Advances in Pure Mathematics, 2011, 1, 204-209
doi:10.4236/apm.2011.14036 Published Online July 2011 (http://www.SciRP.org/journal/apm)

A Modified Averaging Composite Implicit Iteration Process for Common Fixed Points of a Finite Family of *k*–Strictly Asymptotically Pseudocontractive Mappings

Donatus Igbokwe, Oku Ini
Department of Mathematics, University of Uyo, Uyo, Nigeria
E-mail: {*igbokwedi, inioku*}*@yahoo.com*
Received April 4, 2011; *revised June* 3, 2011; *accepted June* 10, 2011

Abstract

The composite implicit iteration process introduced by Su and Li [J. Math. Anal. Appl. 320 (2006) 882-891] is modified. A strong convergence theorem for approximation of common fixed points of finite family of $k-$strictly asymptotically pseudo-contractive mappings is proved in Banach spaces using the modified iteration process.

Keywords: Implicit Iteration Process, $k-$Strictly Asymptotically Pseudo-Contractive Maps, Fixed Points

1. Introduction and Preliminaries

Let E be an arbitrary real Banach space and let J denote normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 ; \|x\|^2 = \|f\|^2 \right\}$$

where E^* denotes the dual space of E and $\langle , \rangle$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single-valued. In the sequel, we shall denote single-valued duality mappings by j. A mapping $T: K \to K$ is called *k-strictly asymptotically pseudocontractive* with sequence $\{a_n\} \subseteq [1,\infty)$, $\lim_{n\to\infty} a_n = 1$ (see, for example [1]) if for all $x, y \in K$, there exists $j(x-y) \in J(x-y)$ and a constant $k \in [0,1)$ such that

$$\begin{aligned}&\left\langle T^n x - T^n y, j(x-y)\right\rangle \\ &\le \frac{1}{2}(1+a_n)\|x-y\|^2 - \frac{1}{2}(1-k)\left\|x - T^n x - \left(y - T^n y\right)\right\|^2\end{aligned} \quad (1)$$

for all $n \in N$. If I denotes the identity operator, then (1) can be written in the form

$$\begin{aligned}&\left\langle \left(I - T^n\right)x - \left(I - T^n\right)y, j(x-y)\right\rangle \\ &\ge \frac{1}{2}(1-k)\left\|\left(I-T^n\right)x - \left(I-T^n\right)y\right\|^2 - \frac{1}{2}(a_n - 1)\|x-y\|^2\end{aligned} \quad (2)$$

The class of $k-$strictly asymptotically pseudocontractive maps was first introduced in Hilbert spaces by Qihou [2]. In Hilbert spaces, j is the identity and it is shown in Osilike [3] that (1) (and hence (2)) is equivalent to the the inequality

$$\left\|T^n - T^n y\right\|^2 \le a_n \|x-y\|^2 + k\left\|\left(I-T^n\right)x - \left(I-T^n\right)y\right\|^2 \quad (3)$$

which is the inequality considered by Qihou [2].

A mapping T with domain $D(T)$ and range $R(T)$ in E is called strictly pseudo-contractive in the terminology of Browder and Petryshyn [4] if there exist $\lambda > 0$ such that

$$\left\langle Tx - Ty, j(x-y)\right\rangle \le \|x-y\|^2 - \lambda\left\|x - y - (Tx - Ty)\right\|^2 \quad (4)$$

for all $x, y \in D(T)$ and for all $j(x-y) \in J(x-y)$. Without loss of generality we may assume $\lambda \in (0,1)$. If I denotes the identity operator, then (1) can be written in the form

$$\left\langle (I-T)x - (I-T)y, j(x-y)\right\rangle \ge \left\|(I-T)x - (I-T)y\right\|^2 \quad (5)$$

In the Hilbert space H, (4) (and hence (5)) is equivalent to the inequality

$$\begin{aligned}&\|Tx - Ty\|^2 \le \|x-y\|^2 + k\left\|(I-T)x - (I-T)y\right\|^2 \\ &k = (1-\lambda) < 1\end{aligned} \quad (6)$$

and we can assume also that $k \ge 0$, so that $k \in [0,1]$.

