{ij} = P_i^{-1}a_{ij} + \sum_{k=1, k \neq i}^n P_i^{-1}L_{ik}a_{kj} \end{cases}$$

Let $Q_{ij} = P_i^{-1}L_{ij}$, then we have

$$(T_i): \begin{cases} 1 = P_i^{-1} + \sum_{j=1, j \neq i}^n Q_{ij}, & i = 1, \dots, n \\ Q_{ij} = P_i^{-1}a_{ij} + \sum_{k=1, k \neq i}^n Q_{ik}a_{kj} \end{cases}$$

This system transforms into the following system

$$(U_i): \begin{cases} 1 = P_i^{-1} + \sum_{j=1, j \neq i}^n Q_{ij}, & i = 1, \dots, n \\ 0 = P_i^{-1}a_{ij} + Q_{ij}(a_{ij} - 1) + \sum_{\substack{k=1, k \neq i, \\ k \neq j}}^n Q_{ik}a_{kj} \end{cases}$$

We show that

$$\sum_{k=1}^n P_k^{-1}a_{k1} \equiv P_1^{-1}$$

Define

$$R = \sum_{k=1}^n P_k^{-1}a_{k1} - P_1^{-1} \quad (6)$$

We show that $R=0$. So Equation (6) is converted into the following Equation

$$R - P_1^{-1}(1-a_{11}) - \sum_{k=2}^n P_k^{-1}a_{k1} = 0 \quad (7)$$

Consider the system of the $n^2 + 1$ Equations: Equation (7) and the n systems $(U_1), \dots, (U_n)$. We have the matricial representation

$$\begin{pmatrix} R, P_1^{-1}, Q_{12}, Q_{13}, \dots, Q_{1n}, P_2^{-1}, Q_{21}, Q_{23}, \dots, Q_{2n}, \\ P_n^{-1}, Q_{n1}, \dots, Q_{n,n-1} \end{pmatrix} \cdot E = \lambda$$

$$E = \left[\begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & \dots & 0 & 0 \\ \hline 1 - a_{11} & 0 & 0 & & 0 & 1 \quad 0 \\ \hline 0 & A_1 & 0 & & 0 & \vdots \quad 0 \\ \hline -a_{21} & 0 & A_2 & & 0 & 1 \quad 0 \\ \hline 0 & 0 & 0 & & 0 & 0 \quad \vdots \quad 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline -a_{n1} & 0 & 0 & & A_n & 0 \quad 1 \\ \hline 0 & 0 & 0 & & 0 & \vdots \quad 1 \end{array} \right]$$

$$A_m = \begin{bmatrix} a_{m1} & a_{m2} & a_{m,m-1} & a_{m,m+1} & a_{mn} \\ a_{11} - 1 & a_{12} & a_{1,m-1} & a_{1,m+1} & a_{1n} \\ a_{21} & a_{22} - 1 & a_{2,m-1} & a_{2,m+1} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & a_{m-1,m-1} - 1 & a_{m-1,m+1} & a_{m-1,n} \\ a_{m+1,1} & a_{m+1,2} & a_{m+1,m-1} & a_{m+1,m+1} - 1 & a_{m+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,m-1} & a_{n,m+1} & a_{nn} - 1 \end{bmatrix}$$

where $m = 1, \dots, n$ and $\lambda = (0, \dots, 0, \underbrace{1, \dots, 1}_{n \text{ times}})$. E is a

square matrix of order $n^2 + 1$. We have $R = \lambda E^{-1} \gamma$,

$\gamma = (1, \underbrace{0, \dots, 0}_{n^2 \text{ times}})^T$ and $F = \begin{bmatrix} E & \gamma \\ \lambda & 0 \end{bmatrix}$. The order of this

matrix is $n^2 + 2$. Moreover, with the stochastic identities, we have

where the B_i are defined by

$$E = \left[\begin{array}{c|c|c|c|c|c} a_{12} + \dots + a_{1n} & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & B_1 & 0 & & 0 & 1 \quad 0 \\ \hline -a_{21} & 0 & B_2 & & 0 & 0 \quad 1 \quad 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline -a_{n1} & 0 & 0 & & B_n & 0 \quad 1 \\ \hline 0 & 0 & 0 & & 0 & \vdots \quad 1 \end{array} \right]$$

