Advances in Pure Mathematics, 2011, 1, 221-227
doi:10.4236/apm.2011.14039 Published Online July 2011 (http://www.SciRP.org/journal/apm)

Multiplication and Translation Operators on the Fock Spaces for the *q*-Modified Bessel Function*

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Received February 28, 2011; *revised April* 8, 2011; *accepted April* 20, 2011

Abstract

We study the multiplication operator M by z^2 and the q-Bessel operator $\Delta_{q,\alpha}$ on a Hilbert spaces $\mathbb{F}_{q,\alpha}$ of entire functions on the disk $D\left(o,\frac{1}{1-q}\right)$, $0<q<1$; and we prove that these operators are adjoint-operators and continuous from $\mathbb{F}_{q,\alpha}$ into itself. Next, we study a generalized translation operators on $\mathbb{F}_{q,\alpha}$.

Keywords: Generalized q-Fock Spaces, q-I_α Modified Bessel Function, q-Bessel Operator, Multiplication Operator, q-Translation Operators

1. Introduction

In 1961, Bargmann [1] introduced a Hilbert space $\mathbb{F}$ of entire functions $f(z)=\sum_{n=0}^{\infty}a_n z^n$ on $\mathbb{C}$ such that

$$\|f\|_{\mathbb{F}}^2 := \sum_{n=0}^{\infty}|a_n|^2\, n! < \infty$$

On this space the author studied the differential operator $D = \mathrm{d}/\mathrm{d}z$ and the multiplication operator by z, and proved that these operators are densely defined, closed and adjoint-operators on $\mathbb{F}$ (see [1]).

Next, the Hilbert space $\mathbb{F}$ is called Segal-Bargmann space or Fock space and it was the aim of many works [2].

In 1984, Cholewinski [3] introduced a Hilbert space $\mathbb{F}_\alpha$ of even entire functions on $\mathbb{C}$, where the inner product is weighted by the modified Macdonald function. On $\mathbb{F}_\alpha$ the Bessel operator

$$\Delta_\alpha := \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{2\alpha+1}{z}\frac{\mathrm{d}}{\mathrm{d}z}, \quad \alpha > -1/2$$

and the multiplication by z^2 are densely defined, closed and adjoint-operators.

In this paper, we consider the q-I_α modified Bessel function:

$$I_\alpha\left(x;q^2\right) := \sum_{n=0}^{\infty}\frac{x^{2n}}{b_{2n}\left(\alpha;q^2\right)}$$

where $b_{2n}\left(\alpha;q^2\right)$ are given later in Section 2. We define the q-Fock space $\mathbb{F}_{q,\alpha}$ as the Hilbert space of even entire functions $f(z)=\sum_{n=0}^{\infty}a_n z^{2n}$ on the disk $D\left(o,\frac{1}{1-q}\right)$ of center o and radius $\frac{1}{1-q}$, and such that

$$\|f\|_{\mathbb{F}_{q,\alpha}}^2 := \sum_{n=0}^{\infty}|a_n|^2\, b_{2n}\left(\alpha;q^2\right) < \infty$$

Let f and g be in $\mathbb{F}_{q,\alpha}$, such that $f(z)=\sum_{n=0}^{\infty}a_n z^{2n}$ and $g(z)=\sum_{n=0}^{\infty}c_n z^{2n}$, the inner product is given by

$$\langle f,g\rangle_{\mathbb{F}_{q,\alpha}} = \sum_{n=0}^{\infty}a_n\overline{c_n}\, b_{2n}\left(\alpha;q^2\right) < \infty$$

Next, we consider the multiplication operator M by z^2 and the q-Bessel operator $\Delta_{q,\alpha}$ on the Fock space $\mathbb{F}_{q,\alpha}$, and we prove that these operators are continuous from $\mathbb{F}_{q,\alpha}$ into itself, and satisfy:

*Author partially supported by DGRST project 04/UR/15-02 and CMCU program 10G 1503.

$$\left\|\Delta_{q,\alpha} f\right\|_{\mathbb{F}_{q,\alpha}} \leq \frac{1}{1-q}\|f\|_{\mathbb{F}_{q,\alpha}}$$

$$\|Mf\|_{\mathbb{F}_{q,\alpha}} \leq \frac{1}{1-q}\|f\|_{\mathbb{F}_{q,\alpha}}$$