It is shown in [5] that a strictly pseudocontractive map is L-Lipschitzian ($\|Tx-Ty\| \le \|x-y\|$, for all $x, y \in D(T)$ and for some $L>0$). It is also shown in [3] that a $k-$strictly asymptotically pseudocontractive mapping is uniformly L–Lipschitzian (*i.e.* for some $L>0$, $\|T^n x - T^n y\| \le L\|x-y\|$, for all $x, y \in K$ and $n \in N$). The class of $k-$strictly asymptotically pseudocontractive mappings and the class of strictly pseudo-contractive mappings are independent (see [1]). The class of $k-$strictly asymptotically pseudocontractive mappings is a natural extension of the class of asymptotically nonexpansive mappings (*i.e.* mappings $T: K \to K$ such that

$$\|T^n x - T^n y\| \le a_n \|x-y\| \forall n \ge 1, \forall x, y \in K \qquad (7)$$

and for some sequence $\{a_n\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} a_n = 1$.) If $k=0$, we have from (3) (and hence (1)) that T is asymptotically nonexpansive. In fact, an asymptotically nonexpansive map is $0-$strictly asymptotically pseudocontractive (see Remark 1 [6]). T is called asymptotically quasi-nonexpansive if there exists a sequence $\{a_n\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} a_n = 1$, and

$$\|T^n x - p\| \le a_n \|x-p\|, \forall n \ge 1 \qquad (8)$$

for all $x \in K$ and $p \in F(T) = \{x \in K : Tx = x\}$

In [7], Xu and Ori introduced an implicit iteration process and proved weak convergence theorem for approximation of common fixed points of a finite family of nonexpansive mappings (*i.e.* a subclass of asymptotically nonexpansive mappings for which $\|Tx-Ty\| \le \|x-y\|$, $\forall x, y \in K$). In [8], Osilike extended the results of [7] from nonexpansive mappings to strictly pseudocontractive mappings. In [9], Su and Li introduced a new implicit iteration process and called it composite implicit iteration process. Using the new implicit iteration process, they proved the results established by Osilike in [8]. In compact form, the composite iteration process introduced in [9] is the sequence $\{x_n\}$ generated from arbitrary $x_0 \in K$ by

$$\left.\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n) T_n y_n \\ y_n &= \beta_n x_{n-1} + (1-\beta_n) T_n x_n \end{aligned}\right\} \qquad (9)$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$. In [10] Sun modified the implicit iteration process of Xu and Ori and applied the modified iteration process for the approximation of fixed points of a finite family of asymptotically quasi-nonexpansive maps. In compact form, the modified implicit iteration process of Sun is the sequence $\{x_n\}$ generated from arbitrary $x_0 \in K$ by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_i^k x_n, n \ge 1 \qquad (10)$$

where $n = (k-1)N + i, i \in I = \{1, 2, \cdots, N\}$.

In this paper, we modify (9) as follows. Let K be a nonempty closed convex subset of E, $\{T_i\}_{i=1}^N$ a finite family of $k-$strictly asymptotically pseudocontractive self-maps of K, then for $x_0 \in K$ and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$.

$$\begin{array}{lll} x_1 &=& \alpha_1 x_0 + (1-\alpha_1) T_1 y_1, \\ x_2 &=& \alpha_2 x_1 + (1-\alpha_2) T_2 y_2, \\ \vdots && \\ x_N &=& \alpha_N x_{N-1} + (1-\alpha_N) T_N y_N, \\ x_{N+1} &=& \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1^2 y_{N+1}, \\ x_{N+2} &=& \alpha_{\alpha_{n+2}} x_{N+1} + (1-\alpha_{N+2}) T_2^2 y_{N+2}, \\ \vdots && \\ x_{2N} &=& \alpha_{2N} x_{2N-1} + (1-\alpha_{2N}) T_N^2 y_{2N}, \\ x_{2N+1} &=& \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1}) T_1^3 y_{2N+1}, \\ x_{2N+2} &=& \alpha_{2N+2} x_{2N+1} + (1-\alpha_{2N+2}) T_2^3 y_{2N+2}, \\ \vdots && \end{array} \qquad \begin{array}{lll} y_1 &=& \beta_1 x_0 + (1-\beta_1) T_1 x_1 \\ y_2 &=& \beta_2 x_1 + (1-\beta_2) T_2 x_2 \\ \vdots && \\ y_N &=& \beta_N x_{N-1} + (1-\beta_N) T_N x_N \\ y_{N+1} &=& \beta_{N+1} x_N + (1-\beta_{N+1}) T_1^2 x_{N+1} \\ y_{N+2} &=& \beta_{N+2} x_N + (1-\beta_{N+2}) T_N^2 x_{N+2} \\ \vdots &=& \\ y_{2N} &=& \beta_{2N} x_{2N-1} + (1-\beta_{2N}) T_N^2 x_{2N} \\ y_{2N+1} &=& \beta_{2N+1} x_{2N} + (1-\beta_{2N+1}) T_1^3 x_{2N+1} \\ y_{2N+1} &=& \beta_{2N+2} x_{2N+1} + (1-\beta_{2N+2}) T_2^3 x_{2N+2} \\ \vdots && \end{array}$$