$$B_m = \begin{bmatrix} a_{m1} & a_{m2} & a_{m,m-1} & a_{m,m+1} & a_{mn} \\ -\sum_{k=1, k \neq 1}^n a_{1k} & a_{12} & a_{1,m-1} & a_{1,m+1} & a_{1n} \\ a_{21} & -\sum_{k=1, k \neq 2}^n a_{2k} & a_{2,m-1} & a_{2,m+1} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & -\sum_{\substack{k=1, k \neq 2, \\ k \neq m-1}}^n a_{m-1,k} & a_{m-1,m+1} & a_{m-1,n} \\ a_{m+1,1} & a_{m+1,2} & a_{m+1,m-1} & -\sum_{k=1, k \neq m-1}^n a_{m+1,k} & a_{m+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,m-1} & a_{n,m+1} & -\sum_{k=1, k \neq n}^n a_{nk} \end{bmatrix}$$

To show that $R = 0$, we show that the matrix F is hollow. For this, we apply elementary operations on the rows and on the columns of F until F contains a $s \times t$ block of 0 such that $s + t \geq n^2 + 3$.

$$F = \begin{bmatrix} a_{12} + \dots + a_{1n} & 0 & 0 & \dots & 0 & 0 & 1 \\ \boxed{0} & B_1 & 0 & \dots & 0 & \boxed{0} & 0 \\ -a_{21} & 0 & B_2 & \dots & 0 & \boxed{1} & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -a_{n1} & 0 & 0 & \dots & B_n & \boxed{0} & 0 \\ 0 & 0 & 0 & \dots & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}$$

Step 1. We eliminate the first row and the last column of F . We obtain the square matrix F_1 of order $n^2 + 1$ ($s + t \geq n^2 + 2$).

$$F_1 = \begin{bmatrix} a_{12} + \dots + a_{1n} & \boxed{0} & B_1 & 0 & \dots & 0 & \boxed{1} & \vdots & 0 \\ -a_{21} & \boxed{0} & 0 & B_2 & \dots & 0 & \boxed{0} & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & \boxed{0} & 0 & 0 & \dots & B_n & \boxed{0} & \vdots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}$$

$$C_m = \begin{bmatrix} -\sum_{k=1, k \neq 1}^n a_{1k} & a_{12} & a_{1,m-1} & a_{1,m+1} & a_{1n} \\ a_{21} & -\sum_{k=1, k \neq 2}^n a_{2k} & a_{2,m-1} & a_{2,m+1} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,m-1} & a_{n,m+1} & -\sum_{k=1, k \neq n}^n a_{nk} \end{bmatrix}$$

$m = 2, \dots, n$. We reduce F_2 in removing the last line and the n -th column if we start from the end. We have the following matrix F_3 of order n^2 , ($s + t \geq n + 1$).

$$F_3 = \begin{bmatrix} a_{12} + \dots + a_{1n} & \boxed{0} & B_1 & C_2 & \dots & C_n & 0 \\ -a_{21} & \boxed{0} & 0 & B_2 & \dots & 0 & \boxed{1} & \vdots & 0 \\ -a_{n1} & \boxed{0} & 0 & 0 & \dots & B_n & \boxed{0} & \vdots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}$$

Step 3. Consider the rows L_i of F_3 , where $i > n$ and $i \not\equiv 0 \pmod n$. We want that the $(n-1)$ last elements of these rows to be equals to 0. Let $i = kn + m$, where $0 \leq m \leq n-1$, then we apply the transformation

Step 2. We want a $n \times n$ block of 0 in the right upper corner of F_1 . For this, we consider the j -th row of F_1 (corresponding to the j -th row of B_1). By construction of B_1 , the first index of the elements of the j -th row of this block is j . Now, we consider the row which passes in the block B_2 such that this first index of its elements is j . Such a row exists by construction of B_2 . We repeat this operation with the row which have the first index equals to j in the blocks B_3, \dots, B_n . We add each one of these rows to the j -th row of F_1 . It follows that the n last elements of the j first rows of this new matrix are all equal to 1. From each row of the j first rows, we subtract the last row of F_1 . We obtain the following matrix