Then, we prove that these operators are adjoint-operators on $\mathbb{F}_{q,\alpha}$:

$$\langle Mf, g\rangle_{\mathbb{F}_{q,\alpha}} = \langle f, \Delta_{q,\alpha} g\rangle_{\mathbb{F}_{q,\alpha}}; \quad f, g \in \mathbb{F}_{q,\alpha}$$

Lastly, we define and study on the Fock space $\mathbb{F}_{q,\alpha}$, the q-translation operators:

$$T_z f(w) := I_\alpha\left(z\Delta_{q,\alpha}^{1/2}; q^2\right) f(w); \quad w, z \in D\left(o, \frac{1}{1-q}\right)$$

and the generalized multiplication operators:

$$M_z f(w) := I_\alpha\left(zM^{1/2}; q^2\right) f(w); \quad w, z \in D\left(o, \frac{1}{1-q}\right).$$

Using the previous results, we deduce that the operators T_z and M_z, for $z \in D\left(o, \frac{1}{1-q}\right)$, are continuous from $\mathbb{F}_{q,\alpha}$ into itself, and satisfy:

$$\|T_z f\|_{\mathbb{F}_{q,\alpha}} \leq I_\alpha\left(\frac{|z|}{\sqrt{1-q}}; q^2\right)\|f\|_{\mathbb{F}_{q,\alpha}}$$

$$\|M_z f\|_{\mathbb{F}_{q,\alpha}} \leq I_\alpha\left(\frac{|z|}{\sqrt{1-q}}; q^2\right)\|f\|_{\mathbb{F}_{q,\alpha}}$$

2. The q-Fock Spaces $\mathbb{F}_{q,\alpha}$

Let a and q be real numbers such that $0 < q < 1$; the q-shifted factorial are defined by

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{i=0}^{n-1}\left(1-aq^i\right), \quad n = 1, 2, \cdots, \infty$$

Jackson [5] defined the q-analogue of the Gamma function as

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, \cdots$$

It satisfies the functional equation

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x), \quad \Gamma_q(1) = 1$$

and tends to $\Gamma(x)$ when q tends to 1^-. In particular, for $n = 1, 2, \cdots$, we have

$$\Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n}$$

The q-combinatorial coefficients are defined for $n, k \in \mathbb{N}$, $k = 0, \cdots, n$, by

$$\binom{n}{k}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \frac{\Gamma_q(n+1)}{\Gamma_q(k+1)\Gamma_q(n-k+1)} \quad (1)$$

The q-derivative $D_q f$ of a suitable function f (see [6]) is given by

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0$$

and $D_q f(0) = f'(0)$ provided $f'(0)$ exists.

If f is differentiable then $D_q f(x)$ tends to $f'(x)$ as $q \to 1^-$.

Taking account of the paper [4] and the same way, we define the q-I_α modified Bessel function by

$$I_\alpha\left(x;q^2\right) := \sum_{n=0}^{\infty}\frac{x^{2n}}{b_{2n}\left(\alpha;q^2\right)}$$

where

$$b_{2n}\left(\alpha;q^2\right) := \frac{(1+q)^{2n}\Gamma_{q^2}(n+1)\Gamma_{q^2}(n+\alpha+1)}{\Gamma_{q^2}(\alpha+1)} \quad (2)$$

If we put $U_n = \frac{1}{b_{2n}\left(\alpha;q^2\right)}$, then

$$\frac{U_n}{U_{n+1}} \to \frac{1}{(1-q)^2}, \quad q \to 1^-$$

Thus, the q-I_α modified Bessel function is defined on $D\left(o, \frac{1}{(1-q)^2}\right)$ and tends to the I_α modified Bessel function as $q \to 1^-$.

In [4], the authors study in great detail the q-Bessel operator denoted by

$$\Delta_{q,\alpha} f(x) := D_q^2 f(x) + \frac{[2\alpha+1]_q}{x} D_q f(qx)$$

where

$$[2\alpha+1]_q := \frac{1-q^{2\alpha+1}}{1-q}$$

The q-Bessel operator tends to the Bessel operator Δ_α as $q \to 1^-$.