Our iteration process can be expressed in a compact form as

$$\left.\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n) T_i^k y_n \\ y_n &= \beta_n x_{n-1} + (1-\beta_n) T_i^k x_n \end{aligned}\right\} n \ge 1 \qquad (11)$$

where $n = (k-1)N + i, i = \{1, 2, \cdots, N\}$. Observe that if $T: K \to K$ is $k-$strictly asymptotically pseudocontractive mapping with sequence $\{a_n\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} a_n = 1$, then for every fixed $u \in K$ and $t, s \in \{L/(1+L), 1\}$, the operator $S_{t,s,n}: K \to K$ defined for all $x \in K$ by $S_{t,s,n} x = tu(1-t) T^n (su + (1-s) T^n x)$ satisfies $\|S_{t,s,n} x - S_{t,s,n} y\| \le (1-t)(1-s) L^2 \|x-y\|$, $\forall x, y \in K$. Since $(1-t)(1-s) L^2 \in (0,1)$, it follows that

$S_{t,s,n}$ is a contraction map and hence has a unique fixed point $x_{t,s,n}$ in K. This implies that there exists a unique $x_{t,s,n} \in K$ such that $x_{t,s,n} = tu + (1-t)T^n\left(su + (1-s)T^n x_{t,s,n}\right)$. Thus our modified composite implicit iteration process (11) is defined in K for the family $\{T_i\}_{i=1}^N$ of N $k-$strictly asymptotically pseudocontractive self maps of a nonempty convex subset K of a Banach space provided $\alpha_n, \beta_n \in (\eta, 1)$ where $\eta = L/(1+L)$ and $L = \max_{1\le N}\{L_i\}$.

The purpose of this paper is to study the convergence of the new modified averaging implicit iteration scheme (11) to a common fixed point of a finite family of $k-$strictly asymptotically pseudocontractive maps in arbitrary Banach spaces. The results presented in this paper, generalize the result of Su and Li [9] and several others in the literature (see for example [8], [11], [10], [7]).

In the sequel, we shall need the following:

Lemma 1.1 ***OAA*** ([3], p. 80):

Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be three sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \le (1+b_n)a_n + \delta_n, n \ge 1 \qquad (12)$$

If $\sum \delta_n < \infty$ and $\sum b_n < \infty$ then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Definition 1.1 [12] *A bounded convex subset K of a real Banach space E is said to have normal structure if every nontrivial convex subset C of K contains at least one nondimetrial point. That is, there exists $x_0 \in E$ such that*

$$sup\{\|x_0 - x\| : x \in C\} < sup\{\|x - y\| : x, y \in C = d(C)\}$$

where $d(C)$ is the diameter of C

Every uniformly convex Banach space and every compact convex subset K of a Banach space E has normal structure. For the definition of modulus of convexity of E and the characteristic of convexity ε_0 of E, see [13].

Theorem 1.1 ([13] *Corollary* 3.6)

Let E be a real Banch space with normal structure $N(E) > max(1, \varepsilon_0)$, $\varepsilon_0 > 0$, K a nonempty bounded closed convex subset of E and $T: K \to K$ a uniformly $L-$Lipschitzian mapping with $L < \alpha$, $\alpha > 1$. Then T has a fixed point.

2. Main Results

Lemma 2.1 *Let E be a real Banach space with normal structure $N(E) > max(1, \varepsilon_0)$ and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N k_i- strictly asymptotically pseudo-contractive self-maps of K with sequences $\{a_{in}\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(a_{in} - 1) < \infty$, and let $F = \cap F(T_i) \ne \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\} \subset (\eta, 1)$ be two real sequences satisfying the conditions*:

(*i*) $\sum_{n=1}^{\infty}(1 - a_n) = \infty$, (*ii*) $\sum_{n=1}^{\infty}(1 - \alpha_n)^2 < \infty$,

(*iii*) $\sum_{n=1}^{\infty}(1 - \beta_n) < \infty$, (*iv*) $(1-\alpha_n)(1-\beta_n)L^2 < 1$,

where $\eta = L/(1+L)$ and $L = \max_{1\le i\le N}\{L_i\}$, L_i the Lipschitzian constants of $\{T_i\}_{i=1}^N$. Let $\{x_n\}$ be the implicit iteration sequence generated by (11). Then

(*i*) $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$.