$$F_2 = \begin{bmatrix} a_{12} + \dots + a_{1n} & \boxed{0} & B_1 & C_2 & \dots & C_n & \boxed{0} & \vdots & 0 \\ -a_{21} & \boxed{0} & 0 & B_2 & \dots & 0 & \boxed{0} & \vdots & 0 \\ -a_{n1} & \boxed{0} & 0 & 0 & \dots & B_n & \boxed{0} & \vdots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}$$

where

$$L_i - L_{(k+1)n} = L_{kn+m} - L_{(k+1)n}, \text{ to obtain the matrix}$$

$$F_4 = \begin{bmatrix} a_{12} + \dots + a_{1n} & \boxed{0} & B_1 & C_2 & \dots & C_n & 0 & \dots & 0 \\ -a_{21} & \boxed{0} & 0 & B_2 & \dots & 0 & \vdots & \ddots & \vdots \\ -a_{n1} & \boxed{0} & 0 & 0 & \dots & B_n & \vdots & \vdots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots \\ -a_{n1} & 0 & 0 & 0 & \dots & D_n & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \ddots & 1 \end{bmatrix}$$

where, for all $m = 2, \dots, n$,

$$D_m = (d_{ij}^m)_{1 \leq i, j \leq n}, B_m = (b_{ij}^m), d_{ij} = b_{ij}^m - b_{nj}^m$$

We remove the $(n-1)$ rows L_i where $i > n$ and $n \nmid i$ and the $(n-1)$ last columns of F_4 to obtain the matrix F_5 of order $(s + t \geq n^2 - n + 2)$.

$$F_5 = \begin{bmatrix} a_{12} + \dots + a_{1n} & B_1 & C_2 & \dots & C_n \\ -a_{21} & & & & \\ \vdots & & & & \\ -a_{n1} & & & & \\ 0 & 0 & D'_2 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & 0 & & D'_n \\ 0 & & & & \end{bmatrix}$$

where D'_m is the matrix obtained from D_m in removing the last row.

Step 4. Denote by C_m^1 the first column of the matrix C_m and by d_m the column of F_5 , which is the extension of C_m^1 . We want C_m^1 to be transformed into a vector of 0, $m=2, \dots, n$. For this, we apply the operation $d_j + c_2 + c_3 + \dots + c_n$, where c_i is the i -th column of F_3 .

Finally we want that the first n elements of the first column of F_5 to be all 0; for this, we apply the transformation $c_1 - c_2 - \dots - c_n$. We obtain the matrix F_6 of order $n^2 - n + 1$ ($s+t \geq n^2 - n + 2$)

$$F_6 = \begin{bmatrix} 0 & B_1 & 0 & C'_2 & \dots & 0 & C'_n \\ -a_{21} & & & & & & \\ 0 & 0 & D'_2 & & & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ -a_{n1} & 0 & 0 & & & & D'_n \\ 0 & & & & & & \end{bmatrix}$$

where C'_m is the matrix obtained from C_m by removing the first column.

Step 5. We want each C'_m will be transformed into a matrix of 0. To each column q of C'_m , denoted $(C'_m)^q$ corresponding to the column $d_q = ((C'_m)^q, *, \dots, *)^T$ of F_6 . By construction of C'_m (hence of C'_m), there exists a column b_q of F_6 such that $b_q = ((C'_m)^q, *, \dots, *)^T$.

Hence, for all q , we calculate $d_q - b_q$. We obtain the matrix F_7 of order $n^2 - n + 1$ ($s+t \geq n^2 - n + 2$).

$$F_7 = \begin{bmatrix} 0 & D_1 & 0 & \dots & 0 \\ -a_{21} & & & & \\ 0 & 0 & D'_2 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & 0 & & D'_n \\ 0 & & & & \end{bmatrix}$$

Step 6. Permuting the first and the n -th column of F_7 , we obtain the matrix F_8 of order $n^2 - n + 1$

($s+t \geq n^2 - n + 2$).

$$F_8 = \begin{bmatrix} B_1^n & B'_1 & & 0 & \\ 0 & 0 & -a_{21} & D'_2 & 0 \\ \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & -a_{n1} & 0 & D'_n \end{bmatrix}$$

where B'_1 is the block obtained from B_1 by removing the first column. We find a $n \times (n-1)^2 + 1$ block of 0 located in the right upper corner and