Lemma 1: 1) *The function* $I_\alpha\left(\lambda.;q^2\right), \lambda \in D\left(o, \frac{1}{1-q}\right)$, *is the unique analytic solution of the q-problem*:

$$\Delta_{q,\alpha} y(x) = \lambda^2 y(x),\ y(0) = 1 \text{ and } D_q y(0) = 0 \quad (3)$$

2) For $n \in \mathbb{N}$, we have

$$\Delta_{q,\alpha} z^{2n} = \frac{b_{2n}(\alpha;q^2)}{b_{2(n-1)}(\alpha;q^2)} z^{2(n-1)}, \quad n \geq 1$$

3) The constants $b_{2n}(\alpha;q^2)$, $n \in \mathbb{N}$ satisfy the following relation:

$$b_{2n+2}(\alpha;q^2) = [2n+2]_q [2n+2\alpha+2]_q b_{2n}(\alpha;q^2)$$

Let $\alpha \geq -1/2$. The q-Fock space $\mathbb{F}_{q,\alpha}$ is the Hilbert space of even entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^{2n}$ on $D\left(o, \frac{1}{1-q}\right)$, such that

$$\|f\|^2_{\mathbb{F}_{q,\alpha}} := \sum_{n=0}^{\infty} |a_n|^2 b_{2n}(\alpha;q^2) < \infty \qquad (4)$$

where $b_{2n}(\alpha;q^2)$ is given by (2).

The inner product in $\mathbb{F}_{q,\alpha}$ is given for $f(z) = \sum_{n=0}^{\infty} a_n z^{2n}$ and $g(z) = \sum_{n=0}^{\infty} c_n z^{2n}$ by

$$\langle f, g \rangle_{\mathbb{F}_{q,\alpha}} = \sum_{n=0}^{\infty} a_n \overline{c_n} b_{2n}(\alpha;q^2) \qquad (5)$$

Remark 1: If $q \to 1^-$, the space $\mathbb{F}_{q,\alpha}$ agrees with the generalized Fock space associated to the Bessel operator (see [3]).

Theorem 1: *The function* $\kappa_{q,\alpha}$ *given for* $w, z \in D\left(o, \frac{1}{1-q}\right)$, *by*

$$\kappa_{q,\alpha}(w,z) = I_\alpha(\overline{w}z;q^2)$$

is a reproducing kernel for the q-Fock space $\mathbb{F}_{q,\alpha}$, that is:

1) For all $w \in D\left(o, \frac{1}{1-q}\right)$, the function $z \to \kappa_{q,\alpha}(w,z)$ belongs to $\mathbb{F}_{q,\alpha}$.

2) For all $w \in D\left(o, \frac{1}{1-q}\right)$ and $f \in \mathbb{F}_{q,\alpha}$, we have

$$\langle f, \kappa_{q,\alpha}(w,.) \rangle_{\mathbb{F}_{q,\alpha}} = f(w)$$

Remark 2: From Theorem 1, 2), for $f \in \mathbb{F}_{q,\alpha}$ and $w \in D\left(o, \frac{1}{1-q}\right)$, we have

$$|f(w)| \leq \|\kappa_{q,\alpha}(w,.)\|_{\mathbb{F}_{q,\alpha}} \|f\|_{\mathbb{F}_{q,\alpha}} = \left[I_\alpha(|w|^2;q^2)\right]^{1/2} \|f\|_{\mathbb{F}_{q,\alpha}}$$

3. Multiplication and q-Bessel Operators on $\mathbb{F}_{q,\alpha}$

On $\mathbb{F}_{q,\alpha}$, we consider the multiplication operators M and N_q given by

$$Mf(z) := z^2 f(z)$$

$$N_q f(z) := zD_q f(z) = \frac{f(z) - f(qz)}{1-q}$$

We denote also by $\Delta_{q,\alpha}$ the q-Bessel operator defined for entire functions on $D\left(o, \frac{1}{1-q}\right)$.

We write

$$[\Delta_{q,\alpha}, M] = \Delta_{q,\alpha} M - M \Delta_{q,\alpha}$$

By straightforward calculation we obtain the following result.

Lemma 2: $[\Delta_{q,\alpha}, M] = (1+q)[2\alpha+2]_q B_{q^2} + W_{q,\alpha}$, where

$$B_q(z) := f(qz)$$

and

$$W_{q,\alpha} f(z) := (1+q)(1+q^{2\alpha}) qzD_q(f)(qz) \qquad (6)$$

Remark 3: The Lemma 2 is the analogous commutation rule of Cholewinski [3]. When $q \to 1^-$, then $[\Delta_{q,\alpha}, M]$ tends to $4(\alpha+1)I + 4z\frac{\mathrm{d}}{\mathrm{d}z}$, where I is the identity operator.