(*ii*) $d(x_n, F)$ exists, where

$d(x_n, F) = \inf_{p\in F}\|x_n - p\|$

(*iii*) $\liminf_{n\to\infty}\|x_n - T_n x_n\| = 0$.

Proof

The existence of fixed point follows from Theorem 1.1. We shall use the well known inequality (see for example [7,14])

$$\|x + y\|^2 \le \|x\|^2 + \langle y, j(x-y)\rangle \qquad (13)$$

which holds for all $x, y \in E$ and for all $j(x-y) \in J(x-y)$. Let $p \in F$, then using (11) and (13) we obtain

$$\begin{aligned}
\|x_n - p\|^2 &= \left\|\alpha_n(x_{n-1} - p) + (1-\alpha_n)(T_i^k y_n - p)\right\|^2 \\
&\le \alpha_n^2\|x_{n-1} - p\|^2 + 2(1-\alpha_n)\langle T_i^k y_n - p, j(x_n - p)\rangle \\
&= \alpha_n^2\|x_{n-1} - p\|^2 + 2(1-\alpha_n) \\
&\cdot\left[\langle T_i^k y_n - T_i^k x_n, j(x_n - p)\rangle + \langle T_i^k x_n, j(x_n - p)\rangle\right] \\
&\le \alpha_n^2\|x_{n-1} - p\|^2 + 2(1-\alpha_n) \\
&\cdot\left[L\|y_n - x_n\|\|x_n - p\| + \|x_n - p\|^2\right] \\
&-2(1-\alpha_n)\langle x_n - T_i^k x_n, j(x_n - p)\rangle
\end{aligned} \qquad (14)$$

Since each $T_i : K \to K$, $i \in I$, is k_i- strictly asymptotically pseudocontractive, then

$$\begin{aligned}
&\langle (I - T_i^n)x - (I - T_i^n)y, j(x-y)\rangle \\
&\ge \frac{1}{2}(1-k_i)\left\|x - T_i^n x - (y - T_i^n y)\right\|^2 - \frac{1}{2}(a_{im} - 1)\|x - y\|^2
\end{aligned}$$

$k_i \in [0,1)$. Let $k = \min_{1\le i\le N}\{k_i\}$. Then

$$\left\langle \left(I-T_i^n\right)x-\left(I-T_i^n\right)y, j\left(x-y\right)\right\rangle \ge \frac{1}{2}(1-k)\left\|x-T_i^n x-\left(y-T_i^n y\right)\right\|^2-\frac{1}{2}\left(a_{im}-1\right)\|x-y\|^2$$

Thus it follows from (14) that

$$\|x_n-p\|^2 \le \alpha_n^2\|x_{n-1}-p\|^2+\left(1-\alpha_n\right)L\|y_n-x_n\|\|x_n-p\| +\left(1-\alpha_n\right)\left[2+\left(\alpha_n-1\right)\right]\|x_n-p\|^2 -\left(1-k\right)\left(1-\alpha_n\right)\left\|x_n-T_i^k x_n\right\|^2 \quad (15)$$

Observe that

$$\|y_n-x_n\| \le \beta_n\left(1-\alpha_n\right)\left\|T_i^k y_n-x_{n-1}\right\|+\left(1-\beta_n\right)\left\|x_n-T_i^k x_n\right\| \quad (16)$$

$$\left\|T_i^k y_n-x_{n-1}\right\| \le \left(L\beta_n+1\right)\|x_{n-1}-p\|+L^2\left(1-\beta_n\right)\|x_n-p\| \quad (17)$$

and

$$\left\|x_n-T_i^k x_n\right\| \le \left(L+1\right)\|x_n-p\| \quad (18)$$

Substituting (16)-(18) into (15), we obtain

$$\Big[1-2\left(1-\alpha_n\right)^2 L^3\beta_n(1-\beta_n)-2\left(1-\alpha_n\right)\left(1-\beta_n\right)L\left(L+1\right) -\left(1-\alpha_n\right)\left(2+\left(a_{ik}-1\right)\right)\Big]\|x_n-p\|^2 \le \alpha_n^2\|x_{n-1}-p\|^2+2\left(1-\alpha_n\right)^2 L\beta_n\left(L\beta_n+1\right) \cdot\|x_{n-1}-p\|\|x_n-p\|-\left(1-k\right)\left(1-\alpha_n\right)\left\|x_n-T_i^k x_n\right\|^2 \quad (19)$$