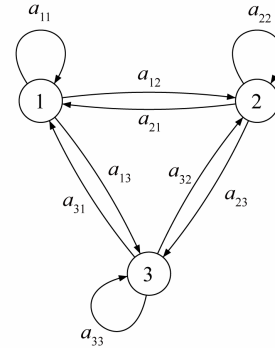
$$n + (n-1)^2 + 1 = n^2 - n + 2 \geq n^2 - n + 1 = \dim(F_8)$$

It follows that F is a hollow. From Corollary 7,

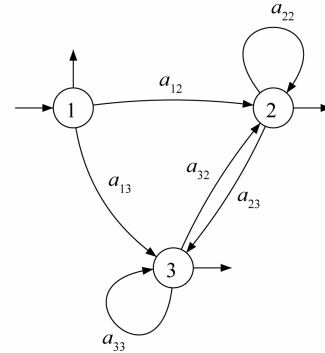
$R = \lambda E^{-1} \gamma = \sum_{k=1}^n P_k^{-1} a_{k1} - P_1^{-1} = 0$. Hence $(P_1^{-1}, \dots, P_n^{-1})$ is the left eigenvector associated to 1. \square

Example 11. Let $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a generic

noncommutative matrix in which the variables satisfy the stochastic identities $a_{i1} + a_{i2} + a_{i3} = 1$, $i=1, 2, 3$. We associate the automaton



Let P_1 be the language recognized by the automaton



Let L_{ij} be the language of words which start from i to j . We have $P_1 = L_{11} + L_{12} + L_{13}$, where

$$L_{11} = 1$$

$$\begin{aligned} L_{12} &= a_{12} + L_{12}a_{22} + L_{13}a_{32} \\ L_{13} &= a_{13} + L_{12}a_{23} + L_{13}a_{33} \end{aligned}$$

We have the system

$$(S_1): \begin{cases} P = 1 + L_{12} + L_{13} \\ L_{12} = a_{12} + L_{12}a_{22} + L_{13}a_{32} \\ L_{13} = a_{13} + L_{12}a_{23} + L_{13}a_{33} \end{cases}$$

Multiplying to left each equation of S_1 by P_1^{-1} , we obtain the system

$$(T_1): \begin{cases} 1 = P_1^{-1} + P_1^{-1}L_{12} + P_1^{-1}L_{13} \\ P_1^{-1}L_{12} = P_1^{-1}a_{12} + P_1^{-1}L_{12}a_{22} + P_1^{-1}L_{13}a_{32} \\ P_1^{-1}L_{13} = P_1^{-1}a_{13} + P_1^{-1}L_{12}a_{23} + P_1^{-1}L_{13}a_{33} \end{cases}$$

Define $Q_{ij} = P_1^{-1}L_{ij}$, then

$$T_1: \begin{cases} 1 = P_1^{-1} + Q_{12} + Q_{13} \\ Q_{12} = P_1^{-1}a_{12} + Q_{12}a_{22} + Q_{13}a_{32} \\ Q_{13} = P_1^{-1}a_{13} + Q_{12}a_{23} + Q_{13}a_{33} \end{cases}$$

We obtain

$$(U_1): \begin{cases} 1 = P_1^{-1} + Q_{12} + Q_{13} \\ 0 = P_1^{-1}a_{12} + Q_{12}(a_{22} - 1) + Q_{13}a_{32} \\ 0 = P_1^{-1}a_{13} + Q_{12}a_{23} + Q_{13}(a_{33} - 1) \end{cases}$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 - a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a_{22} - 1 & a_{23} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a_{32} & a_{33} - 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -a_{21} & 0 & 0 & a_{21} & a_{23} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{11} - 1 & a_{13} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{31} & a_{33} - 1 & 0 & 0 & 0 & 1 & 0 \\ -a_{31} & 0 & 0 & 0 & 0 & a_{31} & a_{32} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_{11} - 1 & a_{12} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_{21} & a_{22} - 1 & 0 & 0 & 1 \end{bmatrix}$$

We have $R = \lambda E^{-1}\gamma$, where $\gamma = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ and E with stochastic identities equal to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{12} + a_{13} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -a_{21} - a_{23} & a_{23} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a_{32} & -a_{31} - a_{32} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -a_{21} & 0 & 0 & a_{21} & a_{23} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -a_{12} - a_{13} & a_{13} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{31} & -a_{31} - a_{32} & 0 & 0 & 0 & 1 & 0 \\ -a_{31} & 0 & 0 & 0 & 0 & a_{31} & a_{32} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -a_{12} - a_{13} & a_{12} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_{21} & -a_{21} - a_{23} & 0 & 0 & 1 \end{bmatrix}$$