Lemma 3: *If* $f \in \mathbb{F}_{q,\alpha}$ *then* $B_q f$, $N_q f$ *and* $W_{q,\alpha} f$ *belong to* $\mathbb{F}_{q,\alpha}$, *and*

1) $\|B_q f\|_{\mathbb{F}_{q,\alpha}} \leq \|f\|_{\mathbb{F}_{q,\alpha}}$,

2) $\|N_q f\|_{\mathbb{F}_{q,\alpha}} \leq \frac{1}{1-q} \|f\|_{\mathbb{F}_{q,\alpha}}$,

3) $\|W_{q,\alpha} f\|_{\mathbb{F}_{q,\alpha}} \leq \frac{(1+q)(1+q^{2\alpha})}{1-q} \|f\|_{\mathbb{F}_{q,\alpha}}$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{2n} \in \mathbb{F}_{q,\alpha}$, then

$$B_q f(z) = f(qz) = \sum_{n=0}^{\infty} a_n q^{2n} z^{2n} \qquad (7)$$

$$N_q f(z) = \frac{f(z) - f(qz)}{1-q} = \sum_{n=0}^{\infty} a_n [2n]_q z^n \qquad (8)$$

and from (4), we obtain

$$\left\|B_q f\right\|^2_{\mathbb{F}_{q,\alpha}} = \sum_{n=0}^{\infty}|a_n|^2 q^{4n} b_{2n}\left(\alpha;q^2\right) \le \sum_{n=0}^{\infty}|a_n|^2 b_{2n}\left(\alpha;q^2\right) = \|f\|^2_{\mathbb{F}_{q,\alpha}}$$

and

$$\left\|N_q f\right\|^2_{\mathbb{F}_{q,\alpha}} = \sum_{n=0}^{\infty}|a_n|^2 \left([2n]_q\right)^2 b_{2n}\left(\alpha;q^2\right)$$

Using the fact that $[2n]_q \le \frac{1}{1-q}$, we deduce

$$\left\|N_q f\right\|^2_{\mathbb{F}_{q,\alpha}} \le \frac{1}{(1-q)^2}\sum_{n=0}^{\infty}|a_n|^2 b_{2n}\left(\alpha;q^2\right) = \frac{1}{(1-q)^2}\|f\|^2_{\mathbb{F}_{q,\alpha}}$$

On the other hand from (6), we have

$$W_{q,\alpha} f(z) = (1+q)\left(1+q^{2\alpha}\right)\sum_{n=1}^{\infty}a_n [2n]_q q^{2n} z^{2n} \qquad (9)$$

and

$$\left\|W_{q,\alpha} f\right\|^2_{\mathbb{F}_{q,\alpha}} = \left[(1+q)\left(1+q^{2\alpha}\right)\right]^2 \cdot \sum_{n=1}^{\infty}|a_n|^2 \left([2n]_q\right)^2 q^{4n} b_{2n}\left(\alpha;q^2\right)$$

Using the fact that $[2n]_q \le \frac{1}{1-q}$, we deduce that

$$\left\|W_{q,\alpha} f\right\|^2_{\mathbb{F}_{q,\alpha}} \le \frac{\left[(1+q)\left(1+q^{2\alpha}\right)\right]^2}{(1-q)^2}\sum_{n=1}^{\infty}|a_n|^2 b_{2n}\left(\alpha;q^2\right)$$

Therefore, we conclude that

$$\left\|W_{q,\alpha} f\right\|_{\mathbb{F}_{q,\alpha}} \le \frac{(1+q)\left(1+q^{2\alpha}\right)}{1-q}\|f\|_{\mathbb{F}_{q,\alpha}}$$

which completes the proof of the Lemma. □

Theorem 2: *If $f \in \mathbb{F}_{q,\alpha}$ then $\Delta_{q,\alpha} f$ and Mf belong to $\mathbb{F}_{q,\alpha}$, and we have*

1) $\left\|\Delta_{q,\alpha} f\right\|_{\mathbb{F}_{q,\alpha}} \le \frac{1}{1-q}\|f\|_{\mathbb{F}_{q,\alpha}}$,

2) $\|Mf\|_{\mathbb{F}_{q,\alpha}} \le \frac{1}{1-q}\|f\|_{\mathbb{F}_{q,\alpha}}$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{2n} \in \mathbb{F}_{q,\alpha}$.