Observe that $\left(a_{ik}-1\right)\le\left(a_{in}-1\right)$, $\forall k=n$, since $n=\left(k-1\right)N+i$, $\forall i\in I=\{1,2,\cdots,N\}$. Setting

$$b_n=2\left(1-\alpha_n\right)^2 L^2\beta_n\left(1-\beta_n\right) +2\left(1-\alpha_n\right)^2\left(1-\beta_n\right)L\left(L+1\right)+\left(1-\alpha_n\right)\left(a_n-1\right)$$

then it follows from (19) that

$$\|x_n-p\|^2 \le \left[1+\frac{\left(1-\alpha_n\right)^2+b_n}{1-2\left(1-\alpha_n\right)-b_n}\right]\|x_{n-1}-p\|^2 +\left[\frac{2\left(1-\alpha_n\right)^2 L\beta_n\left(L\beta_n+1\right)}{1-2\left(1-\alpha_n\right)-b_n}\right]\|x_{n-1}-p\|\|x_n-p\| -\left[\frac{(1-\alpha_n)(1-k)}{1-2(1-\alpha_n)-b_n}\right]\left\|x_n-T_i^k x_n\right\|^2 \quad (20)$$

Since

$$1-2\left(1-\alpha_n\right)-b_n =1-\left(1-\alpha_n\right)\Big[2+\left(a_{in}-1\right) +2\left(1-\alpha_n\right)L^3\beta_n\left(1-\beta_n\right)+2\left(1-\beta_n\right)L\left(L+1\right)\Big]$$

and $\{\alpha_n\}\{\beta_n\}\subseteq\left(\eta,1\right)$, then we obtain that

$$2+\left(a_{in}-1\right)+2\left(1-\alpha_n\right)L^3\beta_n\left(1-\beta_n\right)+2\left(1-\beta_n\right)L\left(L+1\right) \le 2+\left(a_m-1\right)+2L^3+2L\left(L+1\right)$$

Setting $M_1=2+2L^3+2L\left(L+1\right)$, then there must exist a natural number N_1, such that if $n>N_1$ then $\frac{1}{1-2\left(1-\alpha_n\right)-b_n}<2$, (since $\sum_{n=1}^{\infty}\left(1-\alpha_n\right)^2<\infty$ and $\sum_{n=1}^{\infty}\left(a_{in}-1\right)<\infty$). Therefore it follows from (20) that

$$\|x_n-p\|^2 \le \left[1+2\left\{\left(1-\alpha_n\right)^2+b_n\right\}\right]\|x_{n-1}-p\|^2 +2\left[2\left(1-\alpha_n\right)^2 L\beta_n\left(L\beta_n+1\right)\right]\|x_{n-1}-p\|\|x_n-p\| -\left(1-\alpha_n\right)\left(1-k\right)\left\|x_n-T_i^k x_n\right\|^2 -\left(1-\alpha_n\right)\left(1-k\right)\left\|x_n-T_i^k x_n\right\|^2 \quad (21)$$

Observe that,

$$\|x_n-p\|^2=\left\langle x_n-p, j\left(x_n-p\right)\right\rangle =\alpha_n\left\langle x_{n-1}-p, j\left(x_n-p\right)\right\rangle+\left(1-\alpha_n\right) \cdot\left\langle T_i^k y_n-p, j\left(x_n-p\right)\right\rangle =\alpha_n\left\langle x_{n-1}-p, j\left(x_n-p\right)\right\rangle+\left(1-\alpha_n\right) \cdot\left\langle T_i^k y_n-T_i^k x_n, j\left(x_n-p\right)\right\rangle+\left(1-\alpha_n\right) \cdot\left\langle T_i^k x_n-p, j\left(x_n-p\right)\right\rangle \le\alpha_n\|x_{n-1}-p\|\|x_n-p\|+L\left(1-\alpha_n\right) \cdot\|y_n-x_n\|\|x_n-p\|+\left(1-\alpha_n\right)L\|x_n-p\|^2 \quad (22)$$