Permuting the index of the equations of U_1 , we obtain the two following systems

$$(U_2): \begin{cases} 1 = P_2^{-1} + Q_{21} + Q_{23} \\ 0 = P_2^{-1}a_{21} + Q_{12}(a_{11} - 1) + Q_{23}a_{31} \\ 0 = P_2^{-1}a_{23} + Q_{21}a_{13} + Q_{23}(a_{33} - 1) \end{cases}$$

$$(U_3): \begin{cases} 1 = P_3^{-1} + Q_{31} + Q_{32} \\ 0 = P_3^{-1}a_{31} + Q_{31}(a_{11} - 1) + Q_{32}a_{21} \\ 0 = P_3^{-1}a_{32} + Q_{31}a_{12} + Q_{32}(a_{22} - 1) \end{cases}$$

To show that $(P_1^{-1}, P_2^{-1}, P_3^{-1})M = (P_1^{-1}, P_2^{-1}, P_3^{-1})$, it suffices to prove that $P_1^{-1}a_{11} + P_2^{-1}a_{21} + P_3^{-1}a_{31} = P_1^{-1}$.

Define

$$R = P_1^{-1}a_{11} + P_2^{-1}a_{21} + P_3^{-1}a_{31} - P_1^{-1}$$

then

$$R + P_1^{-1}(1 - a_{11}) - P_2^{-1}a_{21} - P_3^{-1}a_{31} = 0$$

Under the matrical system, we have

$$(R, P_1 - 1, Q_{12}, Q_{13}, P_2^{-1}, Q_{21}, Q_{23}, P_3^{-1}, Q_{31}, Q_{32})E = \lambda$$

where

$$\lambda = (0, 0, 0, 0, 0, 0, 0, 1, 1, 1)$$

and

Let $\mathbf{F} = \begin{bmatrix} \mathbf{E} & \boldsymbol{\gamma} \\ \boldsymbol{\lambda} & 0 \end{bmatrix}$, we show that \mathbf{F} is hollow.

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a_{12} + a_{13} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -a_{21} - a_{23} & a_{23} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_{32} & -a_{31} - a_{32} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -a_{21} & 0 & 0 & a_{21} & a_{23} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_{12} - a_{13} & a_{13} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_{31} & -a_{31} - a_{32} & 0 & 0 & 0 & 1 & 0 & 0 \\ -a_{31} & 0 & 0 & 0 & 0 & a_{31} & a_{32} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{12} - a_{13} & a_{12} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{21} & -a_{21} - a_{23} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{F}_1 = \begin{bmatrix} a_{12} + a_{13} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -a_{21} - a_{23} & a_{23} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a_{32} & -a_{31} - a_{32} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -a_{21} & 0 & 0 & a_{21} & a_{23} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -a_{12} - a_{13} & a_{13} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{31} & -a_{31} - a_{32} & 0 & 0 & 0 & 1 & 0 \\ -a_{31} & 0 & 0 & 0 & 0 & a_{31} & a_{32} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -a_{12} - a_{13} & a_{12} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_{21} & -a_{21} - a_{23} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$L_1 + L_5 + L_8 - L_{10}, \quad L_2 + L_4 + L_9 - L_{10}, \quad L_3 + L_6 + L_7 - L_{10},$$