1) From Lemma 1, 2),

$$\Delta_{q,\alpha} f(z) = \sum_{n=1}^{\infty} a_n \frac{b_{2n}\left(\alpha;q^2\right)}{b_{2(n-1)}\left(\alpha;q^2\right)} z^{2(n-1)} = \sum_{n=0}^{\infty} a_{n+1} \frac{b_{2n+2}\left(\alpha;q^2\right)}{b_{2n}\left(\alpha;q^2\right)} z^{2n} \qquad (10)$$

Then from (10), we get

$$\left\|\Delta_{q,\alpha} f\right\|^2_{\mathbb{F}_{q,\alpha}} = \sum_{n=0}^{\infty}|a_{n+1}|^2 \frac{b_{2n+2}\left(\alpha;q^2\right)}{b_{2n}\left(\alpha;q^2\right)} b_{2n+2}\left(\alpha;q^2\right)$$

Using Lemma 1, 3), we obtain

$$\left\|\Delta_{q,\alpha} f\right\|^2_{\mathbb{F}_{q,\alpha}} = \sum_{n=0}^{\infty}|a_{n+1}|^2 [2n+2]_q [2n+2\alpha+2]_q b_{2n+2}\left(\alpha;q^2\right)$$

and consequently,

$$\left\|\Delta_{q,\alpha} f\right\|^2_{\mathbb{F}_{q,\alpha}} = \sum_{n=1}^{\infty}|a_n|^2 [2n]_q [2n+2\alpha]_q b_{2n}\left(\alpha;q^2\right) \qquad (11)$$

Using the fact that $[2n]_q [2n+2\alpha]_q \le \frac{1}{(1-q)^2}$, we obtain

$$\left\|\Delta_{q,\alpha} f\right\|_{\mathbb{F}_{q,\alpha}} \le \frac{1}{1-q}\left[\sum_{n=1}^{\infty}|a_n|^2 b_{2n}\left(\alpha;q^2\right)\right]^{1/2} = \frac{1}{1-q}\|f\|_{\mathbb{F}_{q,\alpha}}$$

2) On the other hand, since

$$Mf(z) = \sum_{n=1}^{\infty} a_{n-1} z^{2n} \qquad (12)$$

then

$$\|Mf\|^2_{\mathbb{F}_{q,\alpha}} = \sum_{n=1}^{\infty}|a_{n-1}|^2 b_{2n}\left(\alpha;q^2\right) = \sum_{n=0}^{\infty}|a_n|^2 b_{2n+2}\left(\alpha;q^2\right)$$

By Lemma 1, 3), we deduce

$$\|Mf\|^2_{\mathbb{F}_{q,\alpha}} = \sum_{n=0}^{\infty}|a_n|^2 [2n+2]_q [2n+2\alpha+2]_q b_{2n}\left(\alpha;q^2\right) \qquad (13)$$

Using the fact that $[2n+2]_q [2n+2\alpha+2]_q \le \frac{1}{(1-q)^2}$, we obtain

$$\|Mf\|_{\mathbb{F}_{q,\alpha}} \le \frac{1}{1-q}\|f\|_{\mathbb{F}_{q,\alpha}}$$

We deduce also the following norm equalities. □

Theorem 3: *If $f \in \mathbb{F}_{q,\alpha}$ then*

1) $\left\langle f, W_{q,\alpha} f\right\rangle_{\mathbb{F}_{q,\alpha}} = (1+q)\left(1+q^{2\alpha}\right)\left\langle N_q f, B_q f\right\rangle_{\mathbb{F}_{q,\alpha}}$,

2) $\left\|\Delta_{q,\alpha} f\right\|^2_{\mathbb{F}_{q,\alpha}} = \left\|N_q f\right\|^2_{\mathbb{F}_{q,\alpha}} + [2\alpha]_q \left\langle N_q f, B_q f\right\rangle_{\mathbb{F}_{q,\alpha}}$,

3) $\|Mf\|^2_{\mathbb{F}_{q,\alpha}} = \left\|N_q f\right\|^2_{\mathbb{F}_{q,\alpha}} + (1+q)[2\alpha+2]_q \left\|B_q f\right\|^2_{\mathbb{F}_{q,\alpha}} + \left(1+q+[2\alpha+2]_q\right)\left\langle N_q f, B_q f\right\rangle_{\mathbb{F}_{q,\alpha}}$,