Substituting (16)-(18) into (21) and simplifying this inequalities, we have

$$\Big[1-\left(1-\alpha_n\right)L-L^3\left(1-\alpha_n\right)^2\beta_n\left(1-\beta_n\right) -L\left(1-\alpha_n\right)\left(1-\beta_n\right)\left(L+1\right)\Big]\|x_n-p\|^2 \le\left[\alpha_n+L\left(1-\alpha_n\right)^2\beta_n\left(L\beta_n+1\right)\right]\|x_{n-1}-p\|\|x_n-p\|$$

$$\|x_n - p\| \le \frac{\alpha_n + L(1-\alpha_n)^2 \beta_n (L\beta_n + 1)}{1-(1-\alpha_n)L - L^3(1-\alpha_n)^2 \beta_n (1-\beta_n) - L(1-\alpha_n)(1-\beta_n)(L+1)} \|x_{n-1} - p\|$$
$$= \left[1 + \frac{L^3(1-\alpha_n)^2 \beta_n (1-\beta_n) + L(1-\alpha_n)(1-\beta_n)(L+1) + L(1-\alpha_n)^2 \beta_n (L\beta_n+1) - (1-\alpha_n)}{1-(1-\alpha_n)L - L^3(1-\alpha_n)^2 \beta_n (1-\beta_n) - L(1-\alpha_n)(1-\beta_n)(L+1)}\right] \|x_{n-1} - p\| \quad (23)$$
$$\le \left[1 + \frac{L^3(1-\alpha_n)^2 \beta_n (1-\beta_n) + L(1-\alpha_n)(1-\beta_n)(L+1) + L(1-\alpha_n)^2 \beta_n (L\beta_n+1)}{1-(1-\alpha_n)L - L^3(1-\alpha_n)^2 \beta_n (1-\beta_n) - L(1-\alpha_n)(1-\beta_n)(L+1)}\right] \|x_{n-1} - p\|$$

Now, we consider the second term on the right-hand side of (23). Since $\{\alpha_n\}\{\beta_n\} \subseteq (\eta, 1)$, then

$$(1-\alpha_n)\left[L + L^3(1-\alpha_n)\beta_n(1-\beta_n) + L(1-\beta_n)(L+1)\right]$$
$$\le (1-\alpha_n)\left[L + L^3 + L(L+1)\right]$$

Set $M_2 = \left[L + L^3 + L(L+1)\right]$. Since $\lim_{n\to\infty}(1-\alpha_n) = 0$, then there exists a natural number N_2, such that if $n > N_2$ then

$$1-(1-\alpha_n)L - L^3(1-\alpha_n)^2 \beta_n(1-\beta_n)$$
$$-L(1-\alpha_n)(1-\beta_n)(L+1) \ge \frac{1}{2}$$

Again it follows from the condition $\{\alpha_n\}\{\beta_n\} \subseteq (\eta, 1)$, that

$$L^3(1-\alpha_n)^2 \beta_n(1-\beta_n) + L(1-\alpha_n)(1-\beta_n)(L+1)$$
$$+L(1-\alpha_n)^2 \beta_n(L\beta_n+1)$$
$$\le L^3(1-\alpha_n)^2 + L(1-\beta_n)(L+1) + L(1-\alpha_n)^2(L+1)$$

Therefore it follows from (23) that

$$\|x_n - p\| \le \Big\{1 + 2\Big[L^3(1-\alpha_n)^2 + L(1-\beta_n)(L+1)$$
$$+L(1-\alpha_n)^2(L+1)\Big]\Big\} \|x_{n-1} - p\| \quad (24)$$
$$= (1+\delta_n)\|x_{n-1} - p\|$$

where

$$\delta_n = 2\left[L^3(1-\alpha_n)^2 + L(1-\beta_n)(L+1) + L(1-\alpha_n)^2(L+1)\right]$$

From conditions (ii), (iii) it is easy to see that

$$\sum_{n=1}^{\infty}\Big\{2\Big[L^3(1-\alpha_n)^2 + L(1-\beta_n)(L+1) + L(1-\alpha_n)^2(L+1)\Big]\Big\}$$
$$< +\infty$$

Thus using Lemma *OOA*, we have $\lim_{n\to\infty}\|x_n - p\|$ exists, completing the proof of (i). Also it follows from (24) that $d(x_n, F) \le [1+\delta_n]\, d(x_{n-1}, F)$, and it again follows from Lemma *OOA* that $\lim_{n\to\infty}$ exists, this completes the proof of (ii).