$$\mathbf{F}_2 = \begin{bmatrix} a_{12} + a_{13} & a_{12} & a_{13} & -a_{12} - a_{13} & a_{13} & -a_{12} - a_{13} & a_{12} & 0 & 0 & 0 \\ -a_{21} & -a_{21} - a_{23} & a_{23} & a_{21} & a_{23} & a_{21} & -a_{21} - a_{23} & 0 & 0 & 0 \\ -a_{31} & a_{32} & -a_{31} - a_{32} & a_{31} & -a_{31} - a_{32} & a_{31} & a_{32} & 0 & 0 & 0 \\ -a_{21} & 0 & 0 & a_{21} & a_{23} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -a_{12} - a_{13} & a_{13} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{31} & -a_{31} - a_{32} & 0 & 0 & 0 & 1 & 0 \\ -a_{31} & 0 & 0 & 0 & 0 & a_{31} & a_{32} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -a_{12} - a_{13} & a_{12} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_{21} & -a_{21} - a_{23} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{F}_3 = \begin{bmatrix} a_{12} + a_{13} & a_{12} & a_{13} & -a_{12} - a_{13} & a_{13} & -a_{12} - a_{13} & a_{12} & 0 & 0 \\ -a_{21} & -a_{21} - a_{23} & a_{23} & a_{21} & a_{23} & a_{21} & -a_{21} - a_{23} & 0 & 0 \\ -a_{31} & a_{32} & -a_{31} - a_{32} & a_{31} & -a_{31} - a_{32} & a_{31} & a_{32} & 0 & 0 \\ -a_{21} & 0 & 0 & a_{21} & a_{23} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -a_{12} - a_{13} & a_{13} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{31} & -a_{31} - a_{32} & 0 & 0 & 1 & 0 \\ -a_{31} & 0 & 0 & 0 & 0 & a_{31} & a_{32} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -a_{12} - a_{13} & a_{12} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_{21} & -a_{21} - a_{23} & 0 & 1 \end{bmatrix}$$

$$L_4 - L_6, \quad L_5 - L_6, \quad L_7 - L_9, \quad L_8 - L_9,$$

$$\mathbf{F}_4 = \begin{bmatrix} a_{12} + a_{13} & a_{12} & a_{13} & -a_{12} - a_{13} & a_{13} & -a_{12} - a_{13} & a_{12} & 0 & 0 \\ -a_{21} & -a_{21} - a_{23} & a_{23} & a_{21} & a_{23} & a_{21} & -a_{21} - a_{23} & 0 & 0 \\ -a_{31} & a_{32} & -a_{31} - a_{32} & a_{31} & -a_{31} - a_{32} & a_{31} & a_{32} & 0 & 0 \\ -a_{21} & 0 & 0 & a_{21} - a_{31} & a_{23} + a_{31} + a_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{12} - a_{13} - a_{31} & a_{13} + a_{31} + a_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{31} & -a_{31} - a_{32} & 0 & 0 & 1 & 0 \\ -a_{31} & 0 & 0 & 0 & 0 & a_{31} - a_{21} & a_{32} + a_{21} + a_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{12} - a_{13} - a_{21} & a_{12} + a_{21} + a_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{21} & -a_{21} - a_{23} & 0 & 1 \end{bmatrix}$$

$$\mathbf{F}_5 = \begin{bmatrix} a_{12} + a_{13} & a_{12} & a_{13} & -a_{12} - a_{13} & a_{13} & -a_{12} - a_{13} & a_{12} & & \\ -a_{21} & -a_{21} - a_{23} & a_{23} & a_{21} & a_{23} & a_{21} & -a_{21} - a_{23} & & \\ -a_{31} & a_{32} & -a_{31} - a_{32} & a_{31} & -a_{31} - a_{32} & a_{31} & a_{32} & & \\ -a_{21} & 0 & 0 & a_{21} - a_{31} & a_{23} + a_{31} + a_{32} & 0 & 0 & & \\ 0 & 0 & 0 & -a_{12} - a_{13} - a_{31} & a_{13} + a_{31} + a_{32} & 0 & 0 & & \\ -a_{31} & 0 & 0 & 0 & 0 & a_{31} - a_{21} & a_{32} + a_{21} + a_{23} & & \\ 0 & 0 & 0 & 0 & 0 & -a_{12} - a_{13} - a_{21} & a_{12} + a_{21} + a_{23} & & \end{bmatrix}$$

$$C_1 - C_2 - C_3, \quad C_4 + C_2 + C_3, \quad C_6 + C_2 + C_3,$$

$$\mathbf{F}_6 = \begin{bmatrix} 0 & a_{12} & a_{13} & 0 & a_{13} & 0 & a_{12} & & \\ 0 & -a_{21} - a_{23} & a_{23} & 0 & a_{23} & 0 & -a_{21} - a_{23} & & \\ 0 & a_{32} & -a_{31} - a_{32} & 0 & -a_{31} - a_{32} & 0 & a_{32} & & \\ -a_{21} & 0 & 0 & a_{21} - a_{31} & a_{23} + a_{31} + a_{32} & 0 & 0 & & \\ 0 & 0 & 0 & -a_{12} - a_{13} - a_{31} & a_{13} + a_{31} + a_{32} & 0 & 0 & & \\ -a_{31} & 0 & 0 & 0 & 0 & a_{31} - a_{21} & a_{32} + a_{21} + a_{23} & & \\ 0 & 0 & 0 & 0 & 0 & -a_{12} - a_{13} - a_{21} & a_{12} + a_{21} + a_{23} & & \end{bmatrix}$$