4) $\|Mf\|^2_{\mathbb{F}_{q,\alpha}} = \left\|\Delta_{q,\alpha} f\right\|^2_{\mathbb{F}_{q,\alpha}} + (1+q)[2\alpha+2]_q \left\|B_q f\right\|^2_{\mathbb{F}_{q,\alpha}} + \left\langle f, W_{q,\alpha} f\right\rangle_{\mathbb{F}_{q,\alpha}}$.

Proof. Let $f(z)=\sum_{n=0}^{\infty}a_n z^{2n}\in\mathbb{F}_{q,\alpha}$.

1) Follows from (7), (8) and (9).

2) From (11), we get

$$\left\|\Delta_{q,\alpha}f\right\|_{\mathbb{F}_{q,\alpha}}^2=\sum_{n=0}^{\infty}|a_n|^2[2n]_q[2n+2\alpha]_q\,b_{2n}(\alpha;q^2)$$

Using the fact $[2n+2\alpha]_q=[2n]_q+q^{2n}[2\alpha]_q$, we deduce

$$\left\|\Delta_{q,\alpha}f\right\|_{\mathbb{F}_{q,\alpha}}^2=\left\|N_qf\right\|_{\mathbb{F}_{q,\alpha}}^2+[2\alpha]_q\left\langle N_qf,B_qf\right\rangle_{\mathbb{F}_{q,\alpha}}$$

3) By (13) and using the fact that

$$\begin{aligned}&[2n+2]_q[2n+2\alpha+2]_q\\&=\left([2n]_q\right)^2+\left(1+q+[2\alpha+2]_q\right)q^{2n}[2n]_q\\&\quad+(1+q)[2\alpha+2]_q\,q^{4n}\end{aligned}$$

we obtain

$$\begin{aligned}\left\|Mf\right\|_{\mathbb{F}_{q,\alpha}}^2&=\left\|N_qf\right\|_{\mathbb{F}_{q,\alpha}}^2+(1+q)[2\alpha+2]_q\left\|B_qf\right\|_{\mathbb{F}_{q,\alpha}}^2\\&\quad+\left(1+q+[2\alpha+2]_q\right)\left\langle N_qf,B_qf\right\rangle_{\mathbb{F}_{q,\alpha}}\end{aligned}$$

4) Follows directly from 1), 2) and 3). □

Remark 4: Let $f(z)=\sum_{n=0}^{\infty}a_n z^{2n}\in\mathbb{F}_{q,\alpha}$. Since $\left\langle f,W_{q,\alpha}f\right\rangle_{\mathbb{F}_{q,\alpha}}\geq 0$, then

$$\left\|Mf\right\|_{\mathbb{F}_{q,\alpha}}^2\geq(1+q)[2\alpha+2]_q\left\|B_qf\right\|_{\mathbb{F}_{q,\alpha}}^2$$

Therefore $Mf=0$ implies that $f=0$. Then $M:\mathbb{F}_{q,\alpha}\to\mathbb{F}_{q,\alpha}$ is an injective continuous operator on $\mathbb{F}_{q,\alpha}$.

Proposition 1: *The operators M and $\Delta_{q,\alpha}$ are adjoint-operators on $\mathbb{F}_{q,\alpha}$; and for all $f,g\in\mathbb{F}_{q,\alpha}$, we have*

$$\left\langle Mf,g\right\rangle_{\mathbb{F}_{q,\alpha}}=\left\langle f,\Delta_{q,\alpha}g\right\rangle_{\mathbb{F}_{q,\alpha}}$$