Now, we consider the second term on the right-hand side of (21). Since $\{x_n\}$ is bounded, $\{\beta_n\} \subseteq (\eta, 1)$, then there exists a constant $M_3 > 0$ such that

$$2\left[2(1-\alpha_n)^2 L\beta_n(L\beta_n+1)\right]\|x_{n-1} - p\|\|x_n - p\|$$
$$\le 4(1-\alpha_n)^2 M_3$$

Thus, it follows from (21) that

$$\|x_n - p\|^2 \le \left[1 + 2\left\{(1-\alpha_n)^2 + b_n\right\}\right]\|x_{n-1} - p\|^2$$
$$+2(1-\alpha_n)^2 M_3 - (1-\alpha_n)(1-k)\left\|x_n - T_i^k x_n\right\| \quad (25)$$

Since $\{x_n\}$ is bounded, then there exists a constant $M_4 > 0$ such that $\|x_n - p\|^2 \le M_4$. It follows from (25) that

$$(1-k)(1-\alpha_n)\left\|x_n - T_i^k x_n\right\|^2$$
$$\le 2M_4\left\{(1-\alpha_n)^2 + b_n\right\} + 4M_3(1-\alpha_n)^2$$
$$+\|x_{n-1} - p\|^2 - \|x_n - p\|^2$$

Hence,

$$(1-k)\sum_{j=N+1}^{n}(1-\alpha_j)\left\|x_j - T_i^k x_j\right\|^2$$
$$\le 2M_4\sum_{j=N+1}^{n}\left\{(1-\alpha_j)^2 + b_n\right\} + 4M_3\sum_{j=N+1}^{n}(1-\alpha_j)^2 \quad (26)$$
$$+\|x_N - p\|^2 < \infty$$

Using condition (ii) and (iii), it follows from (26) that $\sum_{n=1}^{\infty}(1-\alpha_n)\left\|x_n - T_i^k x_n\right\|^2 < \infty$, and using condition (i), $\liminf_{n\to\infty}\left\|x_n - T_i^k x_n\right\| = 0$. Thus $\liminf_{n\to\infty}\left\|x_n - T_n^k x_n\right\| = 0$.

For all $n > N$ we have $T_n = T_{n-N}$ so that

$$\|x_n - T_n x_n\| \le \left\|x_n - T_n^k x_n\right\| + \left\|T_n^k x_n - T_n x_n\right\|$$
$$\le \left\|x_n - T_n^k x_n\right\| + L\left\|T_n^{k-1} x_n - x_n\right\|$$

Thus, $\liminf_{n\to\infty}\|x_n - T_n x_n\| = 0$, completing the proof of (*iii*).

Theorem 2.1 *Let E be a real Banach space with normal structure $N(E) > max(1, \varepsilon_0)$ and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{x_n\}$ be as in Lemma*2.1. *Then $\{x_n\}$ exists in K and converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$ if and only if*

$\liminf_{n\to\infty} d(x_n, F) = 0$ where

$d(x_n, F) = \inf_{p\in F}\|x_n - p\|$.

PROOF

The existence of fixed point follows from Theorem 1.1. If $\{x_n\}$ converges strongly to a common fixed point of of the mappings $\{T_i\}_{i=1}^N$, then $\liminf_{n\to\infty}\|x_n - p\| = 0$. Since $0 \le d(x_n, F) \le \|x_n - p\|$, we have

$\liminf_{n\to\infty}(x_n, F) = 0$.

Conversely, suppose $\liminf_{n\to\infty}(x_n, F) = 0$ then our Lemma implies that $\lim_{n\to\infty} d(x_n, F) = 0$. Thus for arbitrary $\varepsilon > 0$, there exists a positive integer N_3 such that $d(x_n, F) < \varepsilon/4$, $\forall n \ge N_3$. Furthermore $\sum_{n=1}^{\infty}\delta_n < \infty$ implies that there exists a positive integer N_4 such that $\sum_{j=n}^{\infty}\delta_j < \frac{\varepsilon}{4M_4}$, $\forall n \ge N_4$. Choose $N = \max\{N_3, N_4\}$, then $d(x_n, F) \le \varepsilon/4$ and $\sum_{j=N}^{\infty}\delta_j \le \frac{\varepsilon}{4M_4}$. For all $n, m \ge N$ and for all $p \in F$ we have

$$\begin{aligned}\|x_n - x_m\| &\le \|x_n - p\| + \|x_m - p\| \\ &\le \|x_n - p\| + M_4\sum_{j=N+1}^{n}\delta_j + \|x_N - p\| + M_4\sum_{j=N+1}^{m}\delta_j \\ &\le 2\|x_N - p\| + 2M_4\sum_{j=N}^{\infty}\delta_j\end{aligned}$$

Taking infimum over all $p \in F$, we obtain

$$\|x_n - x_m\| \le 2d(x_N, F) + 2M_4\sum_{j=N}^{\infty}\delta_j \le \frac{2\varepsilon}{4} + \frac{2M\varepsilon}{4M} = \varepsilon$$

Thus $\{x_n\}$ is Cauchy. Suppose $\lim_{n\to\infty} x_n = u$. Then $u \in K$ since K is closed. Furthermore, since $F(T_i)$ is closed for all $i \in I$, we have that F is closed. Since $\lim_{n\to\infty} d(x_n, F) = 0$, we must have that $u \in F$.