$$C_5 - C_3, \quad C_7 - C_2,$$

$$F_7 = \begin{bmatrix} 0 & a_{12} & a_{13} & 0 & 0 & 0 & 0 \\ 0 & -a_{21} - a_{23} & a_{23} & 0 & 0 & 0 & 0 \\ 0 & a_{32} & -a_{31} - a_{32} & 0 & 0 & 0 & 0 \\ -a_{21} & 0 & 0 & a_{21} - a_{31} & a_{23} + a_{31} + a_{32} & 0 & 0 \\ 0 & 0 & 0 & -a_{12} - a_{13} - a_{31} & a_{13} + a_{31} + a_{32} & 0 & 0 \\ -a_{31} & 0 & 0 & 0 & 0 & a_{31} - a_{21} & a_{32} + a_{21} + a_{23} \\ 0 & 0 & 0 & 0 & 0 & -a_{12} - a_{13} - a_{21} & a_{12} + a_{21} + a_{23} \end{bmatrix}$$

$$C_1 \leftrightarrow C_3$$

$$F_8 = \begin{bmatrix} a_{13} & a_{12} & 0 & 0 & 0 & 0 & 0 \\ a_{23} & -a_{21} - a_{23} & 0 & 0 & 0 & 0 & 0 \\ -a_{31} - a_{32} & a_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{21} & a_{21} - a_{31} & a_{23} + a_{31} + a_{32} & 0 & 0 \\ 0 & 0 & 0 & -a_{12} - a_{13} - a_{31} & a_{13} + a_{31} + a_{32} & 0 & 0 \\ 0 & 0 & -a_{31} & 0 & 0 & a_{31} - a_{21} & a_{32} + a_{21} + a_{23} \\ 0 & 0 & 0 & 0 & 0 & -a_{12} - a_{13} - a_{21} & a_{12} + a_{21} + a_{23} \end{bmatrix}$$

We have a 3×5 block of zero with $8 > 7$. It follows that F is hollow. By Corollary 7,

$$R = P_1^{-1}a_{11} + P_2^{-1}a_{21} + P_3^{-1}a_{31} - P_1^{-1} = 0.$$

This proof allows us to prove the next result.

Theorem 12. Let P_i be defined as above, then in \mathcal{S} we have

$$\sum_{i=1}^n P_i^{-1} = 1 \quad (8)$$

Proof. Define

$$R' = 1 - \sum_{i=1}^n P_i^{-1}$$

we will show that

$$R' + \sum_{i=1}^n P_i^{-1} = 1$$

We replace the first column of the matrix E by the vector

$$(1, 1, 0, \dots, 0, 1, 0, \dots, 0, \dots, 1, 0, \dots, 0)^T$$

and the first element of λ by 1. Denoted E' and λ' these two matrices. Consider the matrix $F' = \begin{bmatrix} E' & \gamma \\ \lambda' & 0 \end{bmatrix}$.

We repeat each step of the proof of Theorem 9, except for the last operation of step 4, which concerns the first

column. Again, we show that F' is hollow, and hence, by Corollary 7, we have $R' = 0$. This proves Equation (8).

Example 13. (Continued.)

$$\begin{aligned} P_1^{-1} + P_2^{-1} &= P_1^{-1} + P_1^{-1}(1-a)d^* \quad \text{by Equation (5)} \\ &= P_1^{-1} + P_1^{-1}bd^* = P_1^{-1}(1+bd^*) = P_1^{-1}P_1 = 1 \end{aligned}$$

4. Conclusions

One of the contribution of this article is the using of elementary operations on the lines and the columns. This method provides a new way to obtain some results in skew field theory with a minimum of knowledge of this theory. Moreover, Theorems 9 and 12 are proved exactly the same way.

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6. References

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