Proof. Consider $f(z)=\sum_{n=0}^{\infty}a_n z^{2n}$ and $g(z)=\sum_{n=0}^{\infty}c_n z^{2n}$ in $\mathbb{F}_{q,\alpha}$. From (10) and (12),

$$\Delta_{q,\alpha}g(z)=\sum_{n=0}^{\infty}c_{n+1}\frac{b_{2n+2}(\alpha;q^2)}{b_{2n}(\alpha;q^2)}z^{2n}$$

and

$$Mf(z)=\sum_{n=1}^{\infty}a_{n-1}z^{2n}$$

Thus from (5), we get

$$\begin{aligned}\left\langle Mf,g\right\rangle_{\mathbb{F}_{q,\alpha}}&=\sum_{n=1}^{\infty}a_{n-1}\overline{c_n}b_{2n}(\alpha;q^2)\\&=\sum_{n=0}^{\infty}a_n\overline{c_{n+1}}b_{2n+2}(\alpha;q^2)\\&=\left\langle f,\Delta_{q,\alpha}g\right\rangle_{\mathbb{F}_{q,\alpha}}\end{aligned}$$

which gives the result. □

4. Generalized Multiplication and Translation Operators on $\mathbb{F}_{q,\alpha}$

In this section, we study a generalized multiplication and translation operators on $\mathbb{F}_{q,\alpha}$.

Definition 2: For $f\in\mathbb{F}_{q,\alpha}$, and $w,z\in D\left(o,\frac{1}{1-q}\right)$, we define:

-The q-translation operators on $\mathbb{F}_{q,\alpha}$, by

$$\tau_z f(w):=\sum_{n=0}^{\infty}\frac{\Delta_{q,\alpha}^n f(w)}{b_{2n}(\alpha;q^2)}z^{2n}\tag{14}$$

-The generalized multiplication operators on $\mathbb{F}_{q,\alpha}$, by

$$M_z f(w):=\sum_{n=0}^{\infty}\frac{M^n f(w)}{b_{2n}(\alpha;q^2)}z^{2n}\tag{15}$$

For $w,z\in D\left(o,\frac{1}{1-q}\right)$, the function $I(.;q^2)$ satisfies the following product formulas:

$$\tau_z I_\alpha(.;q^2)(w)=I_\alpha(z;q^2)I_\alpha(w;q^2)$$

$$M_z I_\alpha(.;q^2)(w)=I_\alpha(wz;q^2)I_\alpha(w;q^2)$$

Remark 5: If $q\to 1^-$, we obtain the generalized translation operator given in ([3], page 181).

Proposition 2: *Let* $f(z)=\sum_{n=0}^{\infty}a_n z^{2n}\in\mathbb{F}_{q,\alpha}$ *and* $z,w\in D\left(o,\frac{1}{1-q}\right)$. *Then*

1)

$$\tau_z f(w)=\sum_{n=0}^{\infty}a_n\left[\sum_{k=0}^{n}\binom{n}{k}_{q^2}\cdot\frac{\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(n+\alpha+1)}{\Gamma_{q^2}(k+\alpha+1)\Gamma_{q^2}(n-k+\alpha+1)}\left(\frac{z}{w}\right)^{2k}\right]w^{2n}.$$

2) $M_z f(w)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{a_{n-k}}{b_{2k}(\alpha;q^2)}z^{2k}\right]w^{2n}$.

Proof. 1) Let $f(z)=\sum_{n=0}^{\infty}a_n z^{2n}\in\mathbb{F}_{q,\alpha}$. From (14), we have

$$\tau_z f(w)=\sum_{n=0}^{\infty}\frac{\Delta_{q,\alpha}^n f(w)}{b_{2n}(\alpha;q^2)}z^{2n};\quad w,z\in D\left(o,\frac{1}{1-q}\right)$$

Since from Lemma 1, 2),

$$\Delta_{q,\alpha}^n w^{2k}=\frac{b_{2k}(\alpha;q^2)}{b_{2(k-n)}(\alpha;q^2)}w^{2(k-n)},\quad k\ge n$$

we can write

$$\Delta_{q,\alpha}^n f(w)=\sum_{k=n}^{\infty}a_k\frac{b_{2k}(\alpha;q^2)}{b_{2(k-n)}(\alpha;q^2)}w^{2(k-n)}$$

Thus we obtain

$$\tau_z f(w)=\sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\frac{b_{2n}(\alpha;q^2)}{b_{2k}(\alpha;q^2)b_{2(n-k)}(\alpha;q^2)}w^{2(n-k)}z^{2k}$$

On the other hand from (1) and (2), we get

$$\frac{b_{2n}(\alpha;q^2)}{b_{2k}(\alpha;q^2)b_{2(n-k)}(\alpha;q^2)}=\binom{n}{k}_{q^2}\frac{\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(n+\alpha+1)}{\Gamma_{q^2}(k+\alpha+1)\Gamma_{q^2}(n-k+\alpha+1)}$$

which gives the 1).