3. References

[1] M. O. Osilike, A. Udomene, D. I. Igbokwe and B. G. Akuchu, “Demiclosedness Principle and Convergence Theorems for K-Strictly Asymptotically Pseudocontractive Maps,” *Journal of Mathematical Analysis and Applications*, Vol. 326, No. 2, 2007, pp. 1334-1345. doi:10.1016/j.jmaa.2005.12.052

[2] L. Qihou, “Convergence Theorems of the Sequence of Iterates for Asymptotially Demicontractive and Hemi-contractive Mappings,” *Nonlinear Analysis*, Vol. 26, No. 11, 1996, 1835-1842. doi:10.1016/0362-546X(94)00351-H

[3] M. O. Osilike, S. C. Aniagboso and G. B. Akachu, “Fixed Points of Asymptotically Demicontractive Mapping in Arbitrary Banach Space,” *Pan American Mathematical Journal*, Vol. 12, No. 2, 2002, pp. 77-88.

[4] F. E. Browder and W. V. Petryshyn, “Construction of Fixed Points of Nonlinear Mappings in Hilbert Spaces,” *Journal of Mathematical Analysis and Applications*, Vol. 20, No. 2, 1967, pp. 197-228. doi:10.1016/0022-247X(67)90085-6

[5] M. O. Osilike and A. Udomene, “Demiclosedness Principle and Convergence Results for Strictly Pseudocontractive Mappings of Browder-Petryshyn Type,” *Journal of Mathematical Analysis and Applications*, Vol. 256, No. 2, 2001, pp. 431-445. doi:10.1006/jmaa.2000.7257

[6] H. Zhou, “Demiclosedness Principle with Applictions for Asymptotically Pseudocontractions in Hilbert Spaces,” *Nonlinear Analysis*, Vol. 70, No. 9, 2009, pp. 3140-3145. doi:10.1016/j.na.2008.04.017

[7] H. K Xu and R. G. Ori, “An Implicit Iteration Process for Nonexpansive Mappings,” *Numerical Functional Analysis and Optimization*, Vol. 22, No. 5-6, 2001, pp. 767-733. doi:10.1081/NFA-100105317

[8] M. O. Osilike, “Implicit Iteration Process for Common Fixed Point of a Finite Family of Strictly Pseudo-contractive Maps,” *Journal of Mathematical Analysis and Applications*, Vol. 294, No. 1, 2004, pp. 73-81. doi:10.1016/j.jmaa.2004.01.038

[9] Y. Su and S. Li, “Composite Implicit Iteration Process for Common Fixed Points of a Finite Family of Strictly Pseudocontractive Maps,” *Journal of Mathematical Analysis and Applications*, Vol. 320, No. 2, 2006, pp. 882-891. doi:10.1016/j.jmaa.2005.07.038

[10] Z. H. Sun, “Strong Convergence of an Implicit Iteration Process for a Finite Family of Asymptotically Quasi-nonexpansive Mappings,” *Journal of Mathematical Analysis and Applications*, Vol. 286, No. 1, 2003, pp. 351-358. doi:10.1016/S0022-247X(03)00537-7

[11] M. O. Osilike and B. G. Akuchu, “Common Fixed Points of a Finite Family of Asymptotically Pseudocontractive Maps,” *Journal Fixed Points and Applications*, Vol. 2004, No. 2, 2004, pp. 81-88. doi:10.1155/S1687182004312027

[12] C. E. Chidume, “Functional Analysis: Fundamental Theorems with Application,” International Centre for Theoretical Physics, Trieste Italy, 1995.

[13] L. C. Ceng, H. K. Xu and J. C. Yao, “Uniformly Normal Structure and Uniformly Lipschitzian Semigroups,” *Nonlinear Analysis*, Vol. 73, No. 12, 2010, pp. 3742-3750. doi:10.1016/j.na 2010.o7.044.

[14] W. V. Petryshyn, “A Characterization of Strict Convexity of Banach Spaces and Other Uses of Duality Mappings,” *Journal of Functional Analysis*, Vol. 6, No. 2, 1970, pp. 282-291. doi:10.1016/0022-1236(70)90061-3