2) From (15), we have

$$M_z f(w)=\sum_{n=0}^{\infty}\frac{M^n f(w)}{b_{2n}(\alpha;q^2)}z^{2n};\quad w,z\in D\left(o,\frac{1}{1-q}\right)$$

But from (12), we have

$$M^n f(w)=\sum_{k=n}^{\infty}a_{k-n}w^{2k}$$

Thus we obtain

$$M_z f(w)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{a_{n-k}}{b_{2k}(\alpha;q^2)}z^{2k}\right]w^{2n}\qquad\square$$

According to Theorem 2 we study the continuous property of the operators T_z and M_z on $\mathbb{F}_{q,\alpha}$.

Theorem 4: *If* $f\in\mathbb{F}_{q,\alpha}$ *and* $z\in D\left(o,\frac{1}{1-q}\right)$, *then* $T_z f$ *and* $M_z f$ *belong to* $\mathbb{F}_{q,\alpha}$, *and we have*

1) $\|T_z f\|_{\mathbb{F}_{q,\alpha}}\le I_\alpha\left(\frac{|z|}{\sqrt{1-q}};q^2\right)\|f\|_{\mathbb{F}_{q,\alpha}}$,

2) $\|M_z f\|_{\mathbb{F}_{q,\alpha}}\le I_\alpha\left(\frac{|z|}{\sqrt{1-q}};q^2\right)\|f\|_{\mathbb{F}_{q,\alpha}}$.

Proof. From (14) and Theorem 2, 1), we deduce

$$\|T_z f\|_{\mathbb{F}_{q,\alpha}}\le\sum_{n=0}^{\infty}\left\|\Delta_{q,\alpha}^n f\right\|_{\mathbb{F}_{q,\alpha}}\frac{|z|^{2n}}{b_{2n}(\alpha;q^2)}\le\sum_{n=0}^{\infty}\frac{|z|^{2n}}{(1-q)^n b_{2n}(\alpha;q^2)}\|f\|_{\mathbb{F}_{q,\alpha}}$$

Therefore,

$$\|T_z f\|_{\mathbb{F}_{q,\alpha}}\le I_\alpha\left(\frac{|z|}{\sqrt{1-q}};q^2\right)\|f\|_{\mathbb{F}_{q,\alpha}}$$

which gives the first inequality, and as in the same way we prove the second inequality of this theorem. □

From Proposition 1 we deduce the following results.

Proposition 3: *For all* $f,g\in\mathbb{F}_{q,\alpha}$, *we have*

$$\langle M_z f,g\rangle_{\mathbb{F}_{q,\alpha}}=\langle f,T_{\bar z}g\rangle_{\mathbb{F}_{q,\alpha}}$$

$$\langle T_z f,g\rangle_{\mathbb{F}_{q,\alpha}}=\langle f,M_{\bar z}g\rangle_{\mathbb{F}_{q,\alpha}}$$

We denote by R_z the following operator defined on $\mathbb{F}_{q,\alpha}$ by

$$R_z:=T_{\bar z}M_z-M_{\bar z}T_z=I_\alpha\left(\bar z\Delta_{q,\alpha}^{1/2};q^2\right)I_\alpha\left(zM^{1/2};q^2\right)-I_\alpha\left(\bar zM^{1/2};q^2\right)I_\alpha\left(z\Delta_{q,\alpha}^{1/2};q^2\right)$$

Then, we prove the following theorem.

Theorem 5. *For all* $f\in\mathbb{F}_{q,\alpha}$, *we have*

$$\|M_z f\|_{\mathbb{F}_{q,\alpha}}^2=\|T_z f\|_{\mathbb{F}_{q,\alpha}}^2+\langle f,R_z f\rangle_{\mathbb{F}_{q,\alpha}}$$

Proof. From Proposition 3, we get

$$\|M_z f\|_{\mathbb{F}_{q,\alpha}}^2=\langle f,T_{\bar z}M_z f\rangle_{\mathbb{F}_{q,\alpha}}=\langle f,(M_{\bar z}T_z+R_z)f\rangle_{\mathbb{F}_{q,\alpha}}=\|T_z f\|_{\mathbb{F}_{q,\alpha}}^2+\langle f,R_z f\rangle_{\mathbb{F}_{q,\alpha}}\qquad\square$$

